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On entire solutions of abstract degenerate differential equations of higher order

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Abstract. The main purpose of this paper is to continue our previous research study of degenerate k-regularized (C_1, C_2)-existence and uniqueness families in sequentially complete locally convex spaces [13] by investigating the entire solutions of abstract degenerate differential equations of higher order. We apply our results in the analysis of existence and uniqueness of entire solutions of the abstract Boussinesq-Love equation and the abstract Barenblatt-Zheltov-Kochina equation in finite domains.

1. Introduction and preliminaries

The first results on entire solutions of abstract degenerate differential equations of first order in locally convex spaces have been obtained by V. Fedorov [8], who investigated strongly holomorphic groups of linear equations of Sobolev type. In this paper, the author has defined the notion of a relatively spectral regular operator and the notion of a phase space of degenerate differential equations of first order. Applications have been made to degenerate differential equations of first order with regularly elliptic differential operators.

On the other hand, in a series of our recent research studies we have investigated various types of abstract Volterra integro-differential equations and abstract degenerate fractional differential equations in locally convex spaces (cf. the forthcoming monograph [9], written in co-autorship with V. Fedorov and R. Ponce, for a comprehensive survey of results, as well as [11] for non-degenerate case $B = I$). In [13], we have considered various types of degenerate k-regularized (C_1, C_2)-existence and uniqueness families applicable in the analysis of the following abstract degenerate multi-term problem:

$$\begin{aligned} &B\mathbf{D}_t^{\alpha_n} u(t) + \sum_{i=1}^{n-1} A_i \mathbf{D}_t^{\alpha_i} u(t) = A\mathbf{D}_t^{\alpha} u(t) + f(t), \quad t \geq 0; \\ &u^{(j)}(0) = u_j, \ j = 0, \cdots, \lceil \alpha_n \rceil - 1, \end{aligned} \tag{1.1}$$

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where $n \in \mathbb{N} \setminus \{1\}$, $A := A_0$, $B := A_n$ and $A_1, \cdots, A_{n-1}$ are closed linear operators on a sequentially complete locally convex space X, $0 \leq \alpha_1 < \cdots < \alpha_n$, $0 \leq \alpha := \alpha_0 < \alpha_n$, $f(t)$ is a continuous X-valued function on, and $\mathbf{D}_t^{\alpha}$ denotes the Caputo fractional derivative of order α. In this paper, we analyze entire properties of degenerate resolvent operator families introduced in [13, Section 3] and apply obtained results in the study of abstract Boussinesq-Love equation, which is important in the modeling the longitudinal waves in an elastic bar with the transverse inertia, and the abstract Barenblatt-Zheltov-Kochina equation, which is important in the study of fluid filtration in fissured rocks, as well as in the studies of moisture transfer in soil and the process of two-temperature heat conductivity. We reconsider some results obtained by G. A. Sviridyuk and A. A. Zamyshlyaeva in [22, Section 5], and slightly improve the assertion of [19, Theorem 5.1.3(ii)] in L^2 type spaces, concerning the well-posedness of abstract Barenblatt-Zheltov-Kochina equation. The paper is intended to be a note and contains only one theoretical result (Theorem 2.2), in which we essentially apply the structural results on vector-valued Laplace transform of holomorphic functions in the analysis of existence and uniqueness of entire solutions of degenerate differential equations with integer order derivatives.

Henceforth we assume that X is a Hausdorff sequentially complete locally convex space over the field of complex numbers, SCLCS for short. By X^* we denote the dual space of X. If Y is also an SCLCS over the field of complex numbers, then by $L(Y, X)$ we denote the space consisting of all continuous linear mappings from Y into X; $L(X) \equiv L(X, X)$. By $⊛_X$ ($⊛$, if there is no risk for confusion), we denote the fundamental system of seminorms which defines the topology of X; the fundamental system of seminorms which defines the topology on Y is denoted by $⊛_Y$. Let $0 < \tau \leq \infty$. A strongly continuous operator family $(W(t))_{t\in[0,\tau)} \subseteq L(Y, X)$ is said to be locally equicontinuous iff, for every $T \in (0, \tau)$ and for every $p \in ⊛_X$, there exist $q_p \in ⊛_Y$ and $c_p > 0$ such that $p(W(t)y) \leq c_p q_p(y)$, $y \in Y$, $t \in [0, T]$; the notions of equicontinuity of $(W(t))_{t\in[0,\tau)}$ and the exponential equicontinuity of $(W(t))_{t\geq 0}$ are defined similarly. Notice that $(W(t))_{t\in[0,\tau)}$ is locally equicontinuous whenever the space Y is barreled ([16]). By $\mathcal{B}$ we denote the family consisting of all bounded subsets of Y. Define $p_{\mathbb{B}}(T) := \sup_{y\in\mathbb{B}} p(Ty)$, $p \in ⊛_X$, $\mathbb{B} \in \mathcal{B}$, $T \in L(Y, X)$. Then $p_{\mathbb{B}}(\cdot)$ is a seminorm on $L(Y, X)$ and the system $(p_{\mathbb{B}})_{(p,\mathbb{B})\in ⊛_X\times\mathcal{B}}$ induces the Hausdorff locally convex topology on $L(Y, X)$. If X is a Banach space, then we denote by $\|x\|$ the norm of an element $x \in X$. Let A be a closed linear operator with domain and range contained in X. Then, by $D(A)$, $N(A)$, $R(A)$, $\rho(A)$ and $\sigma(A)$ we denote the domain, kernel space, range, the resolvent set and spectrum of A, respectively; $R(\lambda : A) \equiv (\lambda - A)^{-1}$ ($\lambda \in \rho(A)$). Since no confusion seems likely, we will identify A with its graph. Set $p_A(x) := p(x) + p(Ax)$, $x \in D(A)$, $p \in ⊛$. Then the calibration $(p_A)_{p\in⊛}$ induces the Hausdorff sequentially complete locally convex topology on $D(A)$; we denote this space simply by $[D(A)]$.

If V is a general topological vector space, then a function $f : \Omega \to V$, where Ω is an open non-empty subset of $\mathbb{C}$, is said to be analytic iff it is locally expressible in a neighborhood of any point $z \in \Omega$ by a uniformly convergent power series with coefficients in V. We refer the reader to [2], [11, Section 1.1] and references cited there for the basic information about vector-valued analytic functions. In our approach the space X is sequentially complete, so that the analyticity of a mapping $f : \Omega \to X$ is equivalent with its weak analyticity.

We need to introduce the following condition:

(P1) $K(\cdot)$ is Laplace transformable, i.e., it is locally integrable on $[0, \infty)$ and there exists $\beta \in \mathbb{R}$ so that

$$\tilde{K}(\lambda) := \mathcal{L}(K)(\lambda) := \lim_{b\to\infty} \int_0^b e^{-\lambda t} K(t)\, dt := \int_0^\infty e^{-\lambda t} K(t)\, dt$$

exists for all $\lambda \in \mathbb{C}$ with $\text{Re}\lambda > \beta$. Put abs(K) :=inf$\{\text{Re}\lambda : \tilde{K}(\lambda) \text{ exists}\}$, and denote by $\mathcal{L}^{-1}$ the inverse Laplace transform.

We say that a function $h(\cdot)$ belongs to the class $LT - X$ iff there exists a function $f \in C([0, \infty) : X)$ such that for each $p \in ⊛$ there exists $M_p > 0$ satisfying $p(f(t)) \leq M_p e^{at}$, $t \geq 0$ and $h(\lambda) = (\mathcal{L}f)(\lambda), \lambda > a$. We refer the reader to [2], [18], [23, Chapter 1] and [11, Section 1.2] for more details concerning the vector-valued Laplace transform. The basic information about abstract degenerate differential equations can be obtained by consulting [1, 3-4, 6-9, 13-15, 17, 19-22, 24].

Define $\Sigma_\theta := \{\lambda \in \mathbb{C} : \lambda \neq 0, |\arg(\lambda)| < \theta\}$ ($\theta \in (0, \pi]$) and $\lceil \beta \rceil := \inf\{n \in \mathbb{Z} : \beta \leq n\}$ ($\beta \in \mathbb{R}$). The Gamma function is denoted by $\Gamma(\cdot)$ and the principal branch is always used to take the powers; the convolution like mapping $*$ is given by $f * g(t) := \int_0^t f(t-s)g(s)\,ds$. Set $g_\zeta(t) := t^{\zeta-1}/\Gamma(\zeta)$, $0^\zeta := 0$ ($\zeta > 0$, $t > 0$) and $g_0(t) :=$ the Dirac δ-distribution. If $f : [0, \infty) \to X$ is a continuous function, then $g_0 * f \equiv f$. In our considerations, $k(t)$ will be a non-zero continuous scalar-valued function defined for $t \geq 0$.

The theory of abstract degenerate fractional differential equations is still very undeveloped and, because of that, we would like to present here the most important structural results obtained recently in [13] for the problem (1.1) in its general form, with the orders α_i not necessarily being integer numbers (deeply believing that it will not complicate the perception of the reader). We will use these results later on as an auxiliary tool in our analysis. Let $\zeta > 0$. Then the Caputo fractional derivative $\mathbf{D}_t^\zeta u$ ([11]) is defined for those functions $u \in C^{\lceil \zeta \rceil - 1}([0, \infty) : X)$ for which $g_{\lceil \zeta \rceil - \zeta} * (u - \sum_{j=0}^{\lceil \zeta \rceil - 1} u^{(j)}(0) g_{j+1}) \in C^{\lceil \zeta \rceil}([0, \infty) : X)$, by

$$\mathbf{D}_t^\zeta u(t) := \frac{d^{\lceil \zeta \rceil}}{dt^{\lceil \zeta \rceil}} \left[g_{\lceil \zeta \rceil - \zeta} * \left(u - \sum_{j=0}^{\lceil \zeta \rceil - 1} u^{(j)}(0) g_{j+1} \right) \right].$$

Set $\mathbb{N}_n := \{1, 2, \cdots, n\}$, $\mathbb{N}_n^0 := \mathbb{N}_n \cup \{0\}$, $m := \lceil \alpha \rceil$, $\alpha_0 := \alpha$, $m_i := \lceil \alpha_i \rceil$ ($i \in \mathbb{N}_n^0$) and

$$\mathbf{P}_z := B + \sum_{j=1}^{n-1} z^{\alpha_j - \alpha_n} A_j - z^{\alpha - \alpha_n} A, \quad z \in \mathbb{C} \setminus \{0\}.$$

By a strong solution of problem (1.1), we mean any function $u \in C^{m_n - 1}([0, \infty) : X)$ satisfying that all terms $B\mathbf{D}_t^{\alpha_n} u(t)$, $A_1 \mathbf{D}_t^{\alpha_1} u(t)$, $\cdots$, $A_{n-1} \mathbf{D}_t^{\alpha_{n-1}} u(t)$, $A\mathbf{D}_t^{\alpha} u(t)$ are well-defined and continuous for $t \geq 0$, as well as that (1.1) holds.

The following notion plays an important role in the analysis of existence and uniqueness of strong solutions of problem (1.1).

Definition 1.1. (cf. [13, Definition 3.1] with $\tau = \infty$) Suppose $k \in C([0, \infty))$, $C_1 \in L(Y, X)$, and $C_2 \in L(X)$ is injective.

(i) A strongly continuous operator family $(E(t))_{t \geq 0} \subseteq L(Y, X)$ is said to be a k-regularized C_1-existence family for (1.1) iff, for every $y \in Y$, the following holds: $E(\cdot)y \in C^{m_n - 1}([0, \infty) : [D(B)])$, $E^{(i)}(0)y = 0$ for every $i \in \mathbb{N}_0$ with $i < m_n - 1$, $A_j(g_{\alpha_n - \alpha_j} * E^{(m_n - 1)})(\cdot)y \in C([0, \infty) : X)$ for $0 \leq j \leq n$, and

$$BE^{(m_n-1)}(t)y + \sum_{j=1}^{n-1} A_j \Big(g_{\alpha_n - \alpha_j} * E^{(m_n - 1)} \Big)(t)y - A\Big(g_{\alpha_n - \alpha} * E^{(m_n - 1)} \Big)(t)y = k(t)C_1 y, \quad t \geq 0.$$

(ii) A strongly continuous operator family $(U(t))_{t \geq 0} \subseteq L(X)$ is said to be a k-regularized C_2-uniqueness family for (1.1) iff, for every $t \geq 0$ and $x \in \bigcap_{0 \leq j \leq n} D(A_j)$, the following holds:

$$U(t)Bx + \sum_{j=1}^{n-1} \Big(g_{\alpha_n - \alpha_j} * U(\cdot)A_j x \Big)(t) - \Big(g_{\alpha_n - \alpha} * U(\cdot)Ax \Big)(t)y = \Big(k * g_{m_n - 1} \Big)(t) C_2 x.$$

(iii) A strongly continuous family $((E(t))_{t \geq 0}, (U(t))_{t \geq 0}) \subseteq L(Y, X) \times L(X)$ is said to be a k-regularized (C_1, C_2)-existence and uniqueness family for (1.1) iff $(E(t))_{t \geq 0}$ is a k-regularized C_1-existence family for (1.1), and $(U(t))_{t \geq 0}$ is a k-regularized C_2-uniqueness family for (1.1).

In the case that $k(t) = g_{\zeta+1}(t)$, where $\zeta \geq 0$, it is also said that $(E(t))_{t \geq 0}$, resp. $(U(t))_{t \geq 0}$, is a ζ-times integrated C_1-existence family for (1.1), resp., ζ-times integrated C_2-uniqueness family for (1.1); 0-times integrated C_1-existence family for (1.1), resp., 0-times integrated C_2-uniqueness family for (1.1), is also said to be a C_1-existence family for (1.1), resp., C_2-uniqueness family for (1.1).

Suppose $0 \le i \le m_n - 1$. In [13, Definition 3.3], we have introduced the following sets: $D_i' =: \{j \in \mathbb{N}_{n-1}^0 : m_j - 1 \ge i\}$, $D_i'' := \mathbb{N}_{n-1}^0 \setminus D_i'$ and

$$\mathbf{D}_i := \left\{ u_i \in \bigcap_{j \in D_i''} D(A_j) : A_j u_i \in R(C_1),\ j \in D_i'' \right\}.$$

These sets will be important in our further work.

We need the following results from [13]:

Lemma 1.2. *Suppose $k(t)$ satisfies (P1), $(U(t))_{t\ge 0} \subseteq L(X)$, $\omega \ge \max(0, abs(k))$, $C_1 \in L(Y,X)$ and $C_2 \in L(X)$ is injective.*

(i) Let the operator $\mathbf{P}_z$ be injective for every $z > \omega$ with $\tilde{k}(z) \neq 0$. Suppose, additionally, that there exist strongly continuous operator families $(W(t))_{t\ge 0} \subseteq L(Y,X)$ and $(W_j(t))_{t\ge 0} \subseteq L(Y,X)$ such that $\{e^{-\omega t} W(t) : t \ge 0\}$ and $\{e^{-\omega t} W_j(t) : t \ge 0\}$ are equicontinuous ($0 \le j \le n$) as well as that:

$$\int_0^\infty e^{-zt} W(t) y\, dt = \tilde{k}(z) \mathbf{P}_z^{-1} C_1 y \text{ and } \int_0^\infty e^{-zt} W_j(t) y\, dt = \tilde{k}(z) z^{\alpha_j - \alpha_n} A_j \mathbf{P}_z^{-1} C_1 y,$$

for every $z > \omega$ with $\tilde{k}(z) \neq 0$, $y \in Y$ and $j \in \mathbb{N}_n^0$. Then there exists a k-regularized C_1-existence family for (1.1), denoted by $(E(t))_{t\ge 0}$. Furthermore, $E^{(m_n-1)}(t)y = W(t)y$, $t \ge 0$, $y \in Y$ and $A_j(g_{\alpha_n - \alpha_j} * E^{(m_n-1)})(t)y = W_j(t)y$, $t \ge 0$, $y \in Y$, $j \in \mathbb{N}_n^0$.

(ii) Suppose $(U(t))_{t\ge 0}$ is strongly continuous and the operator family $\{e^{-\omega t} U(t) : t \ge 0\}$ is equicontinuous. Then $(U(t))_{t\ge 0}$ is a k-regularized C_2-uniqueness family for (1.1) iff, for every $x \in \bigcap_{j=0}^n D(A_j)$, the following holds:

$$\int_0^\infty e^{-zt} U(t) \mathbf{P}_z x\, dt = \tilde{k}(z) z^{1-m_n} C_2 x, \quad \Re \lambda > \omega.$$

(iii) Suppose $(U(t))_{t\ge 0}$ is a locally equicontinuous k-regularized C_2-uniqueness family for (1.1). Then there exists at most one strong solution of (1.1).

In the subsequent section, the existence of strong solutions of problem (1.1) will be goverened by C_1-existence families for (1.1). Then we will be in a position to essentially apply the following lemma:

Lemma 1.3. *(cf. [13, Theorem 3.6]) Suppose $(E(t))_{t\ge 0}$ is a C_1-existence family for (1.1) satisfying that the operator family $(E^{(m_n-1)}(t))_{t\ge 0} \subseteq L(Y,X)$ is locally equicontinuous, and $u_i \in \mathbf{D}_i$ for $0 \le i \le m_n - 1$. Let $g \in C^1([0,\infty) : Y)$ satisfy $C_1 g(t) = f(t)$, $t \ge 0$, and let $G \in C([0,\infty) : Y)$ satisfy $(g_{\alpha_n - m_n + 1} * g)(t) = (g_1 * G)(t)$, $t \ge 0$. Then the function*

$$u(t) = \sum_{i=0}^{m_n-1} u_i g_{i+1}(t) - \sum_{i=0}^{m_n-1} \sum_{j \in \mathbb{N}_{n-1} \setminus D_i} \Big(g_{\alpha_n - \alpha_j} * E^{(m_n-1-i)} \Big)(t) v_{i,j}$$
$$+ \sum_{i=m}^{m_n-1} \Big(g_{\alpha_n - \alpha} * E^{(m_n-1-i)} \Big)(t) v_{i,0} + \int_0^t E(t-s) G(s)\, ds, \quad t \ge 0, \tag{1.2}$$

is a strong solution of problem (1.1), where $v_{i,j} \in Y$ satisfy $A_j u_i = C_1 v_{i,j}$ for $0 \le j \le n - 1$.

2. The main result and its applications

In this section, we would like to propose a new operator theoretical method for seeking of entire solutions to abstract degenerate differential equations of higher order. Although our method can be used for proving some new results on the existence and uniqueness of analytical solutions to certain classes of abstract degenerate multi-term fractional Cauchy problems, we will consider here only the equations with integer order derivatives, so that our standing hypothesis henceforth will be that the orders $\alpha,\ \alpha_1,\cdot\cdot\cdot,\alpha_n$ of Caputo derivatives $\mathbf{D}_t^{\alpha}u(t),\ \mathbf{D}_t^{\alpha_1}u(t),\cdot\cdot\cdot,\mathbf{D}_t^{\alpha_n}u(t)$, appearing in (1.1), are non-negative integers. For more details about fractional-order case, we refer the reader to the forthcoming monograph [9].

Definition 2.1. Let $\alpha_i \in \mathbb{N}_0$ for all $i \in \mathbb{N}_n^0$, and let the function $u \in C^{\alpha_n-1}([0,\infty) : X)$ be a strong solution of problem (1.1). Then we say that $u(\cdot)$ is an entire solution of problem (1.1) iff the functions $u(\cdot)$ and $Bu^{(\alpha_n)}(\cdot)$, $A_1u^{(\alpha_1)}(\cdot)$, $\cdot\cdot\cdot$, $A_{n-1}u^{(\alpha_{n-1})}(\cdot)$, $Au^{(\alpha)}(\cdot)$ can be analytically extended from the interval $[0,\infty)$ to the whole complex plane.

The main result of this paper reads as follows.

Theorem 2.2. *Suppose $k(t)$ satisfies (P1), $C_1 \in L(Y,X)$ and $C_2 \in L(X)$ is injective. Let $\alpha_i \in \mathbb{N}_0$ for all $i \in \mathbb{N}_n^0$, and let there exist a locally equicontinuous k-regularized C_2-uniqueness family for (1.1). Suppose that there exists a sufficiently large number $R > 0$ such that the operator $\mathbf{P}_z$ is injective for $|z| \geq R$, as well as that the operator families $\{\mathbf{P}_z^{-1}C_1 : |z| \geq R\} \subseteq L(Y,X)$ and $\{z^{\alpha_j-\alpha_n}A_j\mathbf{P}_z^{-1}C_1 : |z| \geq R,\ j \in \mathbb{N}_n^0\} \subseteq L(Y,X)$ are equicontinuous. Let the mappings $z \mapsto \mathbf{P}_z^{-1}C_1y \in X$, $|z| > R$ and $z \mapsto z^{\alpha_j-\alpha_n}A_j\mathbf{P}_z^{-1}C_1y \in X$, $|z| > R$ be analytic for any $y \in Y$, $j \in \mathbb{N}_n^0$, and let there exist operators $D, D_0, D_1,\cdot\cdot\cdot, D_n \in L(Y,X)$ such that $\lim_{z\to\infty}\mathbf{P}_z^{-1}C_1y = Dy$ and $\lim_{z\to\infty}z^{\alpha_j-\alpha_n}A_j\mathbf{P}_z^{-1}C_1y = D_jy$ for any $y \in Y$, $j \in \mathbb{N}_n^0$. Suppose that $u_i \in \mathbf{D}_i$ for $0 \leq i \leq m_n - 1$, $v_{i,j} \in Y$ satisfy $A_ju_i = C_1v_{i,j}$ for $0 \leq j \leq n-1$, as well as that $g \in C^1([0,\infty) : Y)$ and $C_1g(t) = f(t)$, $t \geq 0$. Then there exists a unique strong solution of (1.1). Assume, additionally, that the function $t \mapsto g(t)$, $t \geq 0$ can be analytically extended to the whole complex plane, resp., to a continuously differentiable function $\mathbb{R} \mapsto Y$. Then there exists a unique entire solution $u(\cdot)$ of (1.1), resp., the function $t \mapsto u(t)$, $t \geq 0$ can be extended to an α_n-times continuously differentiable function $\mathbb{R} \mapsto X$ and the functions $Bu^{(\alpha_n)}(\cdot)$, $A_1u^{(\alpha_1)}(\cdot)$, $\cdot\cdot\cdot$, $A_{n-1}u^{(\alpha_{n-1})}(\cdot)$, $Au^{(\alpha)}(\cdot)$ can be extended to continuously differentiable functions $\mathbb{R} \mapsto X$. Furthermore, in any case set out above, the existence of a positive real number $\omega' > 0$ such that the set $\{e^{-\omega's}g(s) : s \geq 0\}$, resp., $\{e^{-\omega'z}g(z) : z \in \mathbb{C}\}$ ($\{e^{-\omega'|s|}g(s) : s \in \mathbb{R}\}$) is bounded in Y, implies the existence of a positive real number $\omega'' > 0$ such that the set $\{e^{-\omega''s}u(s) : s \geq 0\}$, resp., $\{e^{-\omega''z}u(z) : z \in \mathbb{C}\}$ ($\{e^{-\omega''|s|}u(s) : s \in \mathbb{R}\}$) is bounded in X.*

Proof. Let $\beta \in (-\pi,\pi]$. Then, for every $\theta \in (0,\pi/2)$, there exists a sufficiently large number $\omega_{\beta,\theta} > 0$ satisfying that the function $q_{\beta,\theta}(z) := z^{-1}\mathbf{P}_{ze^{-i\beta}}^{-1}C_1 \in L(Y,X)$, $z \in \omega_{\beta,\theta} + \Sigma_{\theta+(\pi/2)}$ is well-defined, strongly analytic and that for each $\theta' \in (0,\theta)$ the operator family $\{z^{-1}(z-\omega_{\beta,\theta})\mathbf{P}_{ze^{-i\beta}}^{-1}C_1 : z \in \omega_{\beta,\theta} + \Sigma_{\theta'+(\pi/2)}\} \subseteq L(Y,X)$ is equicontinuous. By [11, Theorem 1.2.5(i)], we obtain that for each $y \in Y$ there exists an X-valued analytic mapping $z \mapsto W_{\beta,y}(z)$, $z \in \Sigma_{\pi/2}$ satisfying that, for every $\theta \in (0,\pi/2)$, we have that $\int_0^{\infty}e^{-zt}W_{\beta,y}(t)\,dt = z^{-1}\mathbf{P}_{ze^{-i\beta}}^{-1}C_1y$, $\Re z > \omega_{\beta,\theta}$ and the set $\{e^{-\omega_{\beta,\theta}z}W_{\beta,y}(z) : z \in \Sigma_{\theta'}\}$ is bounded in X ($y \in Y$, $\theta' \in (0,\theta)$). Define $W_\beta(z)y := W_{\beta,y}(z)$, $z \in \Sigma_{\pi/2}$, $y \in Y$. By the uniqueness theorem for Laplace transform, it readily follows that $W_\beta(z) : Y \to X$ is a linear mapping ($z \in \Sigma_{\pi/2}$); furthermore, we can argue as in the proofs of [11, Theorem 2.2.5] and [2, Theorem 2.6.1] so as to conclude that, for every $\theta \in (0,\pi/2)$, $\{e^{-\omega_{\beta,\theta}z}W_\beta(z) : z \in \Sigma_{\theta'}\} \subseteq L(Y,X)$ is an equicontinuous operator family ($\theta' \in (0,\theta)$).

Since $\lim_{z\to\infty}\mathbf{P}_z^{-1}C_1y = Dy$ ($y \in Y$), we can apply [11, Theorem 1.2.5(ii)/(iii)] in order to see that, for every $y \in Y$ and $\theta \in (0,\pi/2)$, we have $\lim_{z\to 0, z\in\Sigma_\theta}W_\beta(z)y = Dy$. Now we will prove that, for every $z \in \Sigma_{\pi/2}\cap e^{i\pi/2}\Sigma_{\pi/2}$, we have $W_0(z) = W_{\pi/2}(ze^{-i\pi/2})$. Let $y \in Y$ be fixed, and let $\arg(z) = \theta$. Set $\Gamma_\theta := \{e^{i\theta}t : t \geq 0\}$. Using Cauchy's formula, it is not difficult to see that, for all sufficiently large values of positive real parameter $s > 0$, we have

$$\int_0^{\infty} e^{-st}W_0\big(e^{i\theta}t\big)y\,dt = e^{-i\theta}\int_{\Gamma_\theta} e^{-se^{-i\theta}v}W_0(v)y\,dv = e^{-i\theta}\int_0^{\infty} e^{-se^{-i\theta}v}W_0(v)y\,dv = s^{-1}\mathbf{P}_{se^{-i\theta}}^{-1}C_1y,\quad y \in Y.$$

Similarly, $\int_0^\infty e^{-st}W_{\pi/2}(e^{i(\theta-\pi/2)}t)y\,dt = s^{-1}\mathbf{P}^{-1}_{se^{i\theta}}C_1y$, $y \in Y$ so that the uniqueness theorem for Laplace transform implies that $W_0(e^{i\theta}t) = W_{\pi/2}(e^{i(\theta-\pi/2)}t)$ for all $t \geq 0$. Plugging $t = |z|$, we get that $W_0(z) = W_{\pi/2}(ze^{-i\pi/2})$, as claimed.

A similar line of reasoning shows that the operator family $(W(z))_{z\in\mathbb{C}}$, where

$$W(z) := \begin{cases} W_0(z), \ z \in \Sigma_{\pi/2}, \\ W_{\pi/2}(ze^{-i\pi/2}), \ \text{if } z \in e^{i\pi/2}\Sigma_{\pi/2}, \\ W_{-\pi/2}(ze^{i\pi/2}), \ \text{if } z \in e^{-i\pi/2}\Sigma_{\pi/2}, \\ W_{\pi}(ze^{-i\pi}), \ \text{if } z \in e^{i\pi}\Sigma_{\pi/2}, \\ D, \ \text{if } z = 0, \end{cases}$$

is well-defined. By the foregoing, we obtain that there exists $\omega > 0$ such that the operator family $\{e^{-\omega z}W(z) : z \in \mathbb{C}\} \subseteq L(Y,X)$ is equicontinuous as well as that, for every $y \in Y$, the mapping $z \mapsto W(z)y$, $z \in \mathbb{C}$ is entire (because it is weakly entire; this follows from the fact that for each $x^* \in X^*$ the mapping $z \mapsto \langle x^*, W(z)y\rangle$, $z \in \mathbb{C} \setminus \{0\}$ is analytic and has the limit $\langle x^*, Dy\rangle$ as $z \to 0$).

Replacing the function $z \mapsto q_{\beta,\theta}(z) = z^{-1}\mathbf{P}^{-1}_{ze^{-i\beta}}C_1 \in L(Y,X)$, $z \in \omega_{\beta,\theta} + \Sigma_{\theta+(\pi/2)}$ with the function $z \mapsto q_{\beta,\theta,j}(z) := z^{-1}(ze^{-i\beta})^{\alpha_j-\alpha_n}A_j\mathbf{P}^{-1}_{ze^{-i\beta}}C_1 \in L(Y,X)$, $z \in \omega_{\beta,\theta} + \Sigma_{\theta+(\pi/2)}$ in the first part of proof ($\theta \in (0,\pi/2)$, $j \in \mathbb{N}_n^0$), we can define for each $y \in Y$ an X-valued analytic mapping $z \mapsto W_{\beta,j,y}(z)$, $z \in \Sigma_{\pi/2}$ satisfying that, for every $\theta \in (0,\pi/2)$, we have that $\int_0^\infty e^{-zt}W_{\beta,j,y}(t)\,dt = z^{-1}(ze^{-i\beta})^{\alpha_j-\alpha_n}A_j\mathbf{P}^{-1}_{ze^{-i\beta}}C_1y$, $\Re z > \omega_{\beta,\theta}$ and the set $\{e^{-\omega_{\beta,\theta}z}W_{\beta,j,y}(z) : z \in \Sigma_{\theta'}\}$ is bounded in X ($y \in Y$, $\theta' \in (0,\theta)$). Define now $W_{\beta,j}(z)y := W_{\beta,j,y}(z)$, $z \in \Sigma_{\pi/2}$, $y \in Y$, and $W^j(\cdot)$ by replacing $W_0(\cdot)$, $W_{\pi/2}(\cdot)$, $W_{-\pi/2}(\cdot)$, $W_\pi(\cdot)$ and D in the definition of $W(\cdot)$ with $W_{0,j}(\cdot)$, $W_{\pi/2,j}(\cdot)$, $W_{-\pi/2,j}(\cdot)$, $W_{\pi,j}(\cdot)$ and D_j, respectively ($j \in \mathbb{N}_n^0$). Then there exists $\omega_j > 0$ such that the operator family $\{e^{-\omega_j z}W^j(z) : z \in \mathbb{C}\} \subseteq L(Y,X)$ is equicontinuous and, for every $y \in Y$, the mapping $z \mapsto W^j(z)y$, $z \in \mathbb{C}$ is entire ($j \in \mathbb{N}_n^0$).

By Lemma 1.2, we get that there exists an exponentially equicontinuous C_1-existence family for (1.1), denoted by $(E(t))_{t\geq 0}$. Furthermore, for every $y \in Y$, the mapping $t \mapsto E(t)y$, $t \geq 0$ can be analytically extended to the whole complex plane so that $E^{(\alpha_n-1)}(z)y = W(z)y$, $z \in \mathbb{C}$, $y \in Y$ and $A_j(g_{\alpha_n-\alpha_j} * E^{(\alpha_n-1)})(z)y = W^j(z)y$, $z \in \mathbb{C}$, $y \in Y$, $j \in \mathbb{N}_n^0$. Making use of the closedness of operators A_j for $j \in \mathbb{N}_n^0$, the above implies that the functions $z \mapsto A_jE^{(\alpha_n-1)}(z)y$, $z \in \mathbb{C}$ are well-defined and entire ($y \in Y$, $j \in \mathbb{N}_n^0$). By Lemma 1.3 and Lemma 1.2(iii), we get that the function $t \mapsto u(t)$, $t \geq 0$, given by (1.2), with $g(t) = G(t)$, $t \geq 0$, is a unique strong solution of problem (1.1).

Define $v(t) := u(t) - \int_0^t E(t-s)g(s)\,ds$, $t \geq 0$. By the proof of [13, Theorem 3.6], we have:

$$\mathbf{D}_t^{\alpha_n}v(\cdot) = \sum_{i=m}^{m_n-1}\Big(g_{i-\alpha} * E^{(m_n-1)}\Big)(\cdot)v_{i,0} - \sum_{i=0}^{m_n-1}\sum_{j\in\mathbb{N}_{n-1}\setminus D_i}\Big(g_{i-\alpha_j} * E^{(m_n-1)}\Big)(\cdot)v_{i,j} \in C([0,\infty) : X),$$

$B\mathbf{D}_t^{\alpha_n}v(\cdot) \in C([0,\infty) : X)$ and

$$A_i\mathbf{D}_t^{\alpha_i}v(\cdot) = \sum_{j=m_i}^{m_n-1} g_{j+1-\alpha_i}(\cdot)A_iu_j - \sum_{l=0}^{m_n-1}\sum_{j\in\mathbb{N}_{n-1}\setminus D_l}\Big[g_{l-\alpha_j} * A_i\Big(g_{\alpha_n-\alpha_i} * E^{(m_n-1)}\Big)\Big](\cdot)v_{l,j}$$
$$+ \sum_{l=m}^{m_n-1}\Big[g_{l-\alpha} * A_i\Big(g_{\alpha_n-\alpha_i} * E^{(m_n-1)}\Big)\Big](\cdot)v_{l,0} \in C([0,\infty) : X),$$

for all $i \in \mathbb{N}_{n-1}^0$. These representation formulae imply that the functions $v(\cdot)$ and $Bv^{(\alpha_n)}(\cdot)$, $A_1v^{(\alpha_1)}(\cdot)$, $\cdot\cdot\cdot$, $A_{n-1}v^{(\alpha_{n-1})}(\cdot)$, $Av^{(\alpha)}(\cdot)$ can be analytically extended from the interval $[0,\infty)$ to the whole complex plane. Furthermore, $(u-v)^{(\alpha_n-1)}(t) = \int_0^t E^{(\alpha_n-1)}(t-s)g(s)\,ds$, $t \geq 0$ and $(u-v)^{(\alpha_n)}(t) = \int_0^t E^{(\alpha_n-1)}(t-s)g'(s)\,ds + E^{(\alpha_n-1)}(t)g(0)$, $t \geq 0$. Now it is quite simple to prove that if the function $t \mapsto g(t)$, $t \geq 0$ can be analytically extended to the whole complex plane, resp., to a continuously differentiable function $\mathbb{R} \mapsto Y$, then $u(\cdot)$ is an entire solution of problem (1.1), resp., the function $t \mapsto u(t)$, $t \geq 0$ can be extended to an α_n-times continuously differentiable function $\mathbb{R} \mapsto X$ and the functions $Bu^{(\alpha_n)}(\cdot)$, $A_1u^{(\alpha_1)}(\cdot)$, $\cdots$, $A_{n-1}u^{(\alpha_{n-1})}(\cdot)$, $Au^{(\alpha)}(\cdot)$ can be extended to continuously differentiable functions $\mathbb{R} \mapsto X$. The remaining part of proof can be left to the reader. □

Remark 2.3. (i) Suppose that $Y = X$, $C_1 \in L(X)$ is injective, $C_1A_j \subseteq A_jC_1$, $j \in \mathbb{N}_n^0$, as well as that the operator families $\{\mathbf{P}_z^{-1}C_1 : |z| \geq R\} \subseteq L(X)$ and $\{z^{\alpha_j-\alpha_n}A_j\mathbf{P}_z^{-1}C_1 : |z| \geq R,\ j \in \mathbb{N}_n^0\} \subseteq L(X)$ are equicontinuous and strongly continuous. Then the analyticity of mappings $z \mapsto \mathbf{P}_z^{-1}C_1x \in X$, $|z| > R$ and $z \mapsto z^{\alpha_j-\alpha_n}A_j\mathbf{P}_z^{-1}C_1x \in X$, $|z| > R$ automatically follows for any $x \in X$, $j \in \mathbb{N}_n^0$ (cf. the proof of [14, Lemma 2.3]).

(ii) Suppose that $g \in C^\infty([0,\infty) : Y)$, resp., $g(\cdot)$ can be extended to an infinitely differentiable function $\mathbb{R} \mapsto Y$. Then $u \in C^\infty([0,\infty) : X)$ and $Bu^{(\alpha_n)}$, $A_1u^{(\alpha_1)}$, $\cdots$, $A_{n-1}u^{(\alpha_{n-1})}$, $Au^{(\alpha)} \in C^\infty([0,\infty) : X)$, resp., the functions $u(\cdot)$ and $Bu^{(\alpha_n)}(\cdot)$, $A_1u^{(\alpha_1)}(\cdot)$, $\cdots$, $A_{n-1}u^{(\alpha_{n-1})}(\cdot)$, $Au^{(\alpha)}(\cdot)$ can be extended to infinitely differentiable functions $\mathbb{R} \mapsto X$.

(iii) Let $0 \leq i \leq m_n - 1$, $0 \leq j \leq n-1$ and $i \geq \alpha_j$. If a strong solution $u(\cdot)$ of problem (1.1) has the property that $u \in C^\infty([0,\infty) : X)$ and $Bu^{(\alpha_n)}$, $A_1u^{(\alpha_1)}$, $\cdots$, $A_{n-1}u^{(\alpha_{n-1})}$, $Au^{(\alpha)} \in C^\infty([0,\infty) : X)$, then it can be easily seen that the mapping $t \mapsto A_ju^{(\alpha_j')}(t)$, $t \geq 0$ is well-defined and infinitely differentiable for $\alpha_j' \geq \alpha_j$; hence, $u_i \in D(A_j)$ for $0 \leq i \leq m_n - 1$, $j \in D_i''$ and our result on the well-posedness of (1.1) is optimal provided that $R(C_1) = X$.

Now we would like to present how Theorem 2.2 can be applied in the analysis of abstract Boussinesq-Love equation.

Example. Suppose that $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$. In the cylinder $\mathbb{R} \times \Omega$, we consider the following Cauchy-Dirichlet problem for linearized Boussinesq-Love equation:

$$(\lambda - \Delta)u_{tt}(t,x) - \alpha(\Delta - \lambda')u_t(t,x) = \beta(\Delta - \lambda'')u(t,x) + f(t,x), \quad t \in \mathbb{R},\ x \in \Omega, \tag{2.1}$$

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x),\ (t,x) \in \mathbb{R} \times \Omega; \quad u(t,x) = 0,\ (t,x) \in \mathbb{R} \times \partial\Omega, \tag{2.2}$$

where λ, λ', $\lambda'' \in \mathbb{R}$, α, $\beta \in \mathbb{R}$ and $\beta \neq 0$ (in [22], the standing hypothesis was that $\alpha \neq 0$; as explained later in [24], the case $\alpha = 0$ is worthy of consideration and has a certain physical meaning). By $\{\lambda_k\}$ [$= \sigma(\Delta)$] we denote the eigenvalues of the Dirichlet Laplacian Δ in $L^2(\Omega)$ (cf. [2, Section 6], [22, Section 5] and [19, Section 1.3] for more details) numbered in nonascending order with regard to multiplicities. By $\{\phi_k\} \subseteq C^\infty(\Omega)$ we denote the corresponding set of mutually orthogonal [in the sense of $L^2(\Omega)$] eigenfunctions.

In [22], G. A. Sviridyuk and A. A. Zamyshlyaeva have considered the well-posedness of problem (2.1)-(2.2) in the Sobolev space $W^{p,l}(\Omega)$, where $1 < p < \infty$ and $l \in \mathbb{N}_0$, and the Hölder space $C^{l+\gamma}(\Omega)$, where $0 < \gamma < 1$ and $l \in \mathbb{N}_0$. In order to apply [22, Theorem 4.1], G. A. Sviridyuk and A. A. Zamyshlyaeva imposed the following condition:

(i) $\lambda \in \rho(\Delta)$,

or

(iii) $\lambda \in \sigma(\Delta) \ \wedge\ \lambda = \lambda' \ \wedge\ \lambda \neq \lambda''$.

Although our results on the well-posedness of problem (2.1)-(2.2) in cases (i) or (iii) give some new information about qualitative properties of strong solutions of (2.1)-(2.2), in the remaining part of this example we will completely focus our attention on the following case:

(ii) $\lambda \in \sigma(\Delta) \ \wedge\ \lambda \neq \lambda' \ \wedge\ (\alpha = 0 \Rightarrow \lambda \neq \lambda'')$.

If (ii) holds with $\alpha \neq 0$, then we cannot apply [22, Theorem 4.1] (despite of the validity of requirement stated in the formulation of [22, Lemma 5.1]) because of the violation of condition [22, (A), p. 271]. Here it is also worth noting that the existence and uniqueness of two times continuously differentiable solutions of problem (2.1)-(2.2) on the non-negative real axis (understood in a broader sense of [19, Definition 5.6.2]) have been considered in [19, Example 5.7.1, Lemma 5.7.1(ii), Theorem 5.7.3] provided that $Y = H^2(\Omega) \cap H_0^1(\Omega)$, $X = L^2(\Omega)$, as well as that (ii) holds and that, additionally, $\lambda'' \neq 0$; even in this case, we obtain from Theorem 2.2 and a simple analysis, with $X = Y = L^2(\Omega)$ and C_1 being the identity operator on X, that a strong solution $t \mapsto u(t)$, $t \geq 0$ of this problem has the property that the mapping $t \mapsto \Delta u(t)$, $t \geq 0$ belongs to the space $C^2([0,\infty) : L^2(\Omega))$ provided that $f \in C^1([0,\infty) : L^2(\Omega))$/needless to say that we obtain the existence and uniqueness of entire solutions of problem (2.1)-(2.2) provided that the function $f(\cdot)$ can be extended analytically to the whole complex plane.

With a little abuse of notation, we have that $n = 2$, $B = (\lambda - \Delta)$, $A_1 = -\alpha(\Delta - \lambda')$, $A = \beta(\Delta - \lambda'')$, $\alpha_2 = 2$, $\alpha_1 = 1$ and $\alpha = 0$ (the use of symbols α and β will be clear from the context). Hence,

$$\mathbf{P}_z = z^{-2}\Big[\Big(z^2\lambda + \alpha z\lambda' + \beta\lambda''\Big) + \Big(-z^2 - \alpha z - \beta\Big)\Delta\Big], \quad z \in \mathbb{C} \setminus \{0\}.$$

It is clear that (ii) implies that

$$\lambda \neq \frac{z^2\lambda + \alpha z\lambda' + \beta\lambda''}{z^2 + \alpha z + \beta} \to \lambda \text{ as } |z| \to \infty.$$

We assume that $X = Y = L^p(\Omega)$ for some $p \in (1,\infty)$, C_1 is the identity operator on X, Δ is the Dirichlet Laplacian on $L^p(\Omega)$ acting with domain $D(\Delta) := W^{p,2}(\Omega) \cap W_0^{p,1}(\Omega)$, as well as that the following condition holds:

P. There exist a sufficiently large real number $R > 0$ and a positive real number number $l < 4$, resp., $l < 2$, provided that (ii) holds with $\alpha \neq 0$, resp., $\alpha = 0$, such that

$$\|R(z:\Delta)\| = O\Big(|\lambda - z|^{-l}\Big) \text{ as } z \to \lambda. \tag{2.3}$$

Before proceeding further, it should be observed that the condition P. holds in the case that $p = 2$, with $l = 1$: Suppose that $\lambda = \lambda_{k_0}$ for some $k_0 \in \mathbb{N}$. Then $g = R(z:\Delta)f = \sum_{k=1}^{\infty} \frac{\langle \phi_k, f\rangle}{z-\lambda_k}\phi_k$ as $z \to \lambda_{k_0}$, so that Parseval's equality implies $|z - \lambda_{k_0}|^2\|g\|^2 = \sum_{k=1}^{\infty} \frac{|z-\lambda_{k_0}|^2|\langle\phi_k,f\rangle|^2}{|z-\lambda_k|^2} \leq \text{Const.} \sum_{k=1}^{\infty} |\langle\phi_k, f\rangle|^2 = \|f\|^2$ as $z \to \lambda_{k_0}$ (let us recall that $\lambda_k \to -\infty$ as $k \to \infty$). Using now the condition P., the expression

$$\mathbf{P}_z^{-1} = z^{-2}\Big(z^2 + \alpha z + \beta\Big)^{-1}\left[\frac{z^2\lambda + \alpha z\lambda' + \beta\lambda''}{z^2 + \alpha z + \beta} - \Delta\right]^{-1}, \quad |z| \geq R,$$

and the resolvent equation, it readily follows that there exists a postive real number $\zeta > 0$ such that the operator families $\{(1+|z|)^{\zeta}\mathbf{P}_z^{-1} : |z| \geq R\} \subseteq L(X)$ and $\{(1+|z|)^{\zeta}z^{\alpha_j-\alpha_n}A_j\mathbf{P}_z^{-1} : |z| \geq R,\ j \in \mathbb{N}_2^0\} \subseteq L(X)$ are equicontinuous, as well as that $\lim_{z\to\infty} \mathbf{P}_z^{-1}x = 0$ and $\lim_{z\to\infty} z^{\alpha_j-\alpha_n}A_j\mathbf{P}_z^{-1}x = 0$ for any $x \in X$, $j \in \mathbb{N}_n^0$. The strong analyticity of mappings $z \mapsto \mathbf{P}_z^{-1}$, $|z| > R$ and $z \mapsto z^{\alpha_j-\alpha_n}A_j\mathbf{P}_z^{-1}$, $|z| > R$ follows from Remark 2.3(i), while the existence of an exponentially bounded I-uniqueness family for the corresponding problem (1.1) simply follows from Lemma 1.2(ii) and the above argumentation; here I stands for the identity operator on X.

Hence, there exists a unique entire solution $z \mapsto u(z)$, $z \in \mathbb{C}$ of problem (2.1)-(2.2), provided that $u_0(x) \in W^{p,2}(\Omega) \cap W_0^{p,1}(\Omega)$, $u_1(x) \in W^{p,2}(\Omega) \cap W_0^{p,1}(\Omega)$ and the function $f(\cdot)$ can be analytically extended to the whole complex plane; moreover, we have the existence of a positive real number $\omega' > 0$ such that the set $\{e^{-\omega' z}g(z) : z \in \mathbb{C}\}$ is bounded in $L^p(\Omega)$. Since C_1 is the identity operator on X, this is an optimal result as long as the condition P. holds (cf. Remark 2.3(iii)).

We close the paper with the following illustrative example, in which we analyze the existence and uniqueness of entire solutions to the abstract Barenblatt-Zheltov-Kochina equation in finite domains by using the argumentation contained in the proof of Theorem 2.2 and an old approach of N. H. Abdelaziz, F. Neubrander (cf. [1] and [9, Subsection 2.2.3]); for the sake of simplicity, we will focus our attention completely on homogenous case.

Example. Let $\emptyset \neq \Omega \subseteq \mathbb{R}^n$, $\{\lambda_k\}$, $\{\phi_k\}$ and Δ possess the same meanings as in the previous example, let $X = Y = L^2(\Omega)$, and let C_1 be the identity operator on X. As mentioned above, we analyze entire solutions of the Barenblatt-Zheltov-Kochina equation

$$(\lambda - \Delta)u_t(t,x) = \zeta\Delta u(t,x), \quad t \in \mathbb{R},\ x \in \Omega; \quad u(0,x) = u_0(x),\ x \in \Omega,\ u(t,x) = 0,\ (t,x) \in \mathbb{R} \times \partial\Omega, \qquad (2.4)$$

where $\zeta \in \mathbb{R} \setminus \{0\}$ and $\lambda = \lambda_{k_0} \in \sigma(\Delta)$ (cf. the equation (1.1) with $n = 2$, $B = \lambda - \Delta$, $A_1 = 0$, $A = \zeta\Delta$, $\alpha_2 = 1$ and $\alpha_1 = \alpha = 0$; then we have $\mathbf{P}_z = \lambda - (1 + \zeta z^{-1})\Delta$).

Using Parseval's equality, it can be easily seen that the operator $D : f \mapsto (-1)(\zeta\lambda)^{-1} \sum_{\lambda=\lambda_k} \langle \phi_k, f \rangle \phi_k$ ($f \in L^2(\Omega)$) belongs to the space $L(L^2(\Omega))$. Let $\beta \in (-\pi, \pi]$. Then the equation (2.3) holds with $l = 1$, which enables us to verify that, for every $\theta \in (0, \pi/2)$, there exists a sufficiently large number $\omega_{\beta,\theta} > 0$ satisfying that the function $q_{\beta,\theta}(z) := z^{-2}\mathbf{P}^{-1}_{ze^{-i\beta}} \in L(X)$, $z \in \omega_{\beta,\theta} + \Sigma_{\theta+(\pi/2)}$ is well-defined, strongly analytic and that for each $\theta' \in (0,\theta)$ the operator family $\{z^{-2}(z - \omega_{\beta,\theta})\mathbf{P}^{-1}_{ze^{-i\beta}} : z \in \omega_{\beta,\theta} + \Sigma_{\theta'+(\pi/2)}\} \subseteq L(X)$ is equicontinuous.

As in the proof of Theorem 2.2, we obtain that for each $f \in X$ there exists an X-valued analytic mapping $z \mapsto W^1_{\beta,f}(z)$, $z \in \Sigma_{\pi/2}$ satisfying that, for every $\theta \in (0, \pi/2)$, one has $\int_0^\infty e^{-zt} W^1_{\beta,f}(t)\, dt = z^{-2}\mathbf{P}^{-1}_{ze^{-i\beta}} f$, $\Re z > \omega_{\beta,\theta}$ and the set $\{e^{-\omega_{\beta,\theta} z} W^1_{\beta,f}(z) : z \in \Sigma_{\theta'}\}$ is bounded in X ($f \in X$, $\theta' \in (0,\theta)$). Define $W^1_\beta(z)f := W^1_{\beta,f}(z)$, $z \in \Sigma_{\pi/2}$, $f \in X$. Then, for every $\theta \in (0, \pi/2)$, $\{e^{-\omega_{\beta,\theta} z} W^1_\beta(z) : z \in \Sigma_\theta\} \subseteq L(X)$ is an equicontinuous operator family. On the other hand, there exist finite constants $R > 0$ and $c > 0$ such that the set $\{|z|^{-1}|(\lambda - \lambda_k)z - \zeta\lambda_k| : |z| \geq R,\ k \in \mathbb{N} \setminus \{k_0\}\}$ is bounded from below by c, so that we can apply Parsval's equality once more in order to see that:

$$z^{-1}\mathbf{P}_z^{-1} f = \frac{1}{\zeta + z}\left[\frac{\lambda z}{\zeta + z} - \Delta\right]^{-1} f = \sum_{k=1,k\neq k_0}^{\infty} \frac{\langle \phi_k, f \rangle}{(\lambda - \lambda_k)z - \zeta\lambda_k}\phi_k + Df \to Df, \quad |z| \to \infty \ \ (f \in X);$$

similarly, we have that the operator family $\{z^{-2}B\mathbf{P}^{-1}_{ze^{-i\beta}} : |z| \geq R\} \in L(X)$ is equicontinuous and that $z^{-1}B\mathbf{P}_z^{-1} f \to 0$, $|z| \to \infty$ ($f \in X$), so that we can define a strongly analytic operator family $(W^1_{\beta,B}(z))_{z\in\Sigma_{\pi/2}} \subseteq L(X)$ satisfying that, for every $\theta \in (0, \pi/2)$, the operator family $\{e^{-\omega'_{\beta,\theta} z} W^1_{\beta,B}(z) : z \in \Sigma_\theta\} \subseteq L(X)$ is equicontinuous for some number $\omega'_{\beta,\theta} > 0$. Since $\lim_{|z|\to\infty} z^{-1}\mathbf{P}_z^{-1} f = Df$ ($f \in X$), an application of [11, Theorem 1.2.5(ii)/(iii)] yields that, for every $f \in X$ and $\theta \in (0, \pi/2)$, we have $\lim_{z\to 0, z\in\Sigma_\theta} W^1_\beta(z)f = Df$.

Define

$$W^1(z) := \begin{cases} W^1_0(z),\ z \in \Sigma_{\pi/2}, \\ e^{i\pi/2} W^1_{\pi/2}(ze^{-i\pi/2}),\ \text{if } z \in e^{i\pi/2}\Sigma_{\pi/2}, \\ e^{-i\pi/2} W^1_{-\pi/2}(ze^{i\pi/2}),\ \text{if } z \in e^{-i\pi/2}\Sigma_{\pi/2}, \\ e^{i\pi} W^1_\pi(ze^{-i\pi}),\ \text{if } z \in e^{i\pi}\Sigma_{\pi/2}, \\ D,\ \text{if } z = 0, \end{cases}$$

and $W^1_B(z)$ by replacing the operators $W^1_0(z)$, $W^1_{\pi/2}(ze^{-i\pi/2})$, $W^1_{-\pi/2}(ze^{i\pi/2})$, $W^1_\pi(ze^{-i\pi})$ and D in the above definition by the operators $W^1_{0,B}(z)$, $W^1_{\pi/2,B}(ze^{-i\pi/2})$, $W^1_{-\pi/2,B}(ze^{i\pi/2})$, $W^1_{\pi,B}(ze^{-i\pi})$ and 0, respectively ($z \in \mathbb{C}$). Then there exists a finite constant $\omega > 0$ such that the operator families $\{e^{-\omega z} W^1(z) : z \in \mathbb{C}\} \subseteq L(X)$ and $\{e^{-\omega z} BW^1(z) : z \in \mathbb{C}\} \subseteq L(X)$ are equicontinuous as well as that, for every $f \in X$, the mappings $z \mapsto W(z)f$, $z \in \mathbb{C}$ and $z \mapsto BW(z)f$, $z \in \mathbb{C}$ are entire; cf. also [10, Proposition 2.4.2, Corollary 2.4.3]. Furthermore, it is not difficult to see that $(W^1(t))_{t\geq 0} \subseteq L(X, [D(B)])$ is a once integrated evolution family generated by A, B in the sense of considerations from [1, Section 2].

By [1, Theorem 2.3] and an elementary analysis, we may conclude that for each function $u_0 \in H^2(\Omega) \cap H^1_0(\Omega)$, orthogonal to the eigenfunction(s) ϕ_k for $k = k_0$, there exists a unique strong solution $t \mapsto u(t)$, $t \geq 0$ of problem (2.4) with the property that there exists a finite constant $\omega' > 0$ such that the mappings $t \mapsto u(t)$, $t \geq 0$ and $t \mapsto Bu(t)$, $t \geq 0$ can be analytically extended to the whole complex plane, as well as that the sets $\{e^{-\omega' z} u(z) : z \in \mathbb{C}\}$ and $\{e^{-\omega' z} Bu(z) : z \in \mathbb{C}\}$ are bounded. This result slightly improves the assertion of [19, Theorem 5.1.3(ii)] in L^2 spaces.

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