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Composition Operators on Poisson Weighted Sequence Spaces

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Abstract. The aim of this paper is to foster interaction between operator theory and probability. In this paper, we introduce Poisson weighted sequence space $l^p(\lambda)$ $\{\lambda > 0, 1 \leq p \leq \infty\}$ and observe that it is a Banach space. Also find a necessary and sufficient condition for composition transformation C_ϕ to be bounded. Then we pass to characterize null space and range space of composition operators. We establish a necessary and a sufficient condition for range space of C_ϕ to be closed. Further, we determine condition under which composition operator is injective or surjective. Finally, we report an explicit expression for the adjoint operator C_ϕ^* of composition operators on Hilbert space $l^2(\lambda)$ and study the above mentioned properties for C_ϕ^* on $l^2(\lambda)$.

1. Introduction

The notion of Composition operators appeared implicitly in the work of Hardy and Littlewood [6] in 1925. A systematic study of this class of operators began by Ryff [8] and Nordgren [4]. The term Composition Operators was coined by Nordgren [4] in his paper entitled ‘Composition Operators’. Ever since, this class of operators have enjoyed constant attention. An excellent overview of them is given in [7], [11].

Definition 1.1. *Let X be a non-empty set and $V(X)$ be a linear space of complex valued functions on X under pointwise addition and scalar multiplication. If ϕ is a selfmap on X into itself such that composition $fo\phi$ belongs to $V(X)$ for each $f \in V(X)$, then ϕ induces a linear transformation on $V(X)$ into itself given by $C_\phi f = fo\phi$. The transformation C_ϕ is known as composition transformation. When $V(X)$ is a Banach space and C_ϕ is a bounded linear operator on $V(X)$, then C_ϕ is called composition operator.*

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1.1. *Notation and Terminology*

In this paper, $\mathbb{N}_0$ and $\mathbb{C}$ denote the set of all non-negative integers and the set of all complex numbers respectively. Further $\chi_n : \mathbb{N}_0 \to \mathbb{N}_0$ is defined as

$$\chi_n(m) = \begin{cases} 1, & \text{if } m = n \\ 0, & \textit{otherwise}. \end{cases}$$

Also, whenever p occurs alone we assume that $1 \leq p < \infty$; and whenever p and q occur together, we assume that both are greater than 1 and that $\frac{1}{p} + \frac{1}{q} = 1$.

1.2. *Poisson distribution*

Poisson distribution is named after French Mathematician Simon-Denis Poisson, who introduced it in 1837. For the details of Poisson distribution we refer to [1].

Definition 1.2. *Poisson distributation with parameter $\lambda > 0$ is defined as $w(n) = e^{-\lambda}\frac{\lambda^n}{n!}$, where $n \in \mathbb{N}_0$.*

Definition 1.3. *For $\lambda > 0$ we define Poisson weighted sequence space as*

$$l^p(\lambda) = \{f : \mathbb{N}_0 \to \mathbb{C} | \sum_{n\in\mathbb{N}_0} \frac{e^{-\lambda}\lambda^n}{n!} |f(n)|^p < \infty\}$$

and

$$l^\infty(\lambda) = \{f : \mathbb{N}_0 \to \mathbb{C} | \sup_{n\in\mathbb{N}_0} \frac{e^{-\lambda}\lambda^n}{n!} |f(n)| < \infty\}.$$

2. Main Results

The following proposition shows that $l^p(\lambda)$ is a normed linear space for $\lambda > 0$.

Proposition 2.1. $l^p(\lambda) = \{f : \mathbb{N}_0 \to \mathbb{C} | \sum_{n\in\in\mathbb{N}_0} \frac{e^{-\lambda}\lambda^n}{n!} |f(n)|^p < \infty\}$ *is normed linear space with norm*

$$\|f\|_p = \Big(\sum_{n\in\in\mathbb{N}_0} \frac{e^{-\lambda}\lambda^n}{n!} |f(n)|^p \Big)^{1/p}.$$

Proof. We prove only triangle inequality as verification of other properties is straightforward. For $p = 1$ the result is immediate. Let $1 < p < \infty$ and $f, g \in l^p(\lambda)$. Consider

$$\begin{aligned}
\sum_{n\in\mathbb{N}_0} |f(n) + g(n)|^p \frac{e^{-\lambda}\lambda^n}{n!} &\leq \sum_{n\in\mathbb{N}_0} \frac{e^{-\lambda}\lambda^n}{n!} |f(n) + g(n)|^{p-1} (|f(n)| + |g(n)|) \\
&= \sum_{n\in\mathbb{N}_0} (\frac{e^{-\lambda}\lambda^n}{n!})^{\frac{1}{p}} |f(n)||f(n) + g(n)|^{p-1} (\frac{e^{-\lambda}\lambda^n}{n!})^{\frac{1}{q}} \\
&+ \sum_{n\in\mathbb{N}_0} |g(n)| (\frac{e^{-\lambda}\lambda^n}{n!})^{\frac{1}{p}} |f(n) + g(n)|^{p-1} (\frac{e^{-\lambda}\lambda^n}{n!})^{\frac{1}{q}} \\
&\leq (\sum_{n\in\mathbb{N}_0} (\frac{e^{-\lambda}\lambda^n}{n!} |f(n)|^p)^{\frac{1}{p}} (\sum_{n\in\mathbb{N}_0} |f(n) + g(n)|^p (\frac{e^{-\lambda}\lambda^n}{n!}))^{\frac{1}{q}} \\
&+ (\sum_{n\in\mathbb{N}_0} (\frac{e^{-\lambda}\lambda^n}{n!} |g(n)|^p)^{\frac{1}{p}} (\sum_{n\in\mathbb{N}_0} |f(n) + g(n)|^p (\frac{e^{-\lambda}\lambda^n}{n!}))^{\frac{1}{q}}.
\end{aligned}$$

Last inequality is obtained by Holder's inequality. Thus

$$\sum_{n\in\mathbb{N}_0} |f(n)+g(n)|^p \frac{e^{-\lambda}\lambda^n}{n!} \quad \leq \|f\|_p \big(\sum_{n\in\mathbb{N}_0} |f(n)+g(n)|^p (\tfrac{e^{-\lambda}\lambda^n}{n!})\big)^{\frac{1}{q}} + \|g\|_p \big(\sum_{n\in\mathbb{N}_0} |f(n)+g(n)|^p (\tfrac{e^{-\lambda}\lambda^n}{n!})\big)^{\frac{1}{q}}.$$

This implies

$$\Big(\sum_{n\in\mathbb{N}_0} |f(n)+g(n)|^p \frac{e^{-\lambda}\lambda^n}{n!}\Big)^{1-\frac{1}{q}} \leq \|f\|_p + \|g\|_p$$

i.e.

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p.$$

□

Now we show that $l^p(\lambda)$ is a Banach space for $\lambda > 0$.

Proposition 2.2. *$l^p(\lambda)$ is a Banach space with the norm $\|f\|_p = \Big(\sum_{n\in\mathbb{N}_0} \frac{e^{-\lambda}\lambda^n}{n!} |f(n)|^p\Big)^{1/p}$.*

Proof. Let $\{f_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $l^p(\lambda)$. For given $\epsilon > 0$, there exists a positive integer n_0 such that

$$\|f_n - f_m\|_p < \epsilon \ \forall\ m, n \geq n_0$$

i.e.

$$\Big(\sum_{r\in\mathbb{N}_0} \frac{e^{-\lambda}\lambda^r}{r!} |f_n(r) - f_m(r)|^p\Big)^{1/p} < \epsilon \ \forall\ m, n \geq n_0.$$

This implies that sequence of scalars $\{f_n(r)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{C}$ for each $r \in \mathbb{N}_0$. Since $\mathbb{C}$ is complete, sequence $\{f_n(r)\}_{n\in\mathbb{N}}$ is convergent for each $r \in \mathbb{N}_0$. We denote $f(r) = \lim_{n\to\infty} f_n(r)$. Let us take $f = \sum_{r\in\mathbb{N}_0} f(r)\chi_r$ and consider

$$\begin{aligned}
\Big(\sum_{r=0}^{k} \frac{e^{-\lambda}\lambda^r}{r!} |f(r)|^p\Big)^{\frac{1}{p}} &= \Big(\lim_{n\to\infty} \Big(\sum_{r=0}^{k} \frac{e^{-\lambda}\lambda^r}{r!} |f_n(r)|^p\Big)^{\frac{1}{p}} \\
&\leq \lim_{n\to\infty} \Big(\sum_{r=0}^{k} \frac{e^{-\lambda}\lambda^r}{r!} |f_n(r) - f_{n_0}(r)|^p\Big)^{\frac{1}{p}} + \Big(\sum_{r=0}^{k} \frac{e^{-\lambda}\lambda^r}{r!} |f_{n_0}(r)|^p\Big)^{\frac{1}{p}} \\
&\leq \lim_{n\to\infty} \Big(\sum_{r\in\mathbb{N}_0} \frac{e^{-\lambda}\lambda^r}{r!} |f_n(r) - f_{n_0}(r)|^p\Big)^{\frac{1}{p}} + \Big(\sum_{r\in\mathbb{N}_0} \frac{e^{-\lambda}\lambda^r}{r!} |f_{n_0}(r)|^p\Big)^{\frac{1}{p}} \\
&\leq \epsilon + \|f_{n_0}\|_p \ \forall\ k \in \mathbb{N}_0.
\end{aligned}$$

This implies that $f \in l^p(\lambda)$. Now for $m \geq n_0$, again we consider

$$\begin{aligned}
\sum_{r=0}^{k} \frac{e^{-\lambda}\lambda^r}{r!} |f_m(r) - f(r)|^p &= \lim_{n\to\infty} \sum_{r=0}^{k} \frac{e^{-\lambda}\lambda^r}{r!} |f_m(r) - f_n(r)|^p \\
&\leq \lim_{n\to\infty} \sum_{r\in\mathbb{N}_0} \frac{e^{-\lambda}\lambda^r}{r!} |f_m(r) - f_n(r)|^p \\
&\leq \epsilon^p \ \forall\ k \in \mathbb{N}_0.
\end{aligned}$$

This implies that f_n converges to f in $l^p(\lambda)$. □

Remark 2.3. *It can also be easily shown that $l^\infty(\lambda)$ is a Banach space under the norm $\|f\|_\infty = \sup\limits_{n\in\mathbb{N}_0} \frac{e^{-\lambda}\lambda^n}{n!}|f(n)|$.*
We now give a necessary and sufficient condition for C_ϕ to be bounded.

Theorem 2.4. *C_ϕ is bounded on $l^p(\lambda)$ if and only if there exists a real number $M > 0$ such that*

$$\sum_{m\in\phi^{-1}(n)} \frac{e^{-\lambda}\lambda^m}{m!} \leq M\frac{e^{-\lambda}\lambda^n}{n!}\ \forall\ n \in \mathbb{N}_0.$$

Proof. If C_ϕ is bounded, then

$$\begin{aligned}\sum_{m\in\phi^{-1}(n)} \frac{e^{-\lambda}\lambda^m}{m!} &= \|C_\phi(\chi_n)\|_p^p \\ &\leq \|C_\phi\|_p^p\|\chi_n\|_p^p \\ &= \|C_\phi\|_p^p\frac{e^{-\lambda}\lambda^n}{n!}.\end{aligned}$$

Now put $M = \|C_\phi\|_p^p$, then we get desired condition. Conversely, assume there exists $M > 0$ such that

$$\sum_{m\in\phi^{-1}(n)} \frac{e^{-\lambda}\lambda^m}{m!} \leq M\frac{e^{-\lambda}\lambda^n}{n!}\ \forall n \in \mathbb{N}_0.$$

Then,

$$\begin{aligned}\|C_\phi(f)\|_p^p &= \|f\circ\phi\|_p^p \\ &= \|\sum_{n\in\mathbb{N}_0} f(n)\chi_{\phi^{-1}(n)}\|_p^p \\ &= \sum_{n\in\mathbb{N}_0}|f(n)|^p(\sum_{m\in\phi^{-1}(n)}\frac{e^{-\lambda}\lambda^m}{m!}) \\ &\leq \sum_{n\in\mathbb{N}_0}|f(n)|^p(M\frac{e^{-\lambda}\lambda^n}{n!}) \\ &= M\|f\|_p^p\end{aligned}$$

i.e.

$$\|C_\phi(f)\|_p^p \leq M\|f\|_\lambda^p\ \forall\ f \in l^p(\lambda).$$

□

We now give an example of a selfmap ϕ such that C_ϕ is composition operator.

Example 2.5. *Define*

$$\phi(n) = \begin{cases} n, & \text{if } n \text{ is even} \\ n-1, & \text{if } n \text{ is odd}.\end{cases}$$

Then,

$$\sum_{m\in\phi^{-1}(2n)} \frac{e^{-\lambda}\lambda^m}{m!} = \frac{e^{-\lambda}\lambda^{2n}}{2n!}(1 + \frac{\lambda}{2n+1})$$

Since λ is fixed so, by Archimedian property, there exists a natural number N such that

$$\frac{\lambda}{2n+1} \leq 1 \forall\ n \geq N.$$

so

$$\begin{aligned}\sum_{m\in\phi^{-1}(2n)} \frac{e^{-\lambda}\lambda^m}{m!} &= \sum_{m\in\{2n,2n+1\}} \frac{e^{-\lambda}\lambda^m}{m!} \\ &= \frac{e^{-\lambda}\lambda^{2n}}{2n!}(1+\frac{\lambda}{2n+1}) \\ &\leq 2\frac{e^{-\lambda}\lambda^{2n}}{2n!}.\end{aligned}$$

Now we choose $M = max(1+\lambda, 2)$. Thus

$$\sum_{m\in\phi^{-1}(n)} \frac{e^{-\lambda}\lambda^m}{m!} \leq M\frac{e^{-\lambda}\lambda^n}{n!}.$$

Consider

$$\begin{aligned}\|C_\phi(f)\|_p^p &= \|\sum_{n\in\mathbb{N}_0} f(n)\chi_{\phi^{-1}(n)}\|_p^p \\ &= \sum_{even\ n\in\mathbb{N}_0} |f(n)|^p(\sum_{m\in\phi^{-1}(n)} \frac{e^{-\lambda}\lambda^m}{m!}) \\ &\leq \sum_{even\ n\in\mathbb{N}_0} |f(n)|^p(M\frac{e^{-\lambda}\lambda^n}{n!}) \\ &\leq M\sum_{n\in\mathbb{N}_0} |f(n)|^p\frac{e^{-\lambda}\lambda^n}{n!} \\ &= M\|f\|_p^p.\end{aligned}$$

We also give an example of a selfmap ϕ such that C_ϕ is a composition operator on $l^p(\lambda)$ but C_ϕ is not composition operator on l^p.

Example 2.6. *Define* $\phi : \mathbb{N}_0 \to \mathbb{N}_0$ *such that*

$$\phi(n) = \begin{cases} 0, & \textit{if } n \textit{ is even} \\ n, & \textit{otherwise.}\end{cases}$$

Since cardinality of $\phi^{-1}(0)$ *is not finite so* C_ϕ *is not bounded on* l^p. *However,* C_ϕ *is bounded on* $l^p(\lambda)$ *as follows.*
If $n \in \phi(\mathbb{N}_0)$ *is odd, then*

$$\sum_{m\in\phi^{-1}(n)} \frac{e^{-\lambda}\lambda^m}{m!} = \frac{e^{-\lambda}\lambda^n}{n!}.$$

If $n \in \phi(\mathbb{N}_0)$ *is even that is* $n = 0$, *then*

$$\sum_{m\in\phi^{-1}(0)} \frac{e^{-\lambda}\lambda^m}{m!} \leq 1.$$

Thus for $M = max(1, e^\lambda)$ *we have*

$$\sum_{m\in\phi^{-1}(n)} \frac{e^{-\lambda}\lambda^m}{m!} \leq M\frac{e^{-\lambda}\lambda^n}{n!}\ \forall\ n \in \mathbb{N}_0.$$

3. Null and range spaces of C_ϕ

In this section we determine the null space and range space of C_ϕ. We further determine the conditions on ϕ under which C_ϕ is injective and surjective. Sequence space version of following results can be found in [9].

Proposition 3.1. *Let C_ϕ be a composition operator on l^p_λ induced by a selfmap ϕ on $\mathbb{N}_0$. Then null space $N(C_\phi)$ of C_ϕ is given by*

$$N(C_\phi) = \{f \in l^p(\lambda) : f|\phi(\mathbb{N}_0) = 0\}.$$

Proof.

$$\begin{aligned} f \in N(C_\phi) &\iff C_\phi(f) = 0 \\ &\iff f \circ \phi = 0 \\ &\iff f|\phi(\mathbb{N}_0) = 0 \end{aligned}$$

□

Next, we find the range space of a composition operator.

Theorem 3.2. *Let C_ϕ be a composition operator on $l^p(\lambda)$ induced by a selfmap ϕ on $\mathbb{N}_0$. Then range space $R(C_\phi)$ of C_ϕ is given by*

$$R(C_\phi) = \{f \in l^p(\lambda) : \sum_{\substack{m\in\phi(\mathbb{N}_0)\\ n\in\phi^{-1}(m)}} f(n)\chi_m \in l^p(\lambda) \textit{ and } f|\phi^{-1}(m) \textit{ is constant } \forall m \in \phi(\mathbb{N}_0)\}.$$

Proof. Let $f \in R(C_\phi)$. There exists $g \in l^p(\lambda)$ such that $C_\phi(g) = f$. This implies $f|\phi^{-1}(m)$ *is constant* $\forall m \in \phi(\mathbb{N}_0)$. Now we show that $\sum_{\substack{m\in\phi(\mathbb{N}_0)\\ n\in\phi^{-1}(m)}} f(n)\chi_m \in l^p(\lambda)$.

Consider

$$\begin{aligned} \|\sum_{\substack{m\in\phi(\mathbb{N}_0)\\ n\in\phi^{-1}(m)}} f(n)\chi_m\|_p^p &= \sum_{\substack{m\in\phi(\mathbb{N}_0)\\ n\in\phi^{-1}(m)}} \frac{e^{-\lambda}\lambda^m}{m!}|f(n)|^p \\ &= \sum_{m\in\phi(\mathbb{N}_0)} \frac{e^{-\lambda}\lambda^m}{m!}|g(m)|^p \\ &\le \|g\|_p^p. \end{aligned}$$

Since $g \in l^p(\lambda)$ so $\sum_{\substack{m\in\phi(\mathbb{N}_0)\\ n\in\phi^{-1}(m)}} f(n)\chi_m \in l^p(\lambda)$. Hence,

$$R(C_\phi) \subseteq \{f \in l^p(\lambda) : \sum_{\substack{m\in\phi(\mathbb{N}_0)\\ n\in\phi^{-1}(m)}} f(n)\chi_m \in l^p(\lambda),\ f|\phi^{-1}(m) \textit{ is constant } \forall m \in \phi(\mathbb{N}_0)\}.$$

Conversely, let $f \in l^p(\lambda)$ such that $f|\phi^{-1}(m)$ be constant $\forall m \in \phi(\mathbb{N}_0)$ and $\sum_{\substack{m\in\phi(\mathbb{N}_0)\\ n\in\phi^{-1}(m)}} f(n)\chi_m \in l^p(\lambda)$.

Now define

$$g(m) = \begin{cases} f(n), & \text{if } n \in \phi^{-1}(m) \\ 0, & \text{otherwise .} \end{cases}$$

Clearly, for $n \in \mathbb{N}_0$

$$\begin{aligned} C_\phi(g)(n) &= (g \circ \phi)(n) \\ &= g(\phi(n))) \\ &= f(n). \end{aligned}$$

This implies that $C_\phi(g) = f$. We claim that $g \in l^p(\lambda)$. Clearly,

$$\begin{aligned} \|g\|_p^p &= \sum_{m\in\mathbb{N}_0} \frac{e^{-\lambda}\lambda^m}{m!}|g(m)|^p \\ &= \sum_{m\in\phi(\mathbb{N}_0)} \frac{e^{-\lambda}\lambda^m}{m!}|g(m)|^p \\ &= \sum_{\substack{m\in\phi(\mathbb{N}_0)\\ n\in\phi^{-1}(m)}} \frac{e^{-\lambda}\lambda^m}{m!}|f(n)|^p \\ &= \|\sum_{\substack{m\in\phi(\mathbb{N}_0)\\ n\in\phi^{-1}(m)}} f(n)\chi_m\|_p^p < \infty. \end{aligned}$$

Thus $g \in l^p(\lambda)$. Hence $f \in R(C_\phi)$ $\square$

In [5], Cima, Thomson, and Wogen gave a necessary and sufficient condition for a composition operator on Hardy space $H^2(D)$ to have a closed range. In [10], Zorboska characterized the composition operators with closed range on H^2. In [3], Cao and Sun gave a necessary and sufficient condition for C_ϕ on Hardy space $H^2(B_n)$ to have a closed range. Recently, Guangfu et al [2] determine a necessary condition for C_ϕ to have a closed range on a Banach space of analytic functions which includes the Bloch space. We give a sufficient and a necessary condition for C_ϕ to have a closed range on $l^p(\lambda)$.

Remark 3.3. *It is known that range space of a composition operator C_ϕ on l^p is closed [9]. However it is interesting to note that range space of a composition operator C_ϕ on $l^p(\lambda)$ need not be closed in general. Consider the following example.*
Let

$$\phi(n) = \begin{cases} 0, & \text{if } n = 0, 1 \\ n-1, & \text{otherwise}, \end{cases}$$

$$f(n) = \begin{cases} \left(\frac{(\phi(n)-1)!}{\lambda^{\phi(n)}}\right)^{\frac{1}{p}}, & \text{if } \phi(n) \geq 1 \\ 0, & \text{otherwise} \end{cases}$$

and

$$f_k(n) = \begin{cases} \left(\frac{(\phi(n)-1)!}{\lambda^{\phi(n)}}\right)^{\frac{1}{p}}, & \text{if } 1 \leq \phi(n) \leq k \\ 0, & \text{otherwise.} \end{cases}$$

Then it is easy to see that $f \in l^p(\lambda)$, $f \notin R(C_\phi)$ but sequence $\{f_k\}_{k\in\mathbb{N}}$ is in $R(C_\phi)$ and converges to f in $l^p(\lambda)$. Hence, range space is not closed for the above choice of ϕ.

A sufficient condition for range space of a composition operator C_ϕ on $l^p(\lambda)$ to be closed.

Theorem 3.4. *If $\phi(n) \geq n$ for all but finitely many $n \in \mathbb{N}_0$, then $R(C_\phi)$ is closed.*

Proof. Let $f \in \overline{R(C_\phi)}$. There exists a sequence $\{f_n\}_{n\in\mathbb{N}_0} \in R(C_\phi)$ such that $\|f_n - f\|_p \to 0$ as $n \to \infty$. Since $\{f_n\}_{n\in\mathbb{N}_0}$ is Cauchy, so for given $\epsilon > 0$ there exists a positive integer n_0 such that

$$\|f_n - f_r\|_p < \epsilon \quad \forall n, r \geq n_0.$$

Since f_n is constant on $\phi^{-1}(m)$ so is f . Now it remains to show

$$\sum_{\substack{m\in\phi(\mathbb{N}_0)\\ n\in\phi^{-1}(m)}} f(n)\chi_m \in l^p(\lambda).$$

First notice that we can choose $m_0 \in \phi(\mathbb{N}_0)$ such that

$$\frac{\lambda}{m_0} < 1 \Rightarrow \frac{\lambda^m}{m!} \leq \frac{\lambda^n}{n!} \ \ \forall m \geq n \geq m_0.$$

Now consider

$$\begin{aligned}
\|\sum_{\substack{m\in\phi(\mathbb{N}_0)\\ n\in\phi^{-1}(m)}} f(n)\chi_m\|_p^p &= \sum_{\substack{m\in\phi(\mathbb{N}_0)\\ n\in\phi^{-1}(m)}} \frac{e^{-\lambda}\lambda^m}{m!}|f(n)|^p \\
&= \sum_{\substack{m\in\phi(\mathbb{N}_0)\\ n\in\phi^{-1}(m), m<n}} \frac{e^{-\lambda}\lambda^m}{m!}|f(n)|^p + \sum_{\substack{m\in\phi(\mathbb{N}_0)\\ n\in\phi^{-1}(m), m\geq n}} \frac{e^{-\lambda}\lambda^m}{m!}|f(n)|^p \\
&= \sum_{\substack{m\in\phi(\mathbb{N}_0)\\ n\in\phi^{-1}(m), m<n}} \frac{e^{-\lambda}\lambda^m}{m!}|f(n)|^p + \sum_{\substack{m\in\phi(\mathbb{N}_0), m<m_0\\ n\in\phi^{-1}(m), m\geq n}} \frac{e^{-\lambda}\lambda^m}{m!}|f(n)|^p + \sum_{\substack{m\in\phi(\mathbb{N}_0), m\geq m_0\\ n\in\phi^{-1}(m), m\geq n}} \frac{e^{-\lambda}\lambda^m}{m!}|f(n)|^p \\
&= \sum_{\substack{m\in\phi(\mathbb{N}_0)\\ n\in\phi^{-1}(m), m<n}} \frac{e^{-\lambda}\lambda^m}{m!}|f(n)|^p + \sum_{\substack{m\in\phi(\mathbb{N}_0), m<m_0\\ n\in\phi^{-1}(m), m\geq n}} \frac{e^{-\lambda}\lambda^m}{m!}|f(n)|^p + \sum_{\substack{m\in\phi(\mathbb{N}_0), m\geq m_0\\ n\in\phi^{-1}(m), m\geq m_0>n}} \frac{e^{-\lambda}\lambda^m}{m!}|f(n)|^p \\
&\quad + \sum_{\substack{m\in\phi(\mathbb{N}_0), m\geq m_0\\ n\in\phi^{-1}(m), m\geq n\geq m_0}} \frac{e^{-\lambda}\lambda^m}{m!}|f(n)|^p \\
&\leq \sum_{\substack{m\in\phi(\mathbb{N}_0)\\ n\in\phi^{-1}(m), m<n}} \frac{e^{-\lambda}\lambda^m}{m!}|f(n)|^p + \sum_{\substack{m\in\phi(\mathbb{N}_0), m<m_0\\ n\in\phi^{-1}(m), m\geq n}} \frac{e^{-\lambda}\lambda^m}{m!}|f(n)|^p + \sum_{\substack{m\in\phi(\mathbb{N}_0), m\geq m_0\\ n\in\phi^{-1}(m), m\geq m_0>n}} \frac{e^{-\lambda}\lambda^m}{m!}|f(n)|^p \\
&\quad + \sum_{\substack{m\in\phi(\mathbb{N}_0), m\geq m_0\\ n\in\phi^{-1}(m), m\geq n\geq m_0}} \frac{e^{-\lambda}\lambda^n}{n!}|f(n)|^p.
\end{aligned}$$

In the above expression first sum is finite since $\phi(n) < n$ for only finitely many $n \in \mathbb{N}_0$. Second sum is finite since there are only finitely many $m's$ such that $m < m_0$. Third sum is finite since there are only finitely $n's$ such that $n < m_0$. Fourth sum is finite since $f \in l^p(\lambda)$. Thus $f \in R(C_\phi)$ and hence range space is closed. □

Remark 3.5. *It is interesting to note that example 2.6 also shows that condition taken in theorem 3.4 is not necessary. Clearly for infinitely many even $n \in \mathbb{N}_0$ we have $\phi(n) < n$. Also, range space of composition operator C_ϕ is $R(C_\phi) = \{f \in l^p(\lambda) : f|_{\phi^{-1}(m)} = constant\ \forall\ m \in \phi(\mathbb{N}_0)\}$. Now it is easy to verify that $R(C_\phi)$ is closed.*

Following corollary is an immediate consequence of theorem 3.4.

Corollary 3.6. *Let C_ϕ be a composition operator on $l^p(\lambda)$ induced by an injective selfmap ϕ on $\mathbf{N}_0$. If $\phi(n) \geq n$ for all but finitely many $n \in \mathbb{N}_0$, then C_ϕ is surjective.*

We now give necessary condition for range space of a composition operator C_ϕ on $l^p(\lambda)$ to be closed.

Theorem 3.7. *If range $R(C_\phi)$is closed, then series $\sum\limits_{\substack{m<n \\ n\in\phi^{-1}(m) \\ m\in\phi(\mathbb{N}_0)}} \frac{1}{m}$ is convergent.*

Proof. We proceed by contraposition. Assume there exists a sequence $\{m_k\}_{k\in\mathbb{N}} \subseteq \phi(\mathbb{N}_0)$ with $\phi(n_k) < n_k$ for $n_k \in \phi^{-1}(m_k)$ such that $\sum\limits_{k\geq 1} \frac{1}{\phi(n_k)}$ diverges. Define $f : \mathbb{N}_0 \to \mathbb{C}$ such that

$$f(n) = \begin{cases} \left(\frac{(\phi(n_k)-1)!}{\lambda^{\phi(n_k)}}\right)^{\frac{1}{p}}, & \text{if } n = n_k \ 1 \leq \phi(n_k) \\ 0, & \textit{otherwise}. \end{cases}$$

First we verify that $f \in l^p(\lambda)$. Infact

$$\begin{aligned} \|f\|_p^p &= \sum_{n\in\mathbb{N}_0} |f(n)|^p \frac{e^{-\lambda}\lambda^n}{n!} \\ &= \sum_{k\geq 1} |f(n_k)|^p \frac{e^{-\lambda}\lambda^{n_k}}{n_k!} \\ &= \sum_{k\geq 1} \frac{(\phi(n_k)-1)!}{\lambda^{\phi(n_k)}} \frac{e^{-\lambda}\lambda^{n_k}}{n_k!} \end{aligned} \tag{1}$$

We choose an $n_0 \in \mathbb{N}_0$ such that $\frac{\lambda}{n_0} < 1$. Therefore

$$\frac{\lambda^m}{m!} \leq \frac{\lambda^n}{n!} \quad \forall m \geq n \geq n_0.$$

We split the sum (1) as follows

$$= \sum_{1\leq k<n_0} \frac{(\phi(n_k)-1)!}{\lambda^{\phi(n_k)}} \frac{e^{-\lambda}\lambda^{n_k}}{n_k!} + \sum_{n_0\leq k} \frac{(\phi(n_k)-1)!}{\lambda^{\phi(n_k)}} \frac{e^{-\lambda}\lambda^{n_k}}{n_k!}.$$

Since $\phi(n_k) \leq n_k - 1$ for all $k \geq 1$, we have

$$\begin{aligned} &\leq \sum_{1\leq k<n_0} \frac{(\phi(n_k)-1)!}{\lambda^{\phi(n_k)}} \frac{e^{-\lambda}\lambda^{n_k}}{n_k!} + \sum_{n_0\leq k} \frac{(\phi(n_k)-1)!}{\lambda^{\phi(n_k)}} \frac{\lambda}{n_k} \frac{e^{-\lambda}\lambda^{\phi(n_k)}}{(\phi(n_k))!} \\ &= \sum_{1\leq k<n_0} \frac{(\phi(n_k)-1)!}{\lambda^{\phi(n_k)}} \frac{e^{-\lambda}\lambda^{n_k}}{n_k!} + \lambda e^{-\lambda} \sum_{k\geq 1} \frac{1}{n_k\phi(n_k)} < \infty. \end{aligned}$$

Now claim that $f \notin R(C_\phi)$.

$$\begin{aligned} \Big\| \sum_{\substack{m\in\phi(\mathbb{N}_0) \\ n\in\phi^{-1}(m)}} f(n)\chi_m \Big\|_p^p &= \sum_{\substack{m\in\phi(\mathbb{N}_0) \\ n\in\phi^{-1}(m)}} |f(n)|^p \frac{e^{-\lambda}\lambda^m}{m!} \\ &\geq \sum_{k\geq 1} |f(n_k)|^p \frac{e^{-\lambda}\lambda^{\phi(n_k)}}{\phi(n_k)!} \\ &= \sum_{k\geq 1} \frac{(\phi(n_k)-1)!}{\lambda^{\phi(n_k)}} \frac{e^{-\lambda}\lambda^{\phi(n_k)}}{\phi(n_k)!} \\ &= \sum_{k\geq 1} \frac{e^{-\lambda}}{\phi(n_k)}. \end{aligned}$$

The above series diverges by assumption. Now define a sequence $f_r : \mathbb{N}_0 \to \mathbb{C}$ such that

$$f_r(n) = \begin{cases} \left(\frac{(\phi(n_k)-1)!}{\lambda^{\phi(n_k)}}\right)^{\frac{1}{p}}, & \text{if } n = n_k \ 1 \leq \phi(n_k) \leq \phi(n_r) \\ 0, & \textit{otherwise}. \end{cases}$$

It is easy to see that $f_r \in R(C_\phi)$. Also sequence $\{f_r\}_{r\in\mathbb{N}}$ converges to f since

$$\begin{aligned} \|f_r - f\|_p &= \Big(\sum_{n\in\mathbb{N}_0} \frac{e^{-\lambda}\lambda^n}{n!} |f_r(n) - f(n)|^p \Big)^{\frac{1}{p}} \\ &= \Big(\sum_{k\geq r+1} \frac{e^{-\lambda}\lambda^{n_k}}{n_k!} |f(n_k)|^p \Big)^{\frac{1}{p}} \to 0 \textit{ as } r \to \infty. \end{aligned}$$

Therefore, $f \in \overline{R(C_\phi)}$ but $f \notin R(C_\phi)$. Hence, range space is not closed. □

Remark 3.8. *Following example shows that condition taken in theorem 3.7 is not sufficient. Define* $\phi : \mathbb{N}_0 \to \mathbb{N}_0$ *such that*

$$\phi(n) = \begin{cases} 1, & \textit{if } 0 \leq n \leq 2^2 \\ 2^2, & \textit{if } 2^2 < n \leq 3^2 \\ 3^2, & \textit{if } 3^2 < n \leq 4^2 \\ \dots \\ k^2, & \textit{if } k^2 < n \leq (k+1)^2 \\ \dots, & \dots . \end{cases}$$

Define $f : \mathbb{N}_0 \to \mathbb{C}$ *such that*

$$f(n) = \left(\frac{1}{m \sum\limits_{i\in\phi^{-1}(m)} \frac{e^{-\lambda}\lambda^n}{n!}} \right)^{\frac{1}{p}} \quad \textit{if for some } m \in \phi(\mathbb{N}_0),\ n \in \phi^{-1}(m).$$

To check $f \in l^p(\lambda)$, *consider*

$$\begin{aligned} \|f\|_p^p &= \sum_{n\in\mathbb{N}_0} |f(n)|^p \frac{e^{-\lambda}\lambda^n}{n!} \\ &= \sum_{m\in\phi(\mathbb{N}_0)} \Big(\sum_{n\in\phi^{-1}(m)} |f(n)|^p \frac{e^{-\lambda}\lambda^n}{n!} \Big) \\ &= \sum_{m\in\phi(\mathbb{N}_0)} \frac{1}{m \sum\limits_{n\in\phi^{-1}(m)} \frac{e^{-\lambda}\lambda^n}{n!}} \Big(\sum_{n\in\phi^{-1}(m)} \frac{e^{-\lambda}\lambda^n}{n!} \Big) \\ &= \sum_{m\in\phi(\mathbb{N}_0)} \frac{1}{m} < \infty. \end{aligned}$$

Now we define sequence $f_k : \mathbb{N}_0 \to \mathbb{C}$ *such that*

$$f_k(n) = \begin{cases} f(n), & \textit{if } n \leq k^2 \\ 0, & \textit{otherwise} \end{cases}$$

Then it is easy to see that $f_k \in R(C_\phi)$ *and sequence* $\{f_k\}_{k\in\mathbb{N}}$ *converges to* f *in* $l^p(\lambda)$*. Finally we show that* $f \notin R(C_\phi)$*. Consider*

$$\begin{aligned}
\|\sum_{\substack{m\in\phi(\mathbb{N}_0)\\ n\in\phi^{-1}(m)}} f(n)\chi_m\|_p^p &= \sum_{\substack{m\in\phi(\mathbb{N}_0)\\ n\in\phi^{-1}(m)}} |f(n)|^p \frac{e^{-\lambda}\lambda^m}{m!}\\
&= \sum_{m\in\phi(\mathbb{N}_0)} \frac{1}{m\sum\limits_{n\in\phi^{-1}(m)}\frac{e^{-\lambda}\lambda^n}{n!}}\frac{e^{-\lambda}\lambda^m}{m!}\\
&= \sum_{m\in\phi(\mathbb{N}_0)} \frac{1}{m\sum\limits_{n\in\phi^{-1}(m)}\frac{\lambda^n}{n!}}\frac{\lambda^m}{m!} \qquad (2)
\end{aligned}$$

We choose $k_0^2 \in \phi(\mathbb{N}_0)$ *such that* $\frac{\lambda}{n} < 1\ \forall n \geq k_0$*. Now for* $k^2 \geq k_0^2$ *consider*

$$\begin{aligned}
\frac{1}{k^2\sum\limits_{n\in\phi^{-1}(k^2)}\frac{\lambda^n}{n!}}\frac{\lambda^{k^2}}{k^2!} &= \frac{1}{k^2\left(\frac{\lambda^{k^2+1}}{k^2+1!}+\frac{\lambda^{k^2+2}}{k^2+2!}+\ldots+\frac{\lambda^{(k+1)^2}}{(k+1)^2!}\right)}\frac{\lambda^{k^2}}{k^2!}\\
&\geq \frac{1}{k^2\left(\frac{\lambda^{k^2+1}}{k^2+1!}+\frac{\lambda^{k^2+1}}{k^2+1!}+\ldots(k+2)times\right)}\frac{\lambda^{k^2}}{k^2!}\\
&= \frac{1}{k^2\frac{\lambda^{k^2+1}}{k^2+1!}(k+2)}\frac{\lambda^{k^2}}{k^2!}\\
&= \frac{k^2+1}{\lambda k^2(k+2)} > \frac{1}{\lambda(k+2)}. \qquad (3)
\end{aligned}$$

Now by (3) *and* (2)

$$\begin{aligned}
\sum_{m\in\phi(\mathbb{N}_0)} \frac{1}{m\sum\limits_{n\in\phi^{-1}(m)}\frac{\lambda^n}{n!}}\frac{\lambda^m}{m!} &= \sum_{k=1}^{k_0}\frac{1}{k^2\sum\limits_{n\in\phi^{-1}(k^2)}\frac{\lambda^n}{n!}}\frac{\lambda^{k^2}}{k^2!} + \sum_{k=k_0+1}^{\infty}\frac{1}{k^2\sum\limits_{n\in\phi^{-1}(k^2)}\frac{\lambda^n}{n!}}\frac{\lambda^{k^2}}{k^2!}\\
&> \sum_{k=1}^{k_0}\frac{1}{k^2\sum\limits_{n\in\phi^{-1}(k^2)}\frac{\lambda^n}{n!}}\frac{\lambda^{k^2}}{k^2!} + \sum_{k=k_0+1}^{\infty}\frac{1}{\lambda(k+2)}\\
&= \infty.
\end{aligned}$$

It follows that $f \notin R(C_\phi)$*. Hence, range is not closed for this choice of* ϕ*.*

Following corollary is a natural consequence of theorem 3.7.

Corollary 3.9. *Let* C_ϕ *be a composition operator on* $l^p(\lambda)$ *induced by an injective selfmap* ϕ *on* $\mathbf{N}_0$*. If* C_ϕ *is surjective, then series* $\sum\limits_{\substack{m<n\\ n\in\phi^{-1}(m)\\ m\in\phi(\mathbb{N}_0)}} \frac{1}{m}$ *is convergent.*

Now we characterize injectivity of C_ϕ in terms of selfmap which induces composition operator.

Proposition 3.10. *Let* C_ϕ *be a composition operator on* $l^p(\lambda)$ *induced by an selfmap* ϕ *on* $\mathbf{N}_0$*. Then,* C_ϕ *is injective if and only if* ϕ *is surjective.*

Proof. Suppose ϕ is surjective. Let $C_\phi(f) = C_\phi(g)\ for\ some f, g \in l^p(\lambda)$. This implies

$$\begin{aligned}
&f(\phi(n)) = g(\phi(n))\ for\ each\ n \in \mathbb{N}_0\\
&\Rightarrow f = g \because\ \phi\ is\ surjective\\
&\Rightarrow C_\phi\ is\ one - one.
\end{aligned}$$

Conversely, suppose that C_ϕ is injective. It follows that for each $n \in \mathbb{N}_0$

$$C_\phi(\chi_n) \neq 0 \Rightarrow \chi_{\phi^{-1}(n)} \neq 0.$$

Hence, $\phi^{-1}(n)$ is non empty for each $n \in \mathbb{N}_0$. Thus ϕ is surjective. □

4. Null and range spaces of C^*_ϕ

In this section we determine explicit expression for the adjoint C^*_ϕ of composition operator C_ϕ on Hilbert space $l^2(\lambda)$ with inner product

$$\langle f,\ g\rangle = \sum_{n\in\mathbb{N}_0} f(n)\overline{g(n)}\frac{e^{-\lambda}\lambda^n}{n!}\ \ \forall f, g \in l^2(\lambda).$$

We determine the null space and range space of C^*_ϕ on $l^2(\lambda)$ and prove that range space of composition operator C^*_ϕ is closed. We further determine the conditions on ϕ under which C_ϕ is injective and surjective.

Proposition 4.1. *Let C_ϕ be a composition operator on $l^2(\lambda)$. If $f = \sum_{n\in\mathbb{N}_0} f(n)\chi_n \in l^2(\lambda)$, then $C^*_\phi(f) = \sum_{n\in\mathbb{N}_0} f(n)\xi_n.\chi_{\phi(n)}$, where . denotes point-wise operation and $\xi_n(m) = \frac{\lambda^n}{n!}\frac{m!}{\lambda^m}\ \forall\ m \in \mathbb{N}_0$.*

Proof. By definition of adjoint of an operator, we have

$$\langle f, C^*_\phi(g)\rangle = \langle C_\phi(f), g\rangle\ \forall\ f, g \in l^2(\lambda).$$

In particular, we have

$$\begin{aligned}
\langle \chi_m, C^*_\phi(\chi_n)\rangle &= \langle C_\phi(\chi_m), \chi_n\rangle\ \forall\ m, n \in \mathbb{N}_0\\
\Longrightarrow\ \frac{e^{-\lambda}\lambda^m}{m!}\overline{C^*_\phi(\chi_n)(m)} &= \frac{e^{-\lambda}\lambda^n}{n!}C_\phi(\chi_m)(n)\ \forall\ m, n \in \mathbb{N}_0\\
\Longrightarrow\ \frac{\lambda^m}{m!}\overline{C^*_\phi(\chi_n)(m)} &= \frac{\lambda^n}{n!}C_\phi(\chi_m)(n)\ \forall\ m, n \in \mathbb{N}_0\\
\Longrightarrow\ C^*_\phi(\chi_n)(m) &= \frac{\lambda^n}{n!}\frac{m!}{\lambda^m}\chi_{\phi(n)}(m)\ \forall\ m, n \in \mathbb{N}_0\\
\Longrightarrow\ C^*_\phi(\chi_n) &= \xi_n.\chi_{\phi(n)}\ \forall\ n \in \mathbb{N}_0.
\end{aligned}$$

□

Now we determine null space of the adjoint C^*_ϕ.

Theorem 4.2. *Let C_ϕ be a composition operator on $l^2(\lambda)$, then the null space $N(C^*_\phi)$ of C^*_ϕ is given by*

$$N(C^*_\phi) = \{f \in l^2(\lambda) : \sum_{n\in\phi^{-1}(m)} f(n)\frac{\lambda^n}{n!} = 0,\ m \in \phi(\mathbb{N}_0)\}.$$

Proof. We have

$$\begin{aligned} C_\phi^*(f) &= \sum_{n\in\mathbb{N}_0} f(n)\xi_m.\chi_{\phi(n)} \\ &= \sum_{n\in\mathbb{N}_0} f(n)\frac{\lambda^n}{n!}\frac{\phi(n)!}{\lambda^{\phi(n)}}\chi_{\phi(n)} \\ &= \sum_{m\in\phi(\mathbb{N}_0)}\Big(\sum_{n\in\phi^{-1}(m)} f(n)\frac{\lambda^n}{n!}\Big)\frac{m!}{\lambda^m}\chi_m. \end{aligned} \tag{4}$$

Now if $f \in N(C_\phi^*)$, then $C_\phi^*(f) = 0$. Therefore by (4) we get

$$\sum_{n\in\phi^{-1}(m)} f(n)\frac{\lambda^n}{n!} = 0 \ \ for\ m \in \phi(\mathbb{N}_0).$$

Conversely, if $f \in l^2(\lambda)$ be such that $\sum\limits_{n\in\phi^{-1}(m)} f(n)\frac{\lambda^n}{n!} = 0$. Then it is easy to see that $f \in N(C_\phi^*)$. □

The following result determines range space of C_ϕ^* under some restricted condition. Recall that an operator is said to be bounded below if there exists $M > 0$ such that $\|C_\phi f\| \geq M\|f\|$ for every $f \in l^2(\lambda)$.

Theorem 4.3. *Let C_ϕ be a bounded below composition operator on $l^2(\lambda)$. Then the range space $R(C_\phi^*)$ of C_ϕ^* is given by $R(C_\phi^*) = \{f \in l^2(\lambda) : f|\mathbb{N}_0 \setminus \phi(\mathbb{N}_0) = 0\}$.*

Proof. Suppose $f \in R(C_\phi^*)$. Then there is a function $g \in l^2(\lambda)$ such that $C_\phi^*(g) = f$. Let $g = \sum_{n\in\mathbb{N}_0} g(n)\chi_n$. Then

$$C_\phi^*(g) = \sum_{n\in\mathbb{N}_0} g(n)\xi_n.\chi_{\phi(n)}.$$

Hence for each $m \in \mathbb{N}_0 \setminus \phi(\mathbb{N}_0)$ $f(m) = C_\phi^*(g)(m) = 0$. Conversely, assume that $f \in l^2(\lambda)$ and $f(m) = 0$ for each $m \in \mathbb{N}_0 \setminus \phi(\mathbb{N}_0)$. Let $\alpha_n = \sum\limits_{r\in\phi^{-1}(n)} \xi_r(n)$. Now define

$$g = \sum_{m\in\mathbb{N}_0, \phi(m)=n} \frac{f(n)}{\alpha_n}\chi_m.$$

We claim that $g \in l^2(\lambda)$ *and* $C_\phi^*(g) = f$. Since C_ϕ is bounded below it follows

$$\sum_{r\in\phi^{-1}(n)} \frac{\lambda^r}{r!} = \|C_\phi(\chi_n)\| \geq M\|\chi_n\| = \frac{\lambda^n}{n!}.$$

Consider

$$
\begin{aligned}
\|g\|_2^2 &= \sum_{m\in\mathbb{N}_0,\phi(m)=n} \frac{|f(n)|^2}{{\alpha_n}^2}\frac{e^{-\lambda}\lambda^m}{m!} \\
&= \sum_{n\in\phi(\mathbb{N}_0)} \frac{|f(n)|^2}{{\alpha_n}^2}\Big(\sum_{m\in\phi^{-1}(n)} \frac{e^{-\lambda}\lambda^m}{m!}\Big) \\
&= \sum_{n\in\phi(\mathbb{N}_0)} \frac{|f(n)|^2}{{\alpha_n}^2}\Big(\sum_{m\in\phi^{-1}(n)} \frac{e^{-\lambda}\lambda^m}{m!}\Big) \\
&= \sum_{n\in\phi(\mathbb{N}_0)} |f(n)|^2\frac{\lambda^n}{n!}\frac{\frac{\lambda^n}{n!}}{\Big(\sum\limits_{r\in\phi^{-1}(n)}\frac{\lambda^r}{r!}\Big)^2}\Big(\sum_{m\in\phi^{-1}(n)} \frac{e^{-\lambda}\lambda^m}{m!}\Big) \\
&= \sum_{n\in\phi(\mathbb{N}_0)} |f(n)|^2\frac{\lambda^n}{n!}\frac{\frac{e^{-\lambda}\lambda^n}{n!}}{\sum_{r\in\phi^{-1}(n)}\frac{\lambda^r}{r!}} \\
&\leq M\sum_{n\in\phi(\mathbb{N}_0)} |f(n)|^2\frac{e^{-\lambda}\lambda^n}{n!} \\
&< \infty.
\end{aligned}
$$

Hence, $g \in l^2(\lambda)$. Now consider

$$
\begin{aligned}
C_\phi^*(g) &= \sum_{m\in\mathbb{N}_0,\phi(m)=n} \frac{f(n)}{\alpha_n}\xi_m.\chi_{\phi(m)} \\
&= \sum_{n\in\phi(\mathbb{N}_0)}\Big(\sum_{m\in\phi^{-1}(n)} \frac{f(n)}{\alpha_n}\xi_m.\chi_{\phi(m)}\Big) \\
&= \sum_{n\in\phi(\mathbb{N}_0)} f(n)\left(\frac{\sum\limits_{m\in\phi^{-1}(n)}\xi_m(n)}{\alpha_n}\right)\chi_n \\
&= \sum_{n\in\phi(\mathbb{N}_0)} f(n)\chi_n \\
&= f.
\end{aligned}
$$

□

Corollary 4.4. *Let C_ϕ be a bounded below composition operator on $l^2(\lambda)$. Then $R(C_\phi^*)$ is a closed subspace of $l^2(\lambda)$.*

Proof. Let $f \in \overline{R(C_\phi^*)}$. There exists a sequence $\{f_m\}_{m\in\mathbb{N}} \in R(C_\phi^*)$ such that $\|f_m - f\|_2 \to 0$ as $m \to \infty$. Since

$$f_m|\mathbb{N}_0 \setminus \phi(\mathbb{N}_0) = 0 \;\; \forall m \in \mathbb{N}.$$

Hence, $f|\mathbb{N}_0 \setminus \phi(\mathbb{N}_0) = 0$. □

We now determine the conditions on selfmap ϕ under which C_ϕ^* is injective.

Proposition 4.5. *The adjoint C_ϕ^* of a composition operator C_ϕ is injective if and only if ϕ is injective.*

Proof. Suppose C^*_ϕ is injective. We show that ϕ is injective.
Let $\phi(m) = \phi(n)$ for some $m, n \in \mathbb{N}_0$.

$$\begin{aligned}\phi(m) = \phi(n) &\Rightarrow \xi_{\phi(m)}\chi_{\phi(m)} = \xi_{\phi(n)}\chi_{\phi(n)}\\ &\Rightarrow C^*_\phi(\chi_m) = C^*_\phi(\chi_n)\\ &\Rightarrow \chi_m = \chi_n \quad (\because\ C^*_\phi \text{ is injective})\\ &\Rightarrow m = n\\ &\Rightarrow \phi \text{ is injective.}\end{aligned}$$

Conversely, assume that ϕ is injective. We show that C^*_ϕ is injective. For some $f, g \in l^2(\lambda)$ suppose

$$\begin{aligned}C^*_\phi(f) = C^*_\phi(g) &\Rightarrow \sum_{n\in\mathbb{N}_0} f(n)\xi_n\chi_{\phi(n)} = \sum_{n\in\mathbb{N}_0} g(n)\xi_n\chi_{\phi(n)}\\ &\Rightarrow f(n)\xi_n(\phi(n)) = g(n)\xi_n(\phi(n)) \quad \forall\ n \in \mathbb{N}_0\\ &\Rightarrow f(n) = g(n) \quad \forall\ n \in \mathbb{N}_0\\ &\Rightarrow f = g \quad \forall\ n \in \mathbb{N}_0\\ &\Rightarrow C^*_\phi \text{ is injective.}\end{aligned}$$

□

We now find sufficient condition for C^*_ϕ to be surjective.

Theorem 4.6. *Let $\phi : \mathbf{N}_0 \to \mathbf{N}_0$ be surjective. Then, C^*_ϕ is surjective if $\phi(n) \geq n$ for all but finitely many $n \in \mathbb{N}_0$.*

Proof. Let $f \in l^2(\lambda)$. Define $g : \mathbb{N}_0 \to \mathbb{N}_0$ such that $g(n) = f(\phi(n))\xi_{\phi(n)}(n)\ \forall n \in \mathbb{N}_0$. Clearly, we have

$$\begin{aligned}C^*_\phi(g) &= \sum_{n\in\mathbb{N}_0} g(n)\xi_n(\phi(n))\chi_{\phi(n)}\\ &= \sum_{n\in\mathbb{N}_0} f(\phi(n))\chi_{\phi(n)}\\ &= \sum_{m\in\phi(\mathbb{N}_0)} f(m)\chi_m\\ &= \sum_{m\in\mathbb{N}_0} f(m)\chi_m, \because\ \phi \text{ is surjective}\\ &= f.\end{aligned}$$

We now claim $g \in l^2(\lambda)$. Consider

$$\begin{aligned}\|g\|_2^2 &= \sum_{m\in\mathbb{N}_0} |g(n)|^2 \frac{e^{-\lambda}\lambda^n}{n!}\\ &= \sum_{m\in\mathbb{N}_0} |f(\phi(n))|^2 \xi_{\phi(n)}(n)^2 \frac{e^{-\lambda}\lambda^n}{n!}\\ &= \sum_{n\in N_0} |f(\phi(n))|^2 \frac{e^{-\lambda}\lambda^{\phi(n)}}{\phi(n)!}\frac{\lambda^{\phi(n)}}{\phi(n)!}\frac{n!}{\lambda^n}.\end{aligned} \tag{5}$$

We choose an $n_0 \in \mathbb{N}_0$ such that $\frac{\lambda}{n_0} < 1$. Therefore

$$\frac{\lambda^m}{m!} \leq \frac{\lambda^n}{n!}\ \ \forall m \geq n \geq n_0.$$

So we split the sum (5) as follows

$$\leq \sum_{0\leq n<n_0} |f(\phi(n))|^2 \frac{e^{-\lambda}\lambda^{\phi(n)}}{\phi(n)!}\frac{\lambda^{\phi(n)}}{\phi(n)!}\frac{n!}{\lambda^n} + \sum_{n\geq n_0} |f(\phi(n))|^2 \frac{e^{-\lambda}\lambda^{\phi(n)}}{\phi(n)!}$$

$$\leq \sum_{0\leq n<n_0} |f(\phi(n))|^2 \frac{e^{-\lambda}\lambda^{\phi(n)}}{\phi(n)!}\frac{\lambda^{\phi(n)}}{\phi(n)!}\frac{n!}{\lambda^n} + \|f\|^2 < \infty.$$

Hence, proved that C_ϕ^* is surjective. □

We give a necessary condition for surjectivity of C_ϕ^*.

Theorem 4.7. *Let $\phi : \mathbf{N}_0 \to \mathbf{N}_0$ be surjective. Then, C_ϕ^* is surjective only if series $\sum_{\substack{m<n \\ n\in\phi^{-1}(m) \\ m\in\phi(\mathbb{N}_0)}} \frac{1}{m}$ is convergent.*

Proof. On the contrary assume there exists a sequence $\{n_k\}_{k\in\mathbb{N}} \subseteq \mathbb{N}_0$ such that $\phi(n_k) < n_k$ $\forall k \geq 1$. Define $f : \mathbb{N}_0 \to \mathbb{C}$ such that

$$f(n) = \begin{cases} \left(\frac{(n-2)!}{\lambda^{n-1}}\right)^{\frac{1}{2}}, & \text{if } n \geq 2 \\ 0, & \textit{otherwise.} \end{cases}$$

It is easy to check that $f \in l^2(\lambda)$. Since ϕ is surjective so we have $f = \sum_{n\in\mathbb{N}_0} f(\phi(n))\chi_{\phi(n)}$. Now for every g which satisfy $C_\phi^*(g) = f$. We have

$$\sum_{n\in\mathbb{N}_0} g(n)\xi_n(\phi(n))\chi_{\phi(n)} = \sum_{n\in\mathbb{N}_0} f(\phi(n))\chi_{\phi(n)}.$$

This implies $g(n)\xi_n(\phi(n)) = f(\phi(n))\xi_n(\phi(n))$ $\forall n \in \mathbb{N}_0$. It can be written as $g(n) = f(\phi(n))\xi_{\phi(n)}(n)$ $\forall n \in \mathbb{N}_0$ since $\xi_m(n)\xi_n(m) = 1 \forall m, n \in \mathbb{N}_0$. Now we claim that $g \notin l^2(\lambda)$. Consider

$$\begin{aligned}
\|g\|_2^2 &= \sum_{n\in\phi(\mathbb{N}_0)} |g(n)|^2 \frac{e^{-\lambda}\lambda^n}{n!} \\
&= \sum_{n\in\mathbb{N}_0} |f(\phi(n))\xi_{\phi(n)}(n)|^2 \frac{e^{-\lambda}\lambda^n}{n!} \\
&= \sum_{\phi(n)\geq n} |f(\phi(n))|^2 \frac{e^{-\lambda}\lambda^{\phi(n)}}{\phi(n)!}\frac{n}{\lambda} + \sum_{\phi(n)<n} |f(\phi(n))|^2 \frac{e^{-\lambda}\lambda^{\phi(n)}}{\phi(n)!}\frac{n}{\lambda} \\
&\geq \sum_{k\geq 1} |f(\phi(n_k))|^2 \frac{e^{-\lambda}\lambda^{\phi(n_k)}}{\phi(n_k)!}\frac{n_k}{\lambda} \\
&= \sum_{k\geq 1} \frac{(\phi(n)-2)!}{\lambda^{\phi(n_k)-1}} \frac{e^{-\lambda}\lambda^{\phi(n_k)}}{\phi(n_k)!}\frac{n_k}{\lambda} \\
&= \sum_{k\geq 1} \frac{n_k e^{-\lambda}}{\phi(n_k)(\phi(n_k)-1)} \\
&\geq \sum_{k\geq 1} \frac{e^{-\lambda}}{\phi(n_k)-1} \\
&\geq \sum_{k\geq 1} \frac{e^{-\lambda}}{\phi(n_k)} = \infty.
\end{aligned}$$

This implies that C_ϕ^* is not surjective. □

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