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Split delta shocks – an overview

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Abstract. The paper contains a review of split delta shock solutions to system of conservation laws. That is one of few attempts to incorporate delta function as a part of a solution. The paper is finished with an extension on so called shadow wave solution concept.

1. Introduction

It has been observed that Riemann problem for certain conservation laws cannot be solved with combinations of elementary waves: constant initial states, shock, rarefaction waves and contact discontinuities only. For that reason, the notion of a delta shock wave are introduced starting from the early 90s (even some numerical results from 70s showed such behaviour ([16]). The main examples come from gas dynamics, magnetohydrodynamics, chromatography, nonlinear elasticity and so on. It was shown that a large class of Riemann problems can be solved globally with these additional building blocks. Let us mention just few pioneering papers, [17] and [30].

The aim of this paper is to describe one attempt in modeling delta shocks that is called split delta shocks (SDS for short). These objects are introduced in [24] (one can find an early version of these in [20]). The main mathematical idea behind their definition was to use measure spaces defined over closed domain to catch delta function that was empirically found in some systems from both sides of its support. Both of these closed domains contain a shock curve, each of them lies on one side of the curve and the system is satisfied in a classical sense in their interiors.

Before defining SDS solutions, let us just briefly mention some different approaches.

We start with the ”usual” method for catching delta function that contains a step of prescribing a value

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at a shock curve for an L^∞-function. That is

$$u(x,t) = \begin{cases} u_0, & x < ct \\ u_\delta, & x = ct \\ u_1, & x > ct \end{cases}, \; v(x,t) = \begin{cases} v_0, & x < ct \\ v_1, & x > ct \end{cases} + w_\delta(t)\delta_{x-ct}.$$

Using such definition, a sets of Lebesgue measure zero becomes important (δ is understood as a Borel measure, for example). One can look at [32] for such type of definitions, but there are a lot of other papers using such a way of dealing with delta problem. The main success of this approach was maybe a solution of pressureless gas dynamics type systems. But there are few problematic points. The first one is a proper definition of a measure space used for solving conservation law systems. One has to be very careful to avoid non-mathematical arguments. We point out the paper [1] as a good example for rigorous mathematical arguments. The second one is that the initial data are now

$$u(x,0) = \begin{cases} u_0, & x < 0 \\ u_\delta, & x = 0 \\ u_1, & x > 0, \end{cases}$$

so it means that one use special initial data at zero measure set that is not given in advance (a value of u_δ is determined by a solution). That was one of the reasons why SDS are defined only on non-zero measure sets. But, on the other hand, one cannot find an SDS solution to the very important example – pressureless gas dynamics. One will find a way to accomplish both goals at the end of the paper by using so called shadow waves (see [23]).

A distributional version, weak asymptotic method, of the above idea one can find in [9], [10]. In the similar manner one can define also so called δ'-shocks, see [26].

We have already said that the pressureless model is very important, and one can look at the following papers where it is solved (independently): Using so called variational method in [11] and using so called sticky particles method in [3]. For the generalized pressureless-type system one can look in [14].

Let us mention the paper [15] where one can find even more peculiar type of waves – singular shocks. Their main feature that besides δ function they contains some strange objects that can be described as $\sqrt{\delta}$. Let us just mention that they can be also understood as a special type of shadow waves.

At the end of this introduction, we point out on the paper [18] where one can find a real model from chromatography theory with the proof that there are essential reasons to use delta shock with distinguished left- and right-hand side. That was a kind of practical recognition of the whole idea of splitting delta into two parts.

2. The definition of split delta shocks

Let us now briefly describe what we mean by a solution in the form of a split delta shock wave.

Suppose $\overline{R^2_+}$ is divided into finitely disjoint open sets $\Omega_i \neq \emptyset$, $i = 1, ..., n$ with piecewise smooth boundary curves Γ_i, $i = 1, ..., m$, that is $\Omega_i \cap \Omega_j = \emptyset$, $\bigcup_{i=1}^n \overline{\Omega_i} = \overline{R^2_+}$ where $\overline{\Omega_i}$ denotes the closure of Ω_i. Let $C(\overline{\Omega_i})$ be the space of bounded and continuous real-valued functions on $\overline{\Omega_i}$, equipped with the L^∞-norm. Let $\mathcal{M}(\overline{\Omega_i})$, be the space of measures on $\overline{\Omega_i}$.

We consider the spaces

$$C_\Gamma = \prod_{i=1}^{n} C(\overline{\Omega_i}), \quad \mathcal{M}_\Gamma = \prod_{i=1}^{n} \mathcal{M}(\overline{\Omega_i}).$$

The product of an element $G = (G_1, ..., G_n) \in C_\Gamma$ and $D = (D_1, ..., D_n) \in \mathcal{M}_\Gamma$ is defined as an element $D \cdot G = (D_1G_1, ..., D_nG_n) \in \mathcal{M}_\Gamma$, where each component is defined as the usual product of a continuous function and a measure.

Every measure on $\overline{\Omega}_i$ can be viewed as a measure on $\overline{\mathbb{R}^2_+}$ with support in $\overline{\Omega}_i$. This way we obtain a mapping

$$\begin{aligned} &m : \mathcal{M}_\Gamma \to \mathcal{M}(\overline{\mathbb{R}^2_+}) \\ &m(D) = D_1 + D_2 + ... + D_n. \end{aligned}$$

A typical example is obtained when $\overline{\mathbb{R}^2_+}$ is divided into two regions Ω_1, Ω_2 by a piecewise smooth curve $x = \gamma(t)$. The delta function $\delta(x - \gamma(t)) \in \mathcal{M}(\overline{\mathbb{R}^2_+})$ along the line $x = \gamma(t)$ can be split in a non unique way into a left-hand side $D^- \in \mathcal{M}(\overline{\Omega}_1)$ and the right-hand component $D^+ \in \mathcal{M}(\overline{\Omega}_2)$ such that

$$\begin{aligned} \delta(x - \gamma(t)) &= \alpha_0(t)D^- + \alpha_1(t)D^+ \\ &= m(\alpha_0(t)D^- + \alpha_1(t)D^+) \end{aligned}$$

with $\alpha_0(t) + \alpha_1(t) = 1$. The solution concept which allows to incorporate such two sided delta functions as well as shock waves is modeled along the lines of the classical weak solution concept and proceeds as follows:

Step 1: Perform all nonlinear operations of functions in the space C_Γ.

Step 2: Perform multiplications with measures in the space $\mathcal{M}_\Gamma$.

Step 3: Map the space $\mathcal{M}_\Gamma$ into $\mathcal{M}(\overline{\mathbb{R}^2_+})$ by means of the map m and embed it into the space of distributions.

Step 4: Perform the differentiation in the sense of distributions and require that the equation is satisfied in this sense.

Note that in the case of absence of a measure part (Step 2), this is the precisely the concept of a weak solution to equations in divergence form.

Following the usual reasoning, delta shocks are required to satisfy the condition of over-compressibility, meaning that all characteristic curves run into the delta shock curve from both sides. It may happen that at a certain point on a delta shock curve, over-compressibility is lost. In this case we replace the delta shock by a new type of solution which we call a delta contact discontinuity. This new concept is introduced in Lemma 1 and Definition 1 below.

3. Simplified magnetohydrodynamics model

We start our investigation of different models from the literature that contains solutions with SDSs. The first model equation in the paper [24] is

$$\begin{aligned} u_t + (u^2/2)_x &= 0 \\ v_t + ((u-1)v)_x &= 0 \end{aligned} \tag{1}$$

initiated in [13]. This system is derived from a simplified model of magnetohydrodynamics. The eigenvalues of the above system are $\lambda_1(u, v) = u - 1$, $\lambda_2(u, v) = u$, and the right-hand side eigenvectors are $r_1(u, v) = (0, 1)^T$, $r_2(u, v) = (1, v)^T$. The first characteristic field is linearly degenerate and the second is genuinely nonlinear. Thus, there are three types of solution for the Riemann data

$$(u, v)(x, 0) = \begin{cases} (u_0, v_0), & x < 0 \\ (u_1, v_1), & x > 0 \end{cases}$$

(i) When $u_1 > u_0$ the solution is a contact discontinuity followed by a rarefaction wave,

$$u(x,t) = \begin{cases} u_0, & x \le u_0 t \\ \frac{x}{t}, & u_0 t < x < u_1 t \\ u_1, & x \ge u_1 t \end{cases}$$

$$v(x,t) = \begin{cases} v_0, & x \le (u_0-1)t \\ v_1 \exp(u_0 - u_1), & (u_0-1)t < x < u_0 t \\ v_1 \exp(\frac{x}{t} - u_1), & u_0 t \le x \le u_1 t \\ v_1, & x > u_1 t. \end{cases}$$

(ii) If $u_1 < u_0 < u_1 + 2$, the solution is given in the form of contact discontinuity followed by a shock wave,

$$u(x,t) = \begin{cases} u_0, & x \le ct \\ u_1, & x > ct \end{cases}$$

$$v(x,t) = \begin{cases} v_0, & x \le (u_0-1)t \\ v_*, & (u_0-1)t < x < ct \\ v_1, & x \ge ct, \end{cases}$$

where $v_* = v_1 \dfrac{2 - u_0 - u_1}{2 + u_1 - u_0}$.
(iii) If $u_0 \ge u_1 + 2$ the solution is given in the form of delta shock wave,

$$\begin{aligned} u(x,t) &= \begin{cases} u_0, & x \le ct \\ u_1, & x > ct \end{cases} \\ v(x,t) &= \begin{cases} v_0, & x \le ct \\ v_1, & x > ct \end{cases} + \alpha_0(t) D^- + \alpha_1(t) D^+, \end{aligned} \tag{2}$$

where D^- and D^+ are the left- and right-hand side delta functions with the support on the line $x = ct$ (see below), $c = (u_0 + u_1)/2$,

$$\alpha_0(t) = \frac{st(c - (u_1 - 1))}{u_0 - u_1}, \quad \alpha_1(t) = \frac{st(c - (u_0 - 1))}{u_0 - u_1},$$

$\alpha(t) := \alpha_0(t) + \alpha_1(t)$ is called the strength of the delta shock wave, and

$$s := c(v_1 - v_0) - ((u_1 - 1)v_1 - (u_0 - 1)v_0)$$

is called the Rankine-Hugoniot deficit (see [15]).

We will prove only the third part of the assertion. The first two parts can be proved in the usual way (see [8] or [4] for example).

Let us substitute a wave of the form (2) into the first equation:

$$\begin{aligned} -\langle u, \partial_t \varphi \rangle &= -\int_0^\infty \left(\int_{-\infty}^{ct} u_0 \partial_t \varphi(x,t)\, dx + \int_{ct}^\infty u_1 \partial_t \varphi(x,t)\, dx \right) dt \\ &= -c \int_0^\infty (u_1 - u_0)\, \varphi(ct,t)\, dt = \langle [u]\, \delta(x - ct), \varphi \rangle \end{aligned}$$

$$-\left\langle \frac{1}{2} u^2, \partial_x \varphi \right\rangle = \left\langle \frac{1}{2} \left[u^2 \right] \delta(x - ct), \varphi \right\rangle$$

where $[\cdot] := \cdot_1 - \cdot_0$ denotes a jump, and the second

$$\begin{aligned}
-\langle v, \partial_t \varphi \rangle = & -\int_0^\infty \left(\int_{-\infty}^{ct} v_0 \partial_t \varphi (x,t)\, dx + \int_{ct}^\infty v_1 \partial_t \varphi (x,t)\, dx \right) dt \\
& + \left\langle \alpha_0' (t)\, \delta (x - ct), \varphi \right\rangle - c \left\langle \alpha_0 (t)\, \delta' (x - ct), \varphi \right\rangle \\
& + \left\langle \alpha_1' (t)\, \delta (x - ct), \varphi \right\rangle - c \left\langle \alpha_1 (t)\, \delta' (x - ct), \varphi \right\rangle \\
& - c \left\langle [v] - \left(\alpha_0' (t) + \alpha_1' (t) \right) \right\rangle \delta (x - ct) \\
& - c \left\langle (\alpha_0 (t) + \alpha_1 (t)) \right\rangle \delta' (x - ct) \\
& \langle v (u - 1), \varphi_x \rangle = \langle [v (u - 1)]\, \delta (x - ct), \varphi \rangle \\
& \langle (v_0 (u_0 - 1)\, \alpha_0 (t) + v_1 (u_1 - 1)\, \alpha_1 (t))\, \delta' (x - ct), \varphi \rangle
\end{aligned}$$

Thus, the system would be satisfied if the following equations are satisfied. For the first equation the delta terms imply

$$[u]\, c - \frac{1}{2} \left[u^2 \right] = 0$$

and there are no delta derivative terms,

From the second equation we have the following equations

$$\begin{aligned}
& -[v]\, c + [v (u - 1)] + \left(\alpha_0' (t) + \alpha_1' (t) \right) = 0 \\
& (v_0 (u_0 - 1)\, \alpha_0 (t) + v_1 (u_1 - 1)\, \alpha_1 (t)) = c\, (\alpha_0 (t) + \alpha_1 (t)).
\end{aligned} \tag{3}$$

One can immediately see that

$$\alpha_0' (t) + \alpha_1' (t) = k_1 := c\, [v] - [v (u - 1)].$$

Using that and the second equation above we have the following system

$$\begin{aligned}
\alpha_0 (t) + \alpha_1 (t) &= k_1 t \\
(v_0 (u_0 - 1) - c)\, \alpha_0 (t) + (v_1 (u_1 - 1) - c)\, \alpha_1 (t) &= 0
\end{aligned} \tag{4}$$

(we have used that $\alpha_i(0) = 0$, $i = 0, 1$, since there is no delta function in the initial data) with the solution

$$\begin{aligned}
\alpha_0 (t) &= \frac{k_1 (v_1 (u_1 - 1) - c)\, t}{(v_1 (u_1 - 1) - v_0 (u_0 - 1))} \\
\alpha_1 (t) &= \frac{k_1 (c - v_0 (u_0 - 1))\, t}{(v_1 (u_1 - 1) - v_0 (u_0 - 1))}
\end{aligned}$$

Let us check the over-compressibility condition:

$$\lambda_1 (u_0, v_0) = u_0 - 1 \geq c = \frac{u_0 + u_1}{2} \geq \lambda_2 (u_1, v_1) = u_1.$$

Obviously, it is satisfied if $u_0 - 2 \geq u_1$ That proves the assertion.

3.1. Delta initial data problem

If there is an interaction of two waves among these is at least one split delta shock we have to solve a new initial problem at the interaction time. In that case, the initial data contains the δ-function. With no loss in generality, we shall translate the interaction time to zero, so we have to solve system (1) with the initial data

$$(u, v)(x, 0) = \begin{cases} (u_0, v_0), & x < 0 \\ (u_1, v_1), & x > 0 \end{cases} + \gamma \delta(x, 0)$$

It can be solved in a simple way as above and the result is a single over-compressive delta shock wave. But, when this is not a case, the types of admissible solution known so far are not enough to obtain a solution. Let us first look at the case $u_0 \geq u_1 + 2$. It simply means that we have to solve ODE system (3) with the initial data $\alpha_0(0) + \alpha_1(0) = \gamma$. Then, we have

$$\begin{aligned} \alpha_0(t) + \alpha_1(t) &= k_1 t + \gamma \\ (v_0(u_0-1) - c)\,\alpha_0(t) + (v_1(u_1-1) - c)\,\alpha_1(t) &= 0 \end{aligned}$$

instead of (4). Now we have

$$\begin{aligned} a_0 &= -\frac{((u_1-1)v_1 - c)k_1 t + ((u_1-1)v_1 - c)\gamma}{(u_0-1)v_0 - (u_1-1)v_1}, \\ a_1 &= \frac{((u_0-1)v_0 - c)k_1 t + ((u_0-1)v_0 - c)\gamma}{(u_0-1)v_0 - (u_1-1)v_1}. \end{aligned}$$

The speed of the delta shock stays the same, since it is completely determined by the first equation. Like in the case of the Riemann data, the wave is over-compressive.

Let $u_0 < u_1 + 2$ now. We will have a new type of admissible solution, called delta contact discontinuity, defined below. Its existence is justified by two facts. First, a contact discontinuity emerges in the case when one of the characteristic fields is linearly degenerate. Second, if a linear equation has a delta function as initial data, it propagates along the characteristic lines. These two facts inspired the following lemma and the definition of this new type of elementary waves.

Lemma 3.1. *([24]) Let the initial data for system (1) given by*

$$u|_{t=0} = \begin{cases} u_0, & x < 0 \\ u_1, & x > 0 \end{cases}, \quad v|_{t=0} = \begin{cases} v_0, & x < 0 \\ v_1, & x > 0 \end{cases} + \gamma\delta_{(0,0)},$$

where $u_0 > u_1$, but $u_0 < u_1 + 2$. Then, the function

$$u = \begin{cases} u_0, & x < ct \\ u_1, & x > ct \end{cases} \quad v = \begin{cases} v_0, & x < (u_0-1)t \\ v_*, & (u_0-1)t < x < ct \\ v_1, & x > ct \end{cases} + \gamma\delta_{x=(u_0-1)t},$$

where $c = (u_0 + u_1)/2$ weakly solves the Riemann problem for (1).

Proof. For every $\varphi \in C_0^\infty$, $\operatorname{supp}\varphi \cap \{(x,t) : \ x = (u_0-1)t, \ t > 0\} = \emptyset$, it holds that

$$\begin{aligned} &\langle u_t, \varphi \rangle + \frac{1}{2}\langle (u^2)_x, \varphi \rangle = 0 \\ &\langle v_t, \varphi \rangle + \langle ((u-1)v)_x, \varphi \rangle = 0. \end{aligned}$$

Our aim is to show that this still holds true when it is allowed that $\operatorname{supp}\varphi$ intersects the supports of D^- and D^+, i.e. the line $x = (u_0-1)t$. Let us note that the condition $u_0 < u_1 + 2$ means that $(u_0+u_1)/2 > u_0 - 1$ so the line $x = (u_0-1)t$ is on the left-hand side of the shock line $x = (u_0+u_1)t/2$.

The first equation in (1) does not contain v, so it is still satisfied. From the second equation we have that

$$\begin{aligned} v_t + ((u_0-1)v)_x = {} & -(u_0-1)(v_* - v_0)\delta - \gamma\delta' \\ & + (u_0-1)(v_* - v_0)\delta + (u_0-1)\gamma\delta' = 0 \end{aligned}$$

near the line $x = (u_0-1)t$. □

One can find a detailed analysis of all interaction cases where this lemma is useful in [24].

4. Transport equations

We shall now give another example from literature when split delta shocks can be used – the results are taken from [29]. Let us look at the transport equations in the following form

$$\begin{aligned} \rho_t + (\rho u)_x &=0 \\ u_t + \left(\frac{u^2}{2}\right)_x &=0. \end{aligned} \tag{5}$$

Let us note that that system is essentially the same as the one in [16]

$$\begin{aligned} u_t + \left(\frac{u^2}{2}\right)_x &=0 \\ v_t + \left(\frac{uv}{2}\right)_x &=0 \end{aligned}$$

where delta shocks are observed first (numerically).

System (5) is weakly hyperbolic with one eigenvalue $\lambda = u$ (and linearly degenerate field). For $u_0 \leq u_1$, a solution to the Riemann problem

$$(\rho, u)(x, 0) = \begin{cases} (\rho_0, u_0), & x < 0 \\ (\rho_1, u_1), & x > 0 \end{cases}$$

consists of two contact discontinuities connected with the vacuum state

$$\rho(x,t) = \begin{cases} \rho_0, & x \leq u_0 t \\ 0, & u_0 t < x < u_1 t \\ \rho_1, & u_1 t \leq x, \end{cases}$$

$$u(x,t) = \begin{cases} u_0, & x \leq u_0 t \\ \frac{x}{t}, & u_0 t < x < u_1 t \\ u_1, & x \geq u_1 t. \end{cases}$$

In the case $u_0 > u_1$ we expect the SDS like in the previous case. Substitution of (2) into system (5) gives $c = \frac{1}{2}(u_0 + u_1)$ from the second and

$$\begin{aligned} -[\rho]\, c + [\rho u] + \left(\alpha_0'(t) + \alpha_1'(t)\right) &= 0 \\ (\rho 0 u_0 \alpha_0(t) + \rho_1 u_1 \alpha_1(t)) &= c\,(\alpha_0(t) + \alpha_1(t)) \end{aligned}$$

from the first equation in the system (similarly as (3) before). We can put $\alpha_i(t) = \alpha_i t$, $i = 0, 1$ as above, so we have to solve the system

$$\begin{aligned} \alpha_0 + \alpha_1 &= k_1 := [\rho]\, c - [\rho u] \\ (u_0 - c)\alpha_0 + (u_1 - c)\alpha_1 &= 0. \end{aligned}$$

The solution is now given by

$$\alpha_0 = \frac{(c - u_1)k_1}{u_0 - u_1}, \alpha_1 = -\frac{(c - u_0)k_1}{u_0 - u_1}.$$

The over-compressibility condition holds whenever $u_0 > u_1$, since $c = \frac{1}{2}(u_0 + u_1)$.

Note that one can treat the case with delta in initial data similarly to the previous case. We shall omit it here.

5. Nonlinear chromatography equations

Nest example is from the paper [12] about a model in non-linear chromatography

$$
\begin{aligned}
u_t + \left(\left(1 + \frac{1}{1-u+v}\right)u\right)_x =& 0\\
v_t + \left(\left(1 + \frac{1}{1-u+v}\right)v\right)_x =& 0.
\end{aligned}
\tag{6}
$$

Physical domain for solutions is defined by $1 - u + v >$, or $v - u > -1$ The system has

$$
\lambda_i = \frac{1}{1-u+v}, \ \lambda_j = \frac{1}{(1-u+v)^2}
$$

as eigenvalues. Here, $i = 1$, $j = 2$ if $-1 < v - u < 1$, and $i = 2$, $j = 1$ if $v - u \geq 1$. The system is strictly hyperbolic if $u \neq v$ with i-th field being linearly degenerate and j-th field being genuinely nonlinear. One can look in [12] for a complete solution to Riemann problem

$$
(u,v)(x,0) = \begin{cases} (u_0, v_0) & x < 0 \\ (u_1, v_1), & x > 0 \end{cases}.
$$

We can give here only the case $v_0 - u_0 < 0$, $v_1 - u_1 > 0$ when there are no elementary wave solutions. In that case, we try with the SDS solution

$$
\begin{aligned}
u(x,t) &= \begin{cases} u_0, & x \leq ct \\ u_1, & x > ct \end{cases} + \alpha_0(t)D^- + \alpha_1(t)D^+ \\
v(x,t) &= \begin{cases} v_0, & x \leq ct \\ v_1, & x > ct \end{cases} + \beta_0(t)D^- + \beta_1(t)D^+.
\end{aligned}
\tag{7}
$$

A split delta function can be multiplied only with continuous function on the domains $\{(x,t) : x \leq ct\}$ and $\{(x,t) : x \geq ct\}$. That is, $\frac{1}{1-u+v}$ has to be continuous on these sets. That is possible if delta parts in $1 - u + v$ cancel each other, i.e. when

$$
\alpha_0 + \alpha_1 = \beta_0 + \beta_1. \tag{8}
$$

A substitution of (7) into system (6) gives the following two relations. The first equation is satisfied if the following relation is true

$$
\begin{aligned}
&\left(-c\,[u] + \alpha_0 + \alpha_1 + \left[\left(1 + \frac{1}{1-u+v}\right)u\right]\right)\delta\,(x-ct) \\
&+ \left(-c\,(\alpha_0 + \alpha_1)\,t + \left(\left(1 + \frac{1}{1-u_0+v_0}\right)u_0\alpha_0 + \left(1 + \frac{1}{1-u_1+v_1}\right)u_1\alpha_1\right)t\right) \\
&\delta'\,(x-ct) = 0
\end{aligned}
$$

while the second one holds if

$$
\begin{aligned}
&\left(-c\,[v] + \beta_0 + \beta_1 + \left[\left(1 + \frac{1}{1-u+v}\right)v\right]\right)\delta\,(x-ct) \\
&+ \left(-c\,(\beta_0 + \beta_1)\,t + \left(\left(1 + \frac{1}{1-u_0+v_0}\right)v_0\beta_0 + \left(1 + \frac{1}{1-u_1+v_1}\right)v_1\beta_1\right)t\right) \\
&\delta'\,(x-ct) = 0
\end{aligned}
$$

These relations produce four equations

$$\alpha_0 + \alpha_1 = k_1 := c\,[u] - \left[\left(1 + \frac{1}{1-u+v}\right)u\right] \tag{9}$$

$$\left(\left(1 + \frac{1}{1-u_0+v_0}\right)u_0 - c\right)\alpha_0 + \left(\left(1 + \frac{1}{1-u_1+v_1}\right)u_1 - c\right)\alpha_1 = 0 \tag{10}$$

$$\beta_0 + \beta_1 = k_2 := c\,[v] - \left[\left(1 + \frac{1}{1-u+v}\right)v\right] \tag{11}$$

$$\left(\left(1 + \frac{1}{1-u_0+v_0}\right)v_0 - c\right)\beta_0 + \left(\left(1 + \frac{1}{1-u_1+v_1}\right)v_1 - c\right)\beta_1 = 0. \tag{12}$$

The condition $\alpha_0 + \alpha_1 = \beta_0 + \beta_1$ imply $k_1 = k_2$, and that condition determines the speed by

$$c\,[u] - \left[\left(1 + \frac{1}{1-u+v}\right)u\right] = c\,[v] - \left[\left(1 + \frac{1}{1-u+v}\right)v\right]$$

i.e.

$$c = 1 + \frac{1}{(1-u_0+v_0)\,(1-u_1+v_1)}.$$

Finally, we can find unique values of $\alpha_0,\ \alpha_1,\ \beta_0$ and β_1 that satisfies the following two systems (9-10) and (11,12).

The condition $-1 \le v_0 - u_0 \le 0$ and $v_1 - u_1 \ge 0$ ensures that the wave is over-compressible.

6. Systems linear in one variable

Let us try to find a solution in the SDS-form of a fairy general case

$$\begin{aligned} u_t + (f_1(u)\,v + f_2(u))_x &= 0 \\ v_t + (g_1(u)\,v + g_2(u))_x &= 0 \end{aligned} \tag{13}$$

with the Riemann initial data. Like in all previous cases, let us substitute a wave in SDS-form into the system.

Taking care about coefficients in front of δ and δ', we get

$$(-c\,[u] + [f_1(u)\,v + f_2(u)])\,\delta + (f_1(u_0)\,\alpha_0 + f_1(u_1)\,\alpha_1)\,\delta' = 0.$$

From that relation, we get the wave speed:

$$c = \frac{[f_1(u)\,v + f_2(u)]}{[u]} \tag{14}$$

and the first equation for αs:

$$f_1(u_0)\,\alpha_0 + f_1(u_1)\,\alpha_1 = 0 \tag{15}$$

Substitution into the second equation gives

$$\begin{aligned} &(-c[v] + (\alpha_0 + \alpha_1) - [g_1(u)v + g_2(u)])\delta \\ &\qquad + (-c(\alpha_0 + \alpha_1)t + (g_1(u_0)\alpha_0 + g_1(u_1)\alpha_1)t)\delta' = 0 \end{aligned}$$

That will be true if the following equations are satisfied:

$$\alpha_0 + \alpha_1 = k_1 := c\,[v] - [g_1(u)\,v + g_2(u)] \tag{16}$$

$$(g_1(u_0) - c)\,\alpha_0 + (g_1(u_1) - c)\alpha_1 = 0 \tag{17}$$

In order to obtain α_0 and α_1, (15) and (17) has to be linearly dependent, i.e.

$$\frac{g_1(u_0) - c}{f_1(u_0)} = \frac{g_1(u_1) - c}{f_1(u_1)}$$

or

$$c = \frac{f_1(u_1)\,g_1(u_0) - f_1(u_0)\,g_1(u_1)}{f_1(u_1) - f(u_0)}.$$

Together with (14) we obtained the following condition on left- and right-hand states:

$$\frac{[f_1(u)\,v + f_2(u)]}{[u]} = \frac{f_1(u_1)\,g_1(u_0) - f_1(u_0)\,g_1(u_1)}{[f_1(u)]}$$

A set of all right-hand states (u_1, v_1) that satisfy it is called the delta locus. In general, for a fixed left-hand side it is a curve $v_1 = v_1(u_1)$ in (u, v)-plane passing trough that point. With that condition satisfied, one can solve (16), (17) to get α_0, α_1. The fact that the locus is just a curve and that an admissible SDS should be over-compressive restricts a use of such solutions. Contrary to Hugoniot locus or rarefaction curve, a point at the locus can be joined only with rarefaction wave and only if a speed of an SDS obtained in such a way equals one of the eigenvalues – left for R_1 and right one for R_2. A set of such point is a discrete set in general.

Now we shall give a case when this restrictive fact does not hold.

$$\begin{aligned} u_t + f(u)_x &= 0 \\ v_t + (g_1(u)\,v + g_2(u))_x &= 0 \end{aligned}$$

Immediately, one sees that

$$c = \frac{[f(u)]}{[u]}.$$

From the second equation we have

$$\begin{aligned} &\alpha_0 + \alpha_1 = k_1 := c\,[v] - [g_1(u)\,v + g_2(u)] \\ &- c\,(\alpha_0 + \alpha_1) + g_1(u_0)\,\alpha_0 + g_2(u_1)\,\alpha_1 = 0. \end{aligned}$$

Thus, we have

$$\alpha_0 = \frac{g_1(u_1) - c}{g_1(u_1) - g_1(u_0)},\ \alpha_1 = \frac{c - g_1(u_0)}{g_1(u_1) - g_1(u_0)}.$$

So, one can find a SDS solution for almost any initial data. Let us now check the admissibility condition, i.e. when the wave is over-compressive: Eigenvalues are given by

$$\lambda_i(u, v) = f'(u)\,,\ \lambda_j = g_1(u)\,,\ i, j \in \{1, 2\}\,.$$

We have to check when

$$\lambda_i(u_0, v_0) \geq c \geq \lambda_i(u_1, v_1)\,, i = 1, 2.$$

That means the following

$$\begin{aligned} f'(u_0) &\geq \frac{f(u_1) - f(u_0)}{u_1 - u_0} \geq f'(u_1) \\ {}'g_1(u_0) &\geq \frac{f(u_1) - f(u_0)}{u_1 - u_0} \geq g_1(u_1)\,. \end{aligned}$$

Comparing with the general case above, one can see that a Delta locus is an area now, and there is much more chance to find an SDS solution and connect it with some rarefaction wave.

7. Shadow waves

One of the major obstacles in use of split delta shocks is a missing solution for pressureless gas dynamics system. It was one of the first systems where one could expect delta function in solution. There are at least two valid physical models, see [3] and [11], for example. Also, there is a mathematical evidence of delta shock formation, see [5] (one can also check [19] where it is shown that pressureless system is not isolated case).

Her we shall present a method of shadow waves (SDW) that is found to be very efficient.

Consider a following conservation law system

$$\partial_t f(U) + \partial_x g(U) = 0,\ U : \mathbb{R}^2_+ \to \Omega \subset \mathbb{R}^n, \tag{18}$$

where $f = (f^1, \dots, f^n)$ and $g = (g^1, \dots, g^n)$ are continuous mapping from Ω in $\mathbb{R}^n$. A name of f is density function, while g is called flux function. The functions f and g are continuous mappings from a physical domain $\Omega \subset \mathbb{R}^n$ into $\mathbb{R}^n$.

The following notation will be used trough the section. A parameter ε will be taken as small as needed. In the sequel, relations $\sim$, $\approx$, a "growth order", Landau symbols $O(\cdot)$ and $o(\cdot)$ will always be used assuming $\varepsilon \to 0$. The half-space $\{(x,t) \in \mathbb{R} \times \mathbb{R}_+\}$ is denoted by $\mathbb{R}^2_+$.

All calculations in the paper are based on exploitation of the Rankine-Hugoniot conditions. We will obtain all results by the following basic lemma.

Lemma 7.1. *([23]) Let $f, g \in C(\Omega : \mathbb{R}^n)$ and $U : \mathbb{R}^2_+ \to \Omega \subset \mathbb{R}^n$ be a piecewise constant function given by*

$$U_\varepsilon(x,t) = \begin{cases} U_0, & x < ct - \varepsilon t - x_{1,\varepsilon} \\ U_{1,\varepsilon}, & ct - \varepsilon t - x_{1,\varepsilon} < x < ct \\ U_{2,\varepsilon}, & ct < x < ct + \varepsilon t + x_{2,\varepsilon} \\ U_1, & x > ct + \varepsilon t + x_{2,\varepsilon} \end{cases}. \tag{19}$$

Here $x_{1,\varepsilon}, x_{2,\varepsilon} \sim \varepsilon$. Assume

$$\max_{i=1,2} \{\|f(U_{i,\varepsilon})\|_{L^\infty}, \|g(U_{i,\varepsilon})\|_{L^\infty}\} = O(\varepsilon^{-1}). \tag{20}$$

Then

$$\begin{aligned} \partial_t f(U_\varepsilon) \approx & - c(f(U_1) - f(U_0))\delta - c(\varepsilon f(U_{1,\varepsilon}) + \varepsilon f(U_{2,\varepsilon}))t\delta' \\ & + (\varepsilon f(U_{1,\varepsilon}) + \varepsilon f(U_{2,\varepsilon}))\delta \\ \partial_x g(U_\varepsilon) \approx & (g(U_1) - g(U_0))\delta + (\varepsilon g(U_{1,\varepsilon}) + \varepsilon g(U_{2,\varepsilon}))t\delta'. \end{aligned} \tag{21}$$

The support of δ (and δ' consequently) is the line $x = ct$.

Remark 7.2. *The constants $x_{i,\varepsilon}$, $i = 1, 2$ are useful when initial data contains delta function: If $\sigma := \lim_{\varepsilon \to 0} x_{1,\varepsilon} U_{1,\varepsilon} + x_{2,\varepsilon} U_{2,\varepsilon} \in \mathbb{R}^n$ exists, then the function U from (19) satisfies*

$$U|_{t=0} = \begin{cases} U_0, & x < 0, \\ U_1, & x > 0 \end{cases} + \sigma \delta_{(0,0)}.$$

Proof. We shall use the Taylor expansion formula for a test function and neglect all terms with growth order greater than ε. Thus,

$$\begin{aligned} \phi(ct - \varepsilon t - x_{1,\varepsilon}, t) &\approx \phi(ct,t) + \partial_x \phi(ct,t)\varepsilon t \\ \phi(ct + \varepsilon t + x_{2,\varepsilon}, t) &\approx \phi(ct,t) - \partial_x \phi(ct,t)\varepsilon t. \end{aligned}$$

Using the standard Rankine–Hugoniot shock calculations and the above approximations we have

$$\begin{aligned}
\langle \partial_t f(U_\varepsilon), \phi\rangle \approx &-\int_0^\infty (c-\varepsilon)\,(f(U_{1,\varepsilon}) - f(U_0))\,\phi(ct-\varepsilon t - x_{1,\varepsilon}, t)\,dt,\\
&-\int_0^\infty c\,(f(U_{2,\varepsilon}) - f(U_{1,\varepsilon}))\,\phi(ct,t)\,dt - \int_0^\infty (c+\varepsilon)\,(f(U_1) - f(U_{2,\varepsilon}))\,\phi(ct+\varepsilon t + x_{2,\varepsilon}, t)\,dt\\
\approx &-(f(U_{1,\varepsilon}) - f(U_0))\int_0^\infty (c-\varepsilon)\Big(\phi(ct,t) - \partial_x\phi(ct,t)(\varepsilon t + x_{1,\varepsilon})\Big)\,dt\\
&-(f(U_{2,\varepsilon}) - f(U_{1,\varepsilon}))\int_0^\infty c\,\phi(ct,t)\,dt\\
&-(f(U_1) - f(U_{2,\varepsilon}))\int_0^\infty (c+\varepsilon)\Big(\phi(ct,t) + \partial_x\phi(ct,t)(\varepsilon t + x_{2,\varepsilon})\Big)\,dt.
\end{aligned}$$

The assumptions from Lemma 7.1 imply

$$\begin{aligned}
\langle \partial_t f(U_\varepsilon), \phi\rangle \approx &-(f(U_1) - f(U_0))\int_0^\infty c\phi(ct,t)\,dt + \int_0^\infty \Big(\varepsilon f(U_{1,\varepsilon}) + \varepsilon f(U_{2,\varepsilon})\Big)\phi(ct,t)\,dt\\
&+\int_0^\infty c\Big((\varepsilon t + x_{1,\varepsilon}) f(U_{1,\varepsilon}) + (\varepsilon t + x_{2,\varepsilon}) f(U_{2,\varepsilon})\Big)\partial_x\phi(ct,t)dt\\
\approx &\Big\langle\Big(-c(f(U_1) - f(U_0)) + \varepsilon f(U_{1,\varepsilon}) + \varepsilon f(U_{2,\varepsilon})\Big)\delta(x-ct), \phi(x,t)\Big\rangle\\
&\Big\langle -c\Big((\varepsilon t + x_{1,\varepsilon}) f(U_{1,\varepsilon}) + (\varepsilon t + x_{2,\varepsilon}) f(U_{2,\varepsilon})\Big)\delta'(x-ct), \phi(x,t)\Big\rangle.
\end{aligned}$$

With the same type of reasoning, one sees that the space derivative is given by

$$\begin{aligned}
\langle \partial_x g(U_\varepsilon), \phi\rangle \approx &(g(U_{1,\varepsilon}) - g(U_0))\int_0^\infty \phi(ct,t) - \partial_x\phi(ct,t)\,(\varepsilon t + x_{1,\varepsilon})\,dt + (g(U_{2,\varepsilon}) - g(U_{1,\varepsilon}))\int_0^\infty \phi(ct,t)\,dt\\
&+(g(U_1) - g(U_{2,\varepsilon}))\int_0^\infty \phi(ct,t) + \partial_x\phi(ct,t)\,(\varepsilon t + x_{2,\varepsilon})\,dt\\
\approx &(g(U_1) - g(U_0))\int_0^\infty \phi(ct,t)\,dt - \int_0^\infty \Big((\varepsilon + x_{1,\varepsilon}) g(U_{1,\varepsilon}) + (\varepsilon + x_{2,\varepsilon}) g(U_{2,\varepsilon})\Big)\partial_x\phi(ct,t)\,dt\\
\approx &\Big\langle\Big(g(U_1) - g(U_0)\Big)\delta(x-ct), \phi(x,t)\Big\rangle\\
&+\Big\langle\Big((\varepsilon t + x_{1,\varepsilon}) g(U_{1,\varepsilon}) + (\varepsilon t + x_{2,\varepsilon}) g(U_{2,\varepsilon})\Big)\delta'(x-ct), \phi(x,t)\Big\rangle.
\end{aligned}$$

□

Remark 7.3. *We used only constant mean-states $U_{1,\varepsilon}$, $U_{2,\varepsilon}$ and constant central SDW speed curve $(ct,t)_{t\geq 0}$ in (19). Such SDWs are not good enough for solving an SDW interaction problems in general. The problem can be solved by introducing variable mean-states $U_{1,\varepsilon}(t)$ and $U_{2,\varepsilon}(t)$ and variable speed $c = c(t)$. See [23] for a complete solution.*

Definition 7.4. *Functions of the form (19) are called constant shadow waves or constant SDW for short. We shall drop the word "constant" in the sequel beccause we do not deal with SDW interactions. The value*

$$\sigma_\varepsilon(t) := (\varepsilon t + x_{1,\varepsilon})U_{1,\varepsilon} + (\varepsilon t + x_{2,\varepsilon})U_{2,\varepsilon}$$

is called the strength and c is called the speed of the shadow wave. We assume that $\lim_{\varepsilon\to 0}\sigma_\varepsilon(t) = \sigma(t) \in \mathbb{R}^n$ exists for every $t \geq 0$ and

$$\begin{aligned}
\lim_{\varepsilon\to 0}\int U_\varepsilon(x,t)\phi(x,t)\,dx\,dt =&\langle \sigma(t)\delta(x-ct) + U_0 + [U]\theta(x-ct), \phi(x,t)\rangle\\
=&\int \sigma(t)\phi(ct,t)dt + \int (U_0 + [U]\theta(x-ct)\phi(x,t)dx\,dt,
\end{aligned}$$

where θ is the Heaviside function and $[U] := U_1 - U_0$. The SDW front is given by $x = c(t)$.

A way to find a shadow wave solutions to a system of conservation laws (18) directly follows from Lemma 7.1. We use the following assumption to keep our discussion on a general level. An actual construction of SDW solution highly depends on a particular choice of f and g without it.

Assumption 7.5. *All the components U^i_ε, $i = 1, \dots, n$ of an SDW (19) satisfy*

$$\|U^i_\varepsilon\|_{L^\infty} = \mathcal{O}(\varepsilon^{-1}),\ \textit{if } f \textit{ and } g \textit{ are at most linear with respect to i-th variable}$$

or

$$\|U^i_\varepsilon\|_{L^\infty} \textit{ has a growth order small enough for (20) to hold, otherwise.}$$

Definition 7.6. *Components satisfying the first criteria are called the major components or ε^{-1}-components, while all other are called the minor ones.*

A delta shock is a SDW associated with a δ distribution with all minor components having finite limits as $\varepsilon \to 0$. If some of them are unbounded as $\varepsilon \to 0$, then the wave is called singular shock.

The following definition contains an analogous notion to Hugoniot locus for shocks.

Definition 7.7. *Let U_0 be fixed. The set of all $U_1 \in \Omega$ such that there exists an SDW solution to (18) with the initial data*

$$U|_{t=0} = \begin{cases} U_0, & x < 0 \\ U_1, & x > 0 \end{cases}$$

is called the shadow locus. Points for which the above wave is admissible constitutes the it admissible locus. The admissibility will be defined trough entropy conditions given bellow. In the case when the SDW is delta (singular) shock, the above set is called delta (singular delta) locus.

Let us start a search for SDW solutions of (18). Substitution of the function U from (19) into the i-th equation in (18) yields

$$\begin{aligned}\Big(-c(f^i(U_1) - f^i(U_0)) + \varepsilon f^i(U_{1,\varepsilon}) + \varepsilon f^i(U_{2,\varepsilon})\Big)\delta(x-ct)\\ -ct\Big(\varepsilon f^i(U_{1,\varepsilon}) + \varepsilon f^i(U_{2,\varepsilon})\Big)\delta'(x-ct) + \Big(g^i(U_1) - g^i(U_0)\Big)\delta(x-ct)\\ +t\Big(\varepsilon g^i(U_{1,\varepsilon}) + \varepsilon g^i(U_{2,\varepsilon})\Big)\delta'(x-ct) \approx 0.\end{aligned}$$

That implies

$$\begin{aligned}-c(f^i(U_1) - f^i(U_0)) + \varepsilon f^i(U_{1,\varepsilon}) + \varepsilon f^i(U_{2,\varepsilon}) + g^i(U_1) - g^i(U_0) \approx 0\\ -c(\varepsilon f^i(U_{1,\varepsilon}) + \varepsilon f^i(U_{2,\varepsilon})) + \varepsilon g^i(U_{1,\varepsilon}) + \varepsilon g^i(U_{2,\varepsilon}) \approx 0,\\ i = 1, \dots, n.\end{aligned} \tag{22}$$

Define

$$\kappa^i := c(f^i(U_1) - f^i(U_0) - (g^i(U_1) - g^i(U_0))$$

to be so called *Rankine-Hugoniot deficit* (RH deficit for short) in the i-th equation. Now (22) reads as

$$\begin{aligned}&\varepsilon f^i(U_{1,\varepsilon}) + \varepsilon f^i(U_{2,\varepsilon}) \approx \kappa^i\\ &\varepsilon g^i(U_{1,\varepsilon}) + \varepsilon g^i(U_{2,\varepsilon}) \approx c\kappa^i,\ i = 1, \dots, n.\end{aligned} \tag{23}$$

That was the most general case with Assumption 7.5. Let us take the above to be given in evolutionary form.

If the system of conservation laws (18) is given in the evolutionary form $f^i(y) \equiv y^i$, $i = 1, \dots, n$, then the system (22) reduces to

$$\begin{aligned} -c(U_1^i - U_0^i) + \varepsilon U_{1,\varepsilon}^i + \varepsilon U_{2,\varepsilon}^i + g^i(U_1) - g^i(U_0) &\approx 0 \\ -c(\varepsilon U_{1,\varepsilon}^i + \varepsilon U_{2,\varepsilon}^i) + \varepsilon g^i(U_{1,\varepsilon}) + \varepsilon g^i(U_{2,\varepsilon}) &\approx 0,\ i = 1, \dots, n. \end{aligned} \tag{24}$$

and the system (23) has now a simpler form

$$\begin{aligned} \varepsilon U_{1,\varepsilon}^i + \varepsilon U_{2,\varepsilon}^i &\approx \kappa^i \\ \varepsilon g^i(U_{1,\varepsilon}) + \varepsilon g^i(U_{2,\varepsilon}) &\approx c\kappa^i,\ i = 1, \dots, n. \end{aligned} \tag{25}$$

7.1. Systems linear in one variable

When a given system (18) is linear in one component (say U^1 in the sequel), then we are in position to get additional results concerning the existence of shadow wave solutions to a Riemann problem as it was done for split delta shocks above.

Let the system (18) be linear in U^1. Then the i-th equation of the system is

$$\partial_t\Big(f_1^i(\underline{U})U^1 + f_2^i(\underline{U})\Big) + \partial_x\Big(g_1^i(\underline{U})U^1 + g_2^i(\underline{U})\Big) = 0, \tag{26}$$

where f_i, g_i, $i = 1, 2$ are continuous functions with $\underline{U} := (U^2, \dots, U^n)$. Set $U_{1,\varepsilon}$ and $U_{2,\varepsilon}$ as follows.

$$\underline{U}_{i,\varepsilon} := \underline{U}_{s,i} \in \mathbb{R}^{n-1},\ i = 1, 2,\ \lim_{\varepsilon\to 0} \varepsilon U_{1,\varepsilon}^1 = \xi_1,\ \lim_{\varepsilon\to 0} \varepsilon U_{2,\varepsilon}^1 = \xi_2,$$

where $\underline{U}_{s,i}$ and ξ_i, $i = 1, 2$ will be determined later. For an SDW U_ε given by (19) the difference $f(U_1) - f(U_0)$ is denoted by $[f(U_\varepsilon)]$.

From (26) one derives the following system of algebraic equations with respect to ξ_1 and ξ_2 for each $i = 1, \dots, n$:

$$\begin{aligned} f_1^i(\underline{U}_{s,1})\xi_1 + f_1^i(\underline{U}_{s,2})\xi_2 &= \kappa_i \\ g_1^i(\underline{U}_{s,1})\xi_1 + g_1^i(\underline{U}_{s,2})\xi_2 &= c\kappa_i. \end{aligned} \tag{27}$$

Here $\kappa_i := c[f_1^i(\underline{U})U^1 + f_2^i(\underline{U})] - [g_1^i(\underline{U})U^1 + g_2^i(\underline{U})]$ as before.

The following theorem is proved in [23].

Theorem 7.8. *Assume that the density function f does not depend on U^1 in k equations of the system (26) (i.e. $f_1^i \equiv 0$, $i = i_1, \dots, i_k$). Then the shadow locus is a subset of $n - k + 1$-dimensional manifold intersected by Ω.*

Proof. Suppose $f_1^{n-k+1} = \ldots = f_1^n = 0$. From the first equation in (27) it follows $\kappa_i = 0$ for each $i = n-k+1, \dots, n$. Assume for a moment that $\underline{U}_{s,1}$ and $\underline{U}_{s,2}$ are known. If the left-hand side state U_0 is fixed, then the speed c and $U_1 = (U_1^1, \dots, U_1^n)$ has to satisfy the following system

$$c = \frac{[g_1^i(\underline{U})U^1 + g_2^i(\underline{U})]}{[f_2^i(\underline{U})]},\ i = n - k + 1, \dots, n. \tag{28}$$

There are k equations and $n + 1$ scalar variables: $c, U_1^1, \dots, U_1^n$, so we are free to chose $n - k + 1$ of them provided that $\underline{U}_{s,1}$ and $\underline{U}_{s,2}$ are chosen in a good way.

Thus, the set of all possible values U_1 such that (28) is satisfied lies in an $n - k + 1$-dimensional manifold (if the speed c was excluded from the above free choice).

Now we turn our attention to $U_{s,1}$ and $U_{s,2}$ and the first $n - k$ systems given by (27). Let $i \in \{1, \dots, n - k\}$. Assuming

$$D_s^i(\underline{U}_{s,1}, \underline{U}_{s,2}) := \begin{vmatrix} f_1^i(\underline{U}_{s,1}) & f_1^i(\underline{U}_{s,2}) \\ g_1^i(\underline{U}_{s,1}) & g_1^i(\underline{U}_{s,2}) \end{vmatrix} \neq 0,$$

the solution (ξ_1, ξ_2) for each system (27) is given by

$$\begin{aligned} \xi_1^i(\underline{U}_{s,1}, \underline{U}_{s,2}) &= \frac{\kappa_i(g_1^i(\underline{U}_{s,2}) - cf_1^i(\underline{U}_{s,2}))}{D_s^i}, \\ \xi_2^i(\underline{U}_{s,1}, \underline{U}_{s,2}) &= \frac{\kappa_i(g_1^i(\underline{U}_{s,1}) - cf_1^i(\underline{U}_{s,1}))}{D_s^i}. \end{aligned} \tag{29}$$

A consistency for ξ_1 and ξ_2 found from each system produces the new one

$$\begin{aligned} \xi_1^1(\underline{U}_{s,1}, \underline{U}_{s,2}) &= \ldots = \xi_1^{n-k}(\underline{U}_{s,1}, \underline{U}_{s,2}) \\ \xi_2^1(\underline{U}_{s,1}, \underline{U}_{s,2}) &= \ldots = \xi_2^{n-k}(\underline{U}_{s,1}, \underline{U}_{s,2}) \end{aligned} \tag{30}$$

of $2(n-k-1)$ equations.

Let $i \in \{n-k+1, \ldots, n\}$. We already know that $f_1^i \equiv 0$, and substitution of ξ_1^1 and ξ_2^1 into the second equation in (27) for such i gives the following

$$g_1^i(\underline{U}_{s,1})\xi_1^1(\underline{U}_{s,1}, \underline{U}_{s,1}) + g_1^i(\underline{U}_{s,2})\xi_2^1(\underline{U}_{s,1}, \underline{U}_{s,1}) = 0. \tag{31}$$

So, there are k such equations, and the final conclusion is that the shadow locus is defined by (28) provided that there exist a solution $\underline{U}_{s,1}^2, \ldots, \underline{U}_{s,1}^n, \underline{U}_{s,2}^2, \ldots, \underline{U}_{s,2}^n$ to (30, 31) of $2n-k-2$ equations. Since there are $2n-2$ variables there is a chance for solving the system, and obtain a maximal dimension $n-k+1$ of shadow locus. □

Roughly speaking, each additional density function independent of U^1 reduces the dimension of the locus by 1.

The extreme case is when none of f_1 components vanishes (i.e. all density functions depend on U^1). We then solve (27) with respect to ξ_1 and ξ_2. For each $i = 1, \ldots, n$ all the solutions is given by (29) have to be the same, so we get the system

$$\begin{aligned} \xi_1^1 &= \xi_1^2 = \ldots = \xi_1^n \\ \xi_2^1 &= \xi_2^2 = \ldots = \xi_2^n \end{aligned}$$

of $2(n-1)$ equations with $2n-1$ unknowns: $c, \underline{U}_{s,j}^i$, $i = 2, \ldots, n$, $j = 1, 2$. Also, the condition $D_s^i \neq 0$, $i = 1, \ldots, n$ is assumed as above. There are no conditions on a speed c and right-hand values U_1 with fixed U_0. If the solution of the above really exists, then the shadow locus is whole Ω.

In the other extreme case, $f_1^i \equiv 0$, $i = 2, \ldots, n$ the dimension of a shadow locus is at most 2.

Remark 7.9. *One could see that a dimension of a delta locus for 2×2 system linear in one variable was expected to be one for SDS. A delta locus obtained by using SDWs usually has a dimension equal two in such a case. Thus, a SDW delta locus form the present paper is much richer than delta locus. That is the answer to the problem of relatively small delta locus in general case for 2×2 system and problems with a construction of solution to arbitrary Riemann data posed in [21].*

7.2. Entropy conditions

The definition of an SDW solution permits us to use very important tool for extracting physically relevant soltions ("admissible" ones): Entropy-entropy flux method.

Let $\eta(U)$ be a (strictly) convex or semi-convex entropy function for (18), with entropy-flux function $q(U)$. We shall use entropy condition in the following form. A solution U_ε to the system (18) with initial data $U|_{t=0} = U_{0,\varepsilon}$ is *admissible* if for every $T > 0$ we have

$$\underline{\lim}_{\varepsilon\to 0} \int_{\mathbb{R}} \int_0^T \eta(U_\varepsilon)\partial_t\phi + q(U_\varepsilon)\partial_x\phi \, dt \, dx + \int_{\mathbb{R}} \eta(U_{0,\varepsilon}(x,0))\phi(x,0) \, dx \geq 0, \tag{32}$$

for all non-negative test functions $\phi \in C_0^\infty(\mathbb{R} \times (-\infty, T))$.

Take a simple SDW U_ε from (19) and use from Lemma 7.1 with f substituted by η and g by q. As the delta function is a non-negative distribution, the first condition becomes

$$\overline{\lim}_{\varepsilon\to 0} - c(\eta(U_1) - \eta(U_0)) + \varepsilon\eta(U_{1,\varepsilon}) + \varepsilon\eta(U_{2,\varepsilon}) + q(U_1) - q(U_0) \le 0 \tag{33}$$

But a derivative of the delta function has no constant sign and the second condition becomes

$$\lim_{\varepsilon\to 0} -c(\varepsilon\eta(U_{1,\varepsilon}) + \varepsilon\eta(U_{2,\varepsilon})) + \varepsilon q(U_{1,\varepsilon}) + \varepsilon q(U_{2,\varepsilon}) = 0. \tag{34}$$

Here, U_0, U_1, $U_{1,\varepsilon}$ and $U_{2,\varepsilon}$ are constants.

The entropy condition is connected with a problem of uniqueness for a weak solution of a conservation law system. We give a definition of weak (distributional) uniqueness and some results about it afterward.

Definition 7.10. *An SDW solution is called weakly unique if its distributional image is the unique. More precisely, a speed c of the wave has to be unique as well as the limit*

$$\lim_{\varepsilon\to 0} \varepsilon U_{1,\varepsilon} + \varepsilon U_{2,\varepsilon}.$$

Let $i \in \{1, \dots, n\}$. If a limit $\lim_{\varepsilon\to 0} \varepsilon U^i_{1,\varepsilon} + \varepsilon U^i_{2,\varepsilon}$ is unique, then we say that the i-th component is unique.

Note that all minor components of U_ε are unique by the above definition. The following proposition is a direct consequence of the SDW definition.

Proposition 7.11. *Suppose that (18) has an SDW solution.*

(a) If there exists an equation of the system, say i-th one, such that a density function $f^i(U)$ is independent of major components of U, then a speed of the SDW is uniquely determined by the equation

$$-c[f^i(U)] + [g^i(U)] = 0.$$

(b) If there is an equation in the system, say i-th one, such that $f^i(U) = U^j$, where U^j is a major component, then it is uniquely determined by

$$\varepsilon U^j_{1,\varepsilon} + \varepsilon U^j_{2,\varepsilon} = \kappa_i \in \mathbb{R}.$$

Consequently, if (a) holds and (b) holds for all major components, then a distributional limit of an SDW solution to (18) is unique. Specially, that is the case for a system given in evolutionary form.

Definition 7.12. *We say that a solution to (18) is weakly unique if it consists from a unique combination of standard admissible elementary waves (shocks, rarefactions and contact discontinuities) and admissible SDW.*

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