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Computation on the comparative growth analysis of entire functions depending on their generalized relative orders

Sanjib Kumar Datta[a], Tanmay Biswas[b], Jinarul Haque Shaikh[c]

[a]*Department of Mathematics, University of Kalyani, Nadia-741235, West Bengal, India*
[b]*Rajbari, Rabindrapalli, R. N. Tagore Road, Krishnagar,Nadia-741101, West Bengal, India.*
[c]*Department of Mathematics, University of Kalyani, Nadia-741235, West Bengal, India*

Abstract. In this paper we study some comparative growth properties of composite entire functions in terms of their maximum terms on the basis of their generalized relative orders (generalized relative lower orders) with respect to another entire function. In fact, we improve here some results of Datta, Biswas and Pramanik [9].

1. Introduction, Definitions and Notations.

Let f be an entire function defined in the open complex plane $\mathbb{C}$. The *maximum term* $\mu_f(r)$ of $f = \sum_{n=0}^{\infty} a_n z^n$ is defined by

$$\mu_f(r) = \max_{n\geq 0}\left(|a_n| r^n\right) .$$

On the other hand, the *maximum modulus* $M_f(r)$ of f on $|z| = r$ is defined as

$$M_f(r) = \max_{|z|=r}\left|f(z)\right| .$$

We use the standard notations and definitions in the theory of entire functions which are available in [16]. In the sequel we use the following notation:

$$\log^{[k]} x = \log\left(\log^{[k-1]} x\right), k = 1, 2, 3, ...and\ \log^{[0]} x = x .$$

Taking this into account the *order* (respectively, *lower order*) of an entire function f is given by

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} M_f(r)}{\log r}\left(\text{respectively } \lambda_f = \frac{\log^{[2]} M_f(r)}{\log r}\right) .$$

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Email addresses: sanjib_kr_datta@yahoo.co.in (Sanjib Kumar Datta), tanmaybiswas_math@rediffmail.com (Tanmay Biswas), jnrlhqshkh188@gmail.com (Jinarul Haque Shaikh)

Let us recall that Sato [12] defined the *generalized order* and *generalized lower order* of an entire function f respectively as follows:

$$\rho_f^{[l]} = \limsup_{r\to\infty} \frac{\log^{[l]} M_f(r)}{\log r} \left(\text{respectively } \lambda_f^{[l]} = \liminf_{r\to\infty} \frac{\log^{[l]} M_f(r)}{\log r} \right)$$

where l is any positive integer. These definitions extended the order ρ_f and lower order λ_f of an entire function f since these correspond to the particular case $\rho_f^{[2]} = \rho_f$ and $\lambda_f^{[2]} = \lambda_f$.

Since for $0 \le r < R$,

$$\mu_f(r) \le M_f(r) \le \frac{R}{R-r}\mu_f(R) \ \ \{cf. [14]\}$$

it is easy to see that

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} \mu_f(r)}{\log r} \left(\text{respectively } \lambda_f = \liminf_{r\to\infty} \frac{\log^{[2]} \mu_f(r)}{\log r} \right)$$

and

$$\rho_f^{[l]} = \limsup_{r\to\infty} \frac{\log^{[l]} \mu_f(r)}{\log r} \left(\text{respectively } \lambda_f^{[l]} = \liminf_{r\to\infty} \frac{\log^{[l]} \mu_f(r)}{\log r} \right).$$

Given a non-constant entire function f defined in the open complex plane $\mathbb{C}$, its maximum modulus function M_f is strictly increasing and continuous. Hence there exists its inverse function $M_f^{-1} : \left(\left|f(0)\right|, \infty\right) \to (0, \infty)$ with $\lim_{s\to\infty} M_f^{-1}(s) = \infty$.

Then Bernal {[1], [2]} introduced the definition of *relative order* of f with respect to g, denoted by $\rho_g(f)$ as follows:

$$\begin{aligned} \rho_g(f) &= \inf\left\{\mu > 0 : M_f(r) < M_g(r^\mu) \text{ for all } r > r_0(\mu) > 0\right\} \\ &= \limsup_{r\to\infty} \frac{\log M_g^{-1} M_f(r)}{\log r}. \end{aligned}$$

This definition coincides with the classical one [15] if $g = \exp z$. Similarly, one can define the *relative lower order* of f with respect to g denoted by $\lambda_g(f)$ as

$$\lambda_g(f) = \liminf_{r\to\infty} \frac{\log M_g^{-1} M_f(r)}{\log r}.$$

Datta and Maji [5] reformulated the definition of *relative order* and *relative lower order* in terms of *maximum terms* of entire functions in the following way:

Definition 1.1. *[5] The* relative order $\rho_g(f)$ *and* the relative lower order $\lambda_g(f)$ *of an entire function* f *with respect to another entire function* g *are defined as follows:*

$$\rho_g(f) = \limsup_{r\to\infty} \frac{\log \mu_g^{-1} \mu_f(r)}{\log r} \text{ and } \lambda_g(f) = \liminf_{r\to\infty} \frac{\log \mu_g^{-1} \mu_f(r)}{\log r}.$$

Lahiri and Banerjee [11] gave a more generalized concept of relative order in the following way:

Definition 1.2. *[11] If $l \geq 1$ is a positive integer, then the l- th* generalized relative order *of f with respect to g, denoted by $\rho_g^{[l]}(f)$ is defined by*

$$\begin{aligned}\rho_g^{[l]}(f) &= \inf\left\{\mu > 0 : M_f(r) < M_g\left(\exp^{[l-1]} r^{\mu}\right) \text{ for all } r > r_0(\mu) > 0\right\} \\ &= \limsup_{r\to\infty} \frac{\log^{[l]} M_g^{-1} M_f(r)}{\log r} .\end{aligned}$$

Clearly, $\rho_g^1(f) = \rho_g(f)$ and $\rho_{\exp z}^1(f) = \rho_f$.

Likewise one can define the generalized relative lower order *of f with respect to g denoted by $\lambda_g^{[l]}(f)$ as*

$$\lambda_g^{[l]}(f) = \liminf_{r\to\infty} \frac{\log^{[l]} M_g^{-1} M_f(r)}{\log r} .$$

In the case of *generalized relative order* (respectively *generalized relative lower*), it therefore seems reasonable to state suitably an alternative definition of *generalized relative order* (respectively *generalized relative lower*) of entire function in terms of its maximum terms. Datta, Biswas and Ghosh [10] introduced such a definition in the following way:

Definition 1.3. *[10] For any positive integre $l \geq 1$, the growth indicators $\rho_g^{[l]}(f)$ and $\lambda_g^{[l]}(f)$ of an entire function f are defined as:*

$$\rho_g^{[l]}(f) = \limsup_{r\to\infty} \frac{\log^{[l]} \mu_g^{-1} \mu_f(r)}{\log r} \text{ and } \lambda_g^{[l]}(f) = \liminf_{r\to\infty} \frac{\log^{[l]} \mu_g^{-1} \mu_f(r)}{\log r} .$$

For entire functions, the notions of thier growth indicators such as order is classical in complex analysis and during the past decades, several researchers have already been exploring their studies in the area of comparative growth properties of composite entire functions in different directions using the classical growth indicators. But at that time, the concepts of relative orders and consequently the generalized relative orders of entire functions and as well as their technical advantages of not comparing with the growths of exp z are not at all known to the researchers of this area. Therefore the studies of the growths of composite entire functions in the light of their relative orders are the prime concern of this paper. In fact some light has already been thrown on such type of works by Datta et al. in [3], [4], [5], [6], [7], [8], [9] and [10]. Actually in this paper we wish to establish some results relating to the growth rates of composite entire functions in terms of their maximum terms on the basis of *generalized relative order* (*generalized relative lower*).

2. Lemmas.

In this section we present some lemmas which will be needed in the sequel.

Lemma 2.1. *[13] Let f and g be any two entire functions. Then for every $\alpha > 1$ and $0 < r < R$,*

$$\mu_{f\circ g}(r) \leq \frac{\alpha}{\alpha - 1} \mu_f\left(\frac{\alpha R}{R - r} \mu_g(R)\right) .$$

Lemma 2.2. *[13] If f and g are any two entire functions with $g(0) = 0$. Then for all sufficiently large values of r,*

$$\mu_{f\circ g}(r) \geq \frac{1}{2} \mu_f\left(\frac{1}{8} \mu_g\left(\frac{r}{4}\right)\right) .$$

Lemma 2.3. *[5] If f be an entire function and $\alpha > 1$, $0 < \beta < \alpha$, then for all sufficiently large r,*

$$\mu_f(\alpha r) \geq \beta \mu_f(r) .$$

3. Theorems.

In this section we present the main results of the paper.

Theorem 3.1. *Let f, g and h be any three entire functions such that $g(0) = 0$, $\rho_h^{[l]}(f) > 0$ and $\lambda_g^{[l+1]} > 0$ where $l \geq 1$. Then*

$$\limsup_{r\to\infty} \frac{\log^{[l]} \mu_h^{-1}\left(\mu_{f\circ g}(r)\right)}{\log^{[l]} \mu_h^{-1}\left(\mu_f(r)\right)} = \infty \, .$$

Proof. Suppose $\rho_h^{[l]}(f) > 0$ and $\lambda_g^{[l+1]} > 0$.
As $\mu_h^{-1}(r)$ is an increasing function of r, we get from Lemma 2.2, for all sufficiently large values of r that

$$\log^{[l]} \mu_h^{-1}\mu_{f\circ g}(r) \geq \log^{[l]} \mu_h^{-1}\left\{\mu_f\left(\mu_g\left(\frac{r}{100}\right)\right)\right\}$$

$$i.e.,\ \log^{[l]} \mu_h^{-1}\mu_{f\circ g}(r) \geq \left(\rho_h^{[l]}(f) - \varepsilon\right)\log\mu_g\left(\frac{r}{100}\right)$$

$$i.e.,\ \log^{[l]} \mu_h^{-1}\mu_{f\circ g}(r) \geq \left(\rho_h^{[l]}(f) - \varepsilon\right)\cdot \exp^{[l-1]}\left[\left(\frac{r}{100}\right)^{\left(\lambda_g^{[l+1]}-\varepsilon\right)}\right]$$

$$i.e.,\ \frac{\log^{[l]} \mu_h^{-1}\mu_{f\circ g}(r)}{\log r} \geq \frac{\left(\rho_h^{[l]}(f) - \varepsilon\right)\cdot \exp^{[l-1]}\left[\left(\frac{r}{100}\right)^{\left(\lambda_g^{[l+1]}-\varepsilon\right)}\right]}{\log r}$$

$$i.e.,\ \limsup_{r\to\infty}\frac{\log^{[l]} \mu_h^{-1}\mu_{f\circ g}(r)}{\log r} \geq \liminf_{r\to\infty}\frac{\left(\rho_h^{[l]}(f) - \varepsilon\right)\cdot \exp^{[l-1]}\left[\left(\frac{r}{100}\right)^{\left(\lambda_g^{[l+1]}-\varepsilon\right)}\right]}{\log r}$$

$$i.e.,\ \rho_h^{[l]}(f\circ g) = \infty \, . \tag{1}$$

Now in view of (1), we obtain that

$$\limsup_{r\to\infty} \frac{\log^{[l]} \mu_h^{-1}\left(\mu_{f\circ g}(r)\right)}{\log^{[l]} \mu_h^{-1}\left(\mu_f(r)\right)} \geq \limsup_{r\to\infty} \frac{\log^{[l]} \mu_h^{-1}\left(\mu_{f\circ g}(r)\right)}{\log r} \cdot \liminf_{r\to\infty}\frac{\log r}{\log^{[l]} \mu_h^{-1}\left(\mu_f(r)\right)}$$

$$i.e.,\ \limsup_{r\to\infty} \frac{\log^{[l]} \mu_h^{-1}\left(\mu_{f\circ g}(r)\right)}{\log^{[l]} \mu_h^{-1}\left(\mu_f(r)\right)} \geq \rho_h^{[l]}(f\circ g)\cdot\frac{1}{\rho_h^{[l]}(f)}$$

$$i.e.,\ \limsup_{r\to\infty} \frac{\log^{[l]} \mu_h^{-1}\left(\mu_{f\circ g}(r)\right)}{\log^{[l]} \mu_h^{-1}\left(\mu_f(r)\right)} = \infty \, .$$

Thus the theorem follows. □

Theorem 3.2. *Let f, g and h be any three entire functions satisfying $g(0) = 0$, $\lambda_h^{[l]}(f) > 0$ and $\rho_g^{[l+1]} > 0$ where $l \geq 1$. Then*

$$\limsup_{r\to\infty} \frac{\log^{[l]} \mu_h^{-1}\left(\mu_{f\circ g}(r)\right)}{\log^{[l]} \mu_h^{-1}\left(\mu_f(r)\right)} = \infty \, .$$

Theorem 3.3. *Let f, g and h be any three entire functions such that $g(0) = 0$, $\lambda_h^{[l]}(f) > 0$ and $\lambda_g^{[l+1]} > 0$. Then*

$$\limsup_{r\to\infty} \frac{\log^{[l]} \mu_h^{-1}\left(\mu_{f\circ g}(r)\right)}{\log^{[l]} \mu_h^{-1}\left(\mu_f(r)\right)} = \infty .$$

The proofs of Theorem 3.2 and Theorem 3.3 are omitted as those can be carried out in the line of Theorem 3.1.

Theorem 3.4. *Let f, g and h be any three entire functions with $0 < \lambda_h^{[l]}(g) \leq \rho_h^{[l]}(g) < \infty$ and $g(0) = 0$ and*

$$\limsup_{r\to\infty} \frac{\log^{[l]} \mu_h^{-1}\left(\mu_f(r)\right)}{\log^{[l]} \mu_h^{-1}(r)} = A, \text{ a real number } < \infty.$$

Then

$$\lambda_h^{[l]}(f\circ g) \leq A\lambda_h^{[l]}(g) \leq \rho_h^{[l]}(f\circ g) \leq A\rho_h^{[l]}(g)$$

where $l \geq 1$.

Proof. Since $\mu_h^{-1}(r)$ is an increasing function of r, it follows from Lemma 2.1, Lemma 2.2 and Lemma 2.3 for all sufficiently large values of r that

$$\mu_h^{-1}\mu_{f\circ g}(r) \geq \mu_h^{-1}\left\{\mu_f\left(\mu_g\left(\frac{r}{100}\right)\right)\right\} \tag{2}$$

and

$$\mu_h^{-1}\mu_{f\circ g}(r) \leq \mu_h^{-1}\left\{\mu_f\left(\mu_g(26r)\right)\right\} \tag{3}$$

respectively.
Therefore from (2) , we get for all sufficiently large values of r that

$$\begin{aligned} \log^{[l]} \mu_h^{-1}\mu_{f\circ g}(r) &\geq \log^{[l]} \mu_h^{-1}\left\{\mu_f\left(\mu_g\left(\frac{r}{100}\right)\right)\right\} \\ i.e., \ \frac{\log^{[l]} \mu_h^{-1}\mu_{f\circ g}(r)}{\log r} &\geq \frac{\log^{[l]} \mu_h^{-1}\left\{\mu_f\left(\mu_g\left(\frac{r}{100}\right)\right)\right\}}{\log r} \end{aligned}$$

$$\begin{aligned} & i.e., \ \frac{\log^{[l]} \mu_h^{-1}\mu_{f\circ g}(r)}{\log r} \\ \geq \ & \frac{\log^{[l]} \mu_h^{-1}\left\{\mu_f\left(\mu_g\left(\frac{r}{100}\right)\right)\right\}}{\log^{[l]} \mu_h^{-1}\left(\mu_g\left(\frac{r}{100}\right)\right)} \cdot \frac{\log^{[l]} \mu_h^{-1}\left(\mu_g\left(\frac{r}{100}\right)\right)}{\log r} \end{aligned}$$

$$\begin{aligned} & i.e., \ \limsup_{r\to\infty} \frac{\log^{[l]} \mu_h^{-1}\mu_{f\circ g}(r)}{\log r} \\ \geq \ & \limsup_{r\to\infty} \left[\frac{\log^{[l]} \mu_h^{-1}\left\{\mu_f\left(\mu_g\left(\frac{r}{100}\right)\right)\right\}}{\log^{[l]} \mu_h^{-1}\left(\mu_g\left(\frac{r}{100}\right)\right)} \cdot \frac{\log^{[l]} \mu_h^{-1}\left(\mu_g\left(\frac{r}{100}\right)\right)}{\log r}\right] \\ & i.e., \ \limsup_{r\to\infty} \frac{\log^{[l]} \mu_h^{-1}\mu_{f\circ g}(r)}{\log r} \\ \geq \ & \limsup_{r\to\infty} \frac{\log^{[l]} \mu_h^{-1}\left\{\mu_f\left(\mu_g\left(\frac{r}{100}\right)\right)\right\}}{\log^{[l]} \mu_h^{-1}\left(\mu_g\left(\frac{r}{100}\right)\right)} \liminf_{r\to\infty} \frac{\log^{[l]} \mu_h^{-1}\left(\mu_g\left(\frac{r}{100}\right)\right)}{\log r} \end{aligned}$$

$$i.e.,\ \rho_h^{[l]}(f\circ g)\geq A\cdot\lambda_h^{[l]}(g)\ . \tag{4}$$

Similarly from (3) , it follows for all sufficiently large values of r that

$$\begin{aligned}\log^{[l]}\mu_h^{-1}\mu_{f\circ g}(r) &\leq \log^{[l]}\mu_h^{-1}\left\{\mu_f\left(\mu_g(26r)\right)\right\}\\ i.e.,\ \frac{\log^{[l]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log r} &\leq \frac{\log^{[l]}\mu_h^{-1}\left\{\mu_f\left(\mu_g(26r)\right)\right\}}{\log r}\end{aligned}$$

$$i.e.,\ \frac{\log^{[l]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log r}$$
$$\leq \frac{\log^{[l]}\mu_h^{-1}\left\{\mu_f\left(\mu_g(26r)\right)\right\}}{\log^{[l]}\mu_h^{-1}\left(\mu_g(26r)\right)}\cdot\frac{\log^{[l]}\mu_h^{-1}\left(\mu_g(26r)\right)}{\log r} \tag{5}$$

$$i.e.,\ \liminf_{r\to\infty}\frac{\log^{[l]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log r}$$
$$\leq \liminf_{r\to\infty}\left[\frac{\log^{[l]}\mu_h^{-1}\left\{\mu_f\left(\mu_g(26r)\right)\right\}}{\log^{[l]}\mu_h^{-1}\left(\mu_g(26r)\right)}\cdot\frac{\log^{[l]}\mu_h^{-1}\left(\mu_g(26r)\right)}{\log r}\right]$$

$$i.e.,\ \liminf_{r\to\infty}\frac{\log^{[l]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log r}$$
$$\leq \limsup_{r\to\infty}\frac{\log^{[l]}\mu_h^{-1}\left\{\mu_f\left(\mu_g(26r)\right)\right\}}{\log^{[l]}\mu_h^{-1}\left(\mu_g(26r)\right)}\cdot\liminf_{r\to\infty}\frac{\log^{[l]}\mu_h^{-1}\left(\mu_g(26r)\right)}{\log r}$$

$$i.e.,\ \lambda_h^{[l]}(f\circ g)\leq A.\lambda_h^{[l]}(g)\ . \tag{6}$$

Also from (5) , we obtain for all sufficiently large values of r that

$$\limsup_{r\to\infty}\frac{\log^{[l]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log r}$$
$$\leq \limsup_{r\to\infty}\left[\frac{\log^{[l]}\mu_h^{-1}\left\{\mu_f\left(\mu_g(26r)\right)\right\}}{\log^{[l]}\mu_h^{-1}\left(\mu_g(26r)\right)}\cdot\frac{\log^{[l]}\mu_h^{-1}\left(\mu_g(26r)\right)}{\log r}\right]$$

$$i.e.,\ \limsup_{r\to\infty}\frac{\log^{[l]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log r}$$
$$\leq \limsup_{r\to\infty}\frac{\log^{[l]}\mu_h^{-1}\left\{\mu_f\left(\mu_g(26r)\right)\right\}}{\log^{[l]}\mu_h^{-1}\left(\mu_g(26r)\right)}\cdot\limsup_{r\to\infty}\frac{\log^{[l]}\mu_h^{-1}\left(\mu_g(26r)\right)}{\log r}$$

$$i.e.,\ \rho_h^{[l]}(f\circ g)\leq A.\rho_h^{[l]}(g)\ . \tag{7}$$

Therefore the theorem follows from (4), (6) and (7) . □

Theorem 3.5. *Let f, g and h be any three entire functions satisfying $g(0)=0$ and $0<\lambda_h^{[l]}(g)\leq\rho_h^{[l]}(g)<\infty$ and*

$$\liminf_{r\to\infty}\frac{\log^{[l]}\mu_h^{-1}\left(\mu_f(r)\right)}{\log^{[l]}\mu_h^{-1}(r)}=A\text{, a real number }<\infty.$$

Then

$$\lambda_h^{[l]}(f \circ g) \le A\rho_h^{[l]}(g) \le \rho_h^{[l]}(f \circ g)$$

where $l \ge 1$.

The proof of Theorem 3.5 is omitted because it can be carried out in the line of Theorem 3.4.

Theorem 3.6. *Let* f, g *and* h *be any three entire functions satisfying*

$$\limsup_{r\to\infty} \frac{\log^{[l]} \mu_h^{-1}\left(\mu_g(r)\right)}{\left(\log^{[2]} r\right)^{\alpha}} = A, \text{ a real number } > 0,$$

$$\liminf_{r\to\infty} \frac{\log\left[\frac{\log^{[l]} \mu_h^{-1}\left(\mu_f(r)\right)}{\log^{[l]} \mu_h^{-1}(r)}\right]}{\left[\log^{[l]} \mu_h^{-1}(r)\right]^{\beta}} = B, \text{ a real number } > 0$$

and $g(0) = 0$ *for any* α, β *satisfying* $\alpha > 1,\ 0 < \beta < 1,\ \alpha\beta > 1$ *and* $l \ge 1$. *Then*

$$\rho_h^{[l]}(f \circ g) = \infty\,.$$

Proof. From (i) we have for a sequence of values of r tending to infinity we get that

$$\log^{[l]} \mu_h^{-1}\left(\mu_g(r)\right) \ge (A-\varepsilon)\left(\log^{[2]} r\right)^{\alpha} \tag{8}$$

and from (ii) we obtain for all sufficiently large values of r that

$$\begin{aligned} \log\left[\frac{\log^{[l]} \mu_h^{-1}\left(\mu_f(r)\right)}{\log^{[l]} \mu_h^{-1}(r)}\right] &\ge (B-\varepsilon)\left[\log^{[l]} \mu_h^{-1}(r)\right]^{\beta} \\ i.e.,\ \frac{\log^{[l]} \mu_h^{-1}\left(\mu_f(r)\right)}{\log^{[l]} \mu_h^{-1}(r)} &\ge \exp\left[(B-\varepsilon)\left[\log^{[l]} \mu_h^{-1}(r)\right]^{\beta}\right]. \end{aligned}$$

Since $\mu_g(r)$ is continuous, increasing and unbounded function of r, we get from above for all sufficiently large values of r that

$$\frac{\log^{[l]} \mu_h^{-1}\left(\mu_f(\mu_g(r))\right)}{\log^{[l]} \mu_h^{-1}\left(\mu_g(r)\right)} \ge \exp\left[(B-\varepsilon)\left[\log^{[l]} \mu_h^{-1}\left(\mu_g(r)\right)\right]^{\beta}\right]. \tag{9}$$

Also $\mu_h^{-1}(r)$ is increasing function of r, it follows from (8), (9), Lemma 2.2 and Lemma 2.3 for a sequence of values of r tending to infinity that

$$\begin{aligned} \frac{\log^{[l]} \mu_h^{-1}\mu_{f\circ g}(r)}{\log r} &\ge \frac{\log^{[l]} \mu_h^{-1}\left\{\mu_f\left(\frac{1}{24}\mu_g\left(\frac{r}{4}\right)\right)\right\}}{\log r} \\ i.e.,\ \frac{\log^{[l]} \mu_h^{-1}\mu_{f\circ g}(r)}{\log r} &\ge \frac{\log^{[l]} \mu_h^{-1}\left\{\mu_f\left(\mu_g\left(\frac{r}{100}\right)\right)\right\}}{\log r} \end{aligned}$$

$$\begin{aligned} &i.e.,\ \frac{\log^{[l]} \mu_h^{-1}\mu_{f\circ g}(r)}{\log r} \\ &\ge \frac{\log^{[l]} \mu_h^{-1}\left\{\mu_f\left(\mu_g\left(\frac{r}{100}\right)\right)\right\}}{\log^{[l]} \mu_h^{-1}\left(\mu_g\left(\frac{r}{100}\right)\right)} \cdot \frac{\log^{[l]} \mu_h^{-1}\left(\mu_g\left(\frac{r}{100}\right)\right)}{\log r} \end{aligned}$$

$$\begin{aligned}
&i.e., \ \frac{\log^{[l]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log r}\\
\geq\ &\exp\left[(B-\varepsilon)\left[\log^{[l]}\mu_h^{-1}\left(\mu_g\left(\frac{r}{100}\right)\right)\right]^{\beta}\right]\cdot\frac{(A-\varepsilon)\left(\log^{[2]}\left(\frac{r}{100}\right)\right)^{\alpha}}{\log r}\\
&i.e., \ \frac{\log^{[l]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log r}\\
\geq\ &\exp\left[(B-\varepsilon)(A-\varepsilon)^{\beta}\left(\log^{[2]}\left(\frac{r}{100}\right)\right)^{\alpha\beta}\right]\cdot\frac{(A-\varepsilon)\left(\log^{[2]}\left(\frac{r}{100}\right)\right)^{\alpha}}{\log r}
\end{aligned}$$

$$\begin{aligned}
&i.e., \ \frac{\log^{[l]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log r}\\
\geq\ &\exp\left[(B-\varepsilon)(A-\varepsilon)^{\beta}\left(\log^{[2]}\left(\frac{r}{100}\right)\right)^{\alpha\beta-1}\log^{[2]}\left(\frac{r}{100}\right)\right]\cdot\frac{(A-\varepsilon)\left(\log^{[2]}\left(\frac{r}{100}\right)\right)^{\alpha}}{\log r}
\end{aligned}$$

$$\begin{aligned}
&i.e., \ \frac{\log^{[l]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log r}\\
\geq\ &\left(\log\left(\frac{r}{100}\right)\right)^{(B-\varepsilon)(A-\varepsilon)^{\beta}\left(\log^{[2]}\left(\frac{r}{100}\right)\right)^{\alpha\beta-1}}\cdot\frac{(A-\varepsilon)\left(\log^{[2]}\left(\frac{r}{100}\right)\right)^{\alpha}}{\log r}\\
&i.e., \ \limsup_{r\to\infty}\frac{\log^{[l]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log r}\\
\geq\ &\liminf_{r\to\infty}\left(\log\left(\frac{r}{100}\right)\right)^{(B-\varepsilon)(A-\varepsilon)^{\beta}\left(\log^{[2]}\left(\frac{r}{100}\right)\right)^{\alpha\beta-1}}\cdot\frac{(A-\varepsilon)\left(\log^{[2]}\left(\frac{r}{100}\right)\right)^{\alpha}}{\log r}\ .
\end{aligned}$$

Since $\varepsilon\ (>0)$ is arbitrary and $\alpha > 1$, $\alpha\beta > 1$, the theorem follows from above. □

In the line of Theorem 3.6, one may also state the following two theorems without their proofs :

Theorem 3.7. *Let f, g and h be any three transcendental entire functions such that*

$$\begin{aligned}
\liminf_{r\to\infty}\frac{\log^{[l]}\mu_h^{-1}\left(\mu_g(r)\right)}{\left(\log^{[2]} r\right)^{\alpha}} &= A, \ \textit{a real number} > 0,\\
\limsup_{r\to\infty}\frac{\log\left[\frac{\log^{[l]}\mu_h^{-1}\left(\mu_f(r)\right)}{\log^{[l]}\mu_h^{-1}(r)}\right]}{\left[\log^{[l]}\mu_h^{-1}(r)\right]^{\beta}} &= B, \ \textit{a real number} > 0
\end{aligned}$$

and $g(0) = 0$ for any α, β with $\alpha > 1$, $0 < \beta < 1$, $\alpha\beta > 1$ and $l \geq 1$. Then

$$\rho_h^{[l]}(f\circ g) = \infty\ .$$

Theorem 3.8. *Let f, g and h be any three transcendental entire functions such that*

$$\liminf_{r\to\infty}\frac{\log^{[l]}\mu_h^{-1}\left(\mu_g(r)\right)}{\left(\log^{[2]}r\right)^{\alpha}} = A, \text{ a real number } > 0,$$

$$\liminf_{r\to\infty}\frac{\log\left[\frac{\log^{[l]}\mu_h^{-1}\left(\mu_f(r)\right)}{\log^{[l]}\mu_h^{-1}(r)}\right]}{\left[\log^{[l]}\mu_h^{-1}(r)\right]^{\beta}} = B, \text{ a real number } > 0$$

and $g(0)=0$ *for any* α,β *satisfying* $\alpha>1,\ 0<\beta<1,\ \alpha\beta>1$ *and* $l\geq 1$. *Then*

$$\lambda_h^{[l]}(f\circ g)=\infty\,.$$

Theorem 3.9. *Let f, g and h be any three entire functions such that*

$$(i)\ \limsup_{r\to\infty}\frac{\log^{[l]}\mu_h^{-1}\left(\mu_g(r)\right)}{(\log r)^{\alpha}} = A, \text{ a real number } > 0,$$

$$(ii)\ \liminf_{r\to\infty}\frac{\log^{[l]}\mu_h^{-1}\left(\mu_f(r)\right)}{\left(\log^{[l]}\mu_h^{-1}(r)\right)^{\beta+1}} = B, \text{ a real number } > 0$$

and $g(0)=0$ *for any* α,β *satisfying* $0<\alpha<1,\ \beta>0,\ \alpha(\beta+1)>1$ *and* $l\geq 1$ *Then*

$$\rho_h^{[l]}(f\circ g)=\infty\,.$$

Proof. From (i) we have for a sequence of values of r tending to infinity,

$$\log^{[l]}\mu_h^{-1}\left(\mu_g(r)\right)\geq (A-\varepsilon)(\log r)^{\alpha} \tag{10}$$

and from (ii) we obtain for all sufficiently large values of r that

$$\log^{[l]}\mu_h^{-1}\left(\mu_f(r)\right)\geq (B-\varepsilon)\left(\log^{[l]}\mu_h^{-1}(r)\right)^{\beta+1}\ .$$

Since $\mu_g(r)$ is continuous, increasing and unbounded function of r, we get from above for all sufficiently large values of r that

$$\log^{[l]}\mu_h^{-1}\left(\mu_f(\mu_g(r))\right)\geq (B-\varepsilon)\left(\log^{[l]}\mu_h^{-1}\left(\mu_g(r)\right)\right)^{\beta+1}\ . \tag{11}$$

Also $\mu_h^{-1}(r)$ is an increasing function of r, it follows from Lemma 2.2, Lemma 2.3, (10) and (11) for a sequence of values of r tending to infinity that

$$\begin{aligned}
\log^{[l]}\mu_h^{-1}\mu_{f\circ g}(r) &\geq \log^{[l]}\mu_h^{-1}\left\{\mu_f\left(\frac{1}{24}\mu_g\left(\frac{r}{4}\right)\right)\right\}\\
\text{i.e., } \log^{[l]}\mu_h^{-1}\mu_{f\circ g}(r) &\geq \log^{[l]}\mu_h^{-1}\left\{\mu_f\left(\mu_g\left(\frac{r}{100}\right)\right)\right\}\\
\text{i.e., } \log^{[l]}\mu_h^{-1}\mu_{f\circ g}(r) &\geq (B-\varepsilon)\left(\log^{[l]}\mu_h^{-1}\left(\mu_g\left(\frac{r}{100}\right)\right)\right)^{\beta+1}\\
\text{i.e., } \log^{[l]}\mu_h^{-1}\mu_{f\circ g}(r) &\geq (B-\varepsilon)\left[(A-\varepsilon)\left(\log\left(\frac{r}{100}\right)\right)^{\alpha}\right]^{\beta+1}\\
\text{i.e., } \log^{[l]}\mu_h^{-1}\mu_{f\circ g}(r) &\geq (B-\varepsilon)(A-\varepsilon)^{\beta+1}(\log r+O(1))^{\alpha(\beta+1)}\\
\text{i.e., } \frac{\log^{[l]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log r} &\geq \frac{(B-\varepsilon)(A-\varepsilon)^{\beta+1}(\log r+O(1))^{\alpha(\beta+1)}}{\log r}
\end{aligned}$$

$$i.e., \limsup_{r\to\infty} \frac{\log^{[l]} \mu_h^{-1} \mu_{f\circ g}(r)}{\log r} \geq \liminf_{r\to\infty} \frac{(B-\varepsilon)(A-\varepsilon)^{\beta+1}(\log r + O(1))^{\alpha(\beta+1)}}{\log r} .$$

Since $\varepsilon\,(>0)$ is arbitrary and $\alpha(\beta+1)>1$, it follows from above that

$$\rho_h^{[l]}(f\circ g) = \infty,$$

which proves the theorem. □

In the line of Theorem 3.9, one may state the following two theorems without their proofs :

Theorem 3.10. *Let f, g and h be any three entire functions such that*

$$\liminf_{r\to\infty} \frac{\log^{[l]} \mu_h^{-1}\left(\mu_g(r)\right)}{(\log r)^{\alpha}} = A, \text{ a real number } > 0,$$
$$\limsup_{r\to\infty} \frac{\log^{[l]} \mu_h^{-1}\left(\mu_f(r)\right)}{\left(\log^{[l]} \mu_h^{-1}(r)\right)^{\beta+1}} = B, \text{ a real number } > 0$$

and $g(0)=0$ for any α,β satisfying $0<\alpha<1$, $\beta>0$, $\alpha(\beta+1)>1$ and $l\geq 1$. Then

$$\rho_h^{[l]}(f\circ g) = \infty .$$

Theorem 3.11. *Let f, g and h be any three entire functions with*

$$\liminf_{r\to\infty} \frac{\log^{[l]} \mu_h^{-1}\left(\mu_g(r)\right)}{(\log r)^{\alpha}} = A, \text{ a real number } > 0,$$
$$\liminf_{r\to\infty} \frac{\log^{[l]} \mu_h^{-1}\left(\mu_f(r)\right)}{\left(\log^{[l]} \mu_h^{-1}(r)\right)^{\beta+1}} = B, \text{ a real number } > 0$$

and $g(0)=0$ for any α,β with $0<\alpha<1$, $\beta>0$, $\alpha(\beta+1)>1$ and $l\geq 1$. Then

$$\lambda_h^{[l]}(f\circ g) = \infty .$$

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