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Convergence results of iterative scheme for asymptotically quasi-I-nonexpansive mappings in Banach spaces

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Abstract. A new $\mathcal{R}$-generated Ishikawa iteration with errors is considered for quasi-nonexpansive mapping, asymptotically quasi-I-nonexpansive mapping $\mathcal{T}$ and asymptotically quasi-nonexpansive mapping I in Banach space. We prove the weak and strong convergence results for considered iteration to common fixed point of such mappings in frame work of real Banach spaces. A comparison table is prepared using a numeric example which shows that the proposed iterative algorithm is faster than some known iterative algorithms. Our main results improve and compliment some known results.

1. Introduction

Let D be a nonempty subset of a real normed linear space E and let $\mathcal{T} : D \to D$ be a mapping. Throughout this article, we assume that $\mathbb{N}$ is the set of natural numbers, we consider that E is real Banach space and $F(\mathcal{T})$ is nonempty. Now, let us recall some known definitions.

Definition 1.1. *Let D be a nonempty closed convex subset of real Banach space E. A mapping $\mathcal{T} : D \to D$ is said to be:*

(i) *nonexpansive* [7] if for all $x,\ y \in D$ and $F(\mathcal{T}) = \{x \in D : \mathcal{T}x = x\}$, the set of fixed points of $\mathcal{T}$,

$$\|\mathcal{T}x - \mathcal{T}y\| \leq \|x - y\|,$$

(ii) *quasi-nonexpansive* [18] if for all $x \in D$ and $q \in F(\mathcal{T}) = \{x \in D : \mathcal{T}x = x\}$, the set of fixed points of $\mathcal{T}$,

$$\|\mathcal{T}x - q\| \leq \|x - q\|,$$

(iii) *uniformly L-Lipschitzian* if there exists a constant $L > 0$ such that, for any $x,\ y \in D$,

$$\|\mathcal{T}^n x - \mathcal{T}^n y\| \leq L\|x - y\|, \quad \forall\ n \in \mathbb{N},$$

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(iv) *asymptotically nonexpansive* [9] with a sequence $\{k_n\} \subset [1,\infty)$ and $\lim_{n\to\infty} k_n = 1$ such that, for all $x,\ y \in D$,

$$\|\mathcal{T}^n x - \mathcal{T}^n y\| \leq k_n\|x - y\|, \quad \forall\ n \in \mathbb{N},$$

(v) *asymptotically quasi-nonexpansive* [15] with a sequence $\{k_n\} \subset [1,\infty)$ and $\lim_{n\to\infty} k_n = 1$ if, for all $x \in D$ and $q \in F(\mathcal{T}) = \{x \in D : \mathcal{T}x = x\}$, the set of fixed points of $\mathcal{T}$,

$$\|\mathcal{T}^n x - q\| \leq k_n\|x - q\| \quad \forall\ n \in \mathbb{N}.$$

In 1916, Tricomi [18] introduced quasi-nonexpansive for real functions and later studied by Diaz and Metcalf [5] for mappings in Banach spaces. Ghosh and Debnath [8] established a necessary and sufficient condition for convergence of Ishikawa iterates of a quasi-nonexpansive mapping on a closed convex subset of a Banach space. In 1972, the class of asymptotically nonexpansive mappings was introduced as a generalization of the class of nonexpansive mappings by Goebel and Kirk [9]. In 2001, the class of asymptotically quasi-nonexpansive mapping was introduced as a generalization of the class of asymptotically nonexpansive mappings by Qihou [15]. Furthermore, it is easy to observe that, if $F(T) \neq \emptyset$, then a nonexpansive mapping must be quasi-nonexpansive and an asymptotically nonexpansive mapping must be asymptotically quasi-nonexpansive mapping. But the converse implications need not be true.

There are many concepts which generalize a notion of nonexpansive mapping. One of such is *I*-nonexpansivity of a mapping $\mathcal{T}$ [12]. Let us recall some notions.

Definition 1.2. *Let D be a nonempty closed convex subset of real Banach space E. A mapping $\mathcal{T}$, $I : D \to D$ be two mappings of nonempty subset D of a real normed linear space E. Then $\mathcal{T}$ is said to be:*

(i) I-nonexpansive if for all $x, y \in D$ and $F(\mathcal{T}) = \{x \in D : \mathcal{T}x = x\}$, the set of fixed points of $\mathcal{T}$,

$$\|\mathcal{T}x - \mathcal{T}y\| \leq \|Ix - Iy\|,$$

(ii) asymptotically-I-nonexpansive with a sequence $\{k_n\} \subset [1,\infty)$ and $\lim_{n\to\infty} k_n = 1$ such that, for all $x,\ y \in D$,

$$\|\mathcal{T}^n x - \mathcal{T}^n y\| \leq k_n\|I^n x - I^n y\|, \quad \forall\ n \in \mathbb{N},$$

(iii) asymptotically quasi-I-nonexpansive with a sequence $\{k_n\} \subset [1,\infty)$ and $\lim_{n\to\infty} k_n = 1$ if, for all $x \in D$ and $q \in F(\mathcal{T}) = \{x \in D : \mathcal{T}x = x\}$, the set of fixed points of $\mathcal{T}$,

$$\|\mathcal{T}^n x - q\| \leq k_n\|I^n x - q\| \quad \forall\ n \in \mathbb{N}.$$

Remark 1.3. *If $F(\mathcal{T}) \cap F(I)$ is nonempty then an asymptotically I-nonexpansive mapping is a asymptotically quasi-I-nonexpansive. But, there exists a nonlinear continuous asymptotically quasi I-nonexpansive mappings which is asymptotically I-nonexpansive.*

In 2012, Purtas and Kiziltunc [14] used the following explicit iterative scheme to prove weak and strong convergence results for asymptotically quasi I-nonexpansive in Banach space,

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n I^n x_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n \mathcal{T}^n y_n. \end{cases} \tag{1}$$

In 2014, Deng et al. [4] used the following iterative scheme to prove a strong convergence result for asymptotically I-nonexpansive mapping in Banach spaces,

$$\begin{cases} y_n = \beta_n x_n + \beta_n(1-\beta_n)\mathcal{T}^n x_n, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n) I^n y_n. \end{cases} \tag{2}$$

2. Preliminaries

We proposed a new $\mathcal{R}$-generated Ishikawa iteration with errors. For arbitrary $x_1 \in D$, the sequence $\{x_n\}$ in D defined by

$$\begin{cases} y_n = \mathcal{R}^n[(1 - b_n - d_n)x_n + b_n I^n x_n + d_n v_n], \\ x_{n+1} = \mathcal{R}^n[(1 - a_n - c_n)x_n + a_n \mathcal{T}^n y_n + c_n u_n], \quad \forall\, n \geq 1. \end{cases} \tag{3}$$

where $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}$ are four real sequences in $[0, 1]$ satisfying $a_n + c_n \leq 1,\ b_n + d_n \leq 1$, and $\{u_n\},\ \{v_n\}$ are any bounded sequences in D.

Recall that a Banach space E is said to satisfy Opial condition [13] if for each sequence $\{x_n\}$ in E such that $\{x_n\}$ converges weakly to x implies that

$$\lim_{n\to\infty} \inf \|x_n - x\| < \lim_{n\to\infty} \inf \|x_n - y\| \tag{4}$$

for all $y \in E$ with $y \neq x$. It is well know that [6] inequality (4) is equivalent to

$$\lim_{n\to\infty} \sup \|x_n - x\| < \lim_{n\to\infty} \sup \|x_n - y\|. \tag{5}$$

Definition 2.1. *Let E be a closed subset of a real Banach space E and let $\mathcal{T} : D \to D$ be a mapping.*

(i) A mapping $\mathcal{T}$ is said to be semi-closed(demi-closed) at zero, if for each bounded sequence $\{x_n\}$ in D, the conditions x_n converges weakly to $x \in D$ and $\mathcal{T}x_n$ converges strongly to zero imply $\mathcal{T}x = 0$.

(ii) A mapping $\mathcal{T}$ is said to be semicompact, if for any bounded sequence $\{x_n\}$ in D such that $\|x_n - \mathcal{T}x_n\| \to 0,\ n \to \infty$, then there exists a subsequence $\{x_{n_p}\} \subset \{x_n\}$ such that $x_{n_p} \to x^* \in D$ strongly.

The following Lemmas play an important role in this paper:

Lemma 2.2. *[16] Let D be a uniformly convex Banach space and let $0 < \beta < \gamma < 1$. Suppose that $\{t_n\}$ is a sequence in $[\beta, \gamma]$ and $\{x_n\},\ \{y_n\}$ are two sequence in D such that*

$$\lim_{n\to\infty} \|t_n x_n + (1 - t_n)y_n\| = d, \quad \lim_{n\to\infty} \sup \|x_n\| \leq d, \ \lim_{n\to\infty} \sup \|y_n\| \leq d, \tag{6}$$

holds some $d \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 2.3. *[17] Let $\{\alpha_n\},\ \{\beta_n\}$ and $\{\gamma_n\}$ be three sequences of nonnegative real numbers with $\sum_{n=1}^{\infty} \beta_n < \infty$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$. If the following conditions is satisfied :*

$$\alpha_{n+1} \leq (1 + \beta_n)\alpha_n + \gamma_n, \ \ n \geq 1,$$

then the limit $\lim_{n\to\infty} \alpha_n$ exists.

3. Main Results

In this section we will prove our main results concerning weak and strong convergence of the sequence defined by (3). To formulate ones, we need some auxiliary results.

Lemma 3.1. *Let E be a real Banach space and let D be a nonempty closed convex subset of E. Let $\mathcal{R} : D \to D$ be a quasi-nonexpansive mapping, $\mathcal{T} : D \to D$ be an asymptotically quasi-I-nonexpansive mapping with a sequence $\{k_n\} \subset [1,\ \infty)$ and $I : D \to D$ be an asymptotically quasi-nonexpansive mapping with a sequence $\{\sigma_n\} \subset [1,\ \infty)$. Suppose $\mathcal{F} = F(\mathcal{R}) \cap F(\mathcal{T}) \cap F(I)$ is nonempty and $q \in \mathcal{F}$. Let $\rho = \sup_n a_n,\ \theta = \sup_n k_n \geq 1,\ \xi = \sup_n \sigma_n \geq 1$ and $\{a_n\},\ \{b_n\},\ \{c_n\},\ \{d_n\}$ are four real sequences in $[0, 1]$ which satisfy the following conditions:*

(1) $\sum_{n=1}^{\infty}(k_n\sigma_n - 1)a_n < \infty$,
(2) $\sum_{n=1}^{\infty} c_n < \infty$, $\sum_{n=1}^{\infty} d_n < \infty$,
(3) $\rho < \frac{1}{\theta^2\xi^2}$ *or* $\rho\theta\xi < 1$.

For some $x_1 \in D$, *let* $\{x_n\}$ *be* $\mathcal{R}$*- generated Ishikawa iterative with errors defined by (3), then for each* $q \in \mathcal{F}$ *the limit* $\lim_{n\to\infty} \|x_n - q\|$ *exists.*

Proof. As $q \in \mathcal{F}$, it follows from (3) that

$$\begin{aligned}
\|x_{n+1} - q\| &= \|\mathcal{R}^n[(1 - a_n - c_n)x_n + a_n\mathcal{T}^n y_n + c_n u_n] - q\| \\
&= \|\mathcal{R}(\mathcal{R}^{n-1}[(1 - a_n - c_n)x_n + a_n\mathcal{T}^n y_n + c_n u_n]) - q\| \\
&\le \|\mathcal{R}^{n-1}[(1 - a_n - c_n)x_n + a_n\mathcal{T}^n y_n + c_n u_n] - q\| \\
&\vdots \\
&\le \|(1 - a_n - c_n)x_n + a_n\mathcal{T}^n y_n + c_n u_n - q\| \\
&\le (1 - a_n - c_n)\|x_n - q\| + a_n\|\mathcal{T}^n y_n - q\| + c_n\|u_n - q\| \\
&\le (1 - a_n - c_n)\|x_n - q\| + a_n k_n\|I^n y_n - q\| + c_n\|u_n - q\| \\
&\le (1 - a_n)\|x_n - q\| + a_n k_n\sigma_n\|y_n - q\| + c_n\|u_n - q\|. \qquad (7)
\end{aligned}$$

Again from (3), we obtain

$$\begin{aligned}
\|y_n - q\| &= \|\mathcal{R}^n[(1 - b_n - d_n)x_n + b_n I^n x_n + d_n v_n] - q\| \\
&= \|\mathcal{R}(\mathcal{R}^{n-1}[(1 - b_n - d_n)x_n + b_n I^n x_n + d_n v_n] - q\| \\
&\le \|\mathcal{R}^{n-1}[(1 - b_n - d_n)x_n + b_n I^n x_n + d_n v_n] - q\| \\
&\vdots \\
&\le \|(1 - b_n - d_n)x_n + b_n I^n x_n + d_n v_n - q\| \\
&\le (1 - b_n - d_n)\|x_n - q\| + b_n\|I^n x_n - q\| + d_n\|v_n - q\| \\
&\le (1 - b_n - d_n)\|x_n - q\| + b_n\sigma_n\|x_n - q\| + d_n\|v_n - q\| \\
&\le (1 - b_n)\sigma_n\|x_n - q\| + b_n\sigma_n\|x_n - q\| + d_n\|v_n - q\| \\
&= \sigma_n\|x_n - q\| + d_n\|v_n - q\| \\
&\le k_n\sigma_n\|x_n - q\| + d_n\|v_n - q\|. \qquad (8)
\end{aligned}$$

Then by (7), we get

$$\begin{aligned}
\|x_{n+1} - q\| &\le (1 - a_n)\|x_n - q\| + a_n k_n^2\sigma_n^2\|x_n - q\| + c_n\|u_n - q\| + a_n k_n\sigma_n d_n\|v_n - q\| \\
&= [1 + a_n(k_n^2\sigma_n^2 - 1)]\|x_n - q\| + c_n\|u_n - q\| + a_n k_n\sigma_n d_n\|v_n - q\| \\
&= (1 + \lambda_n)\|x_n - q\| + \beta_n, \qquad (9)
\end{aligned}$$

where $\lambda_n = a_n(k_n^2\sigma_n^2 - 1)$ and $\beta_n = c_n\|u_n - q\| + a_n k_n\sigma_n d_n\|v_n - q\|$.
By assumption, we get

$$\begin{aligned}
\sum_{n=1}^{\infty}\lambda_n = \sum_{n=1}^{\infty}(k_n^2\sigma_n^2 - 1)a_n &= \sum_{n=1}^{\infty}(k_n\sigma_n + 1)(k_n\sigma_n - 1)a_n \\
&\le (\theta\xi + 1)\sum_{n=1}^{\infty}(k_n\sigma_n - 1)a_n < \infty.
\end{aligned}$$

and boundedness of the sequences $\{\|u_n - q\|\}$, $\{\|v_n - q\|\}$ with (2) condition implies

$$\sum_{n=1}^{\infty} \beta_n = \sum_{n=1}^{\infty} c_n\|u_n - q\| + a_n k_n \sigma_n d_n \|v_n - q\|$$
$$\leq \sum_{n=1}^{\infty} c_n\|u_n - q\| + \rho\theta\xi \sum_{n=1}^{\infty} d_n\|v_n - q\| < \infty.$$

Now taking $\gamma_n = \|x_n - q\|$ in (9) we find

$$\gamma_{n+1} \leq (1 + \lambda_n)\gamma_n + \beta_n,$$

and according to Lemma (2.3) the limit $\lim_{n\to\infty} \gamma_n$ exists. In this way the limit

$$\lim_{n\to\infty} \|x_n - q\| = d \tag{10}$$

exists, where $d \geq 0$ is a constant. This completes the proof. □

Theorem 3.2. *Let E be a real Banach space and let D be a nonempty closed convex subset of E. Let $\mathcal{R} : D \to D$ be a quasi-nonexpansive mapping, $\mathcal{T} : D \to D$ be a uniformly L_1- Lipschitzian asymptotically quasi-I-nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$ and $I : D \to D$ be a uniformly L_2- Lipschitzian asymptotically quasi-nonexpansive mapping with a sequence $\{\sigma_n\} \subset [1, \infty)$. Suppose $\mathcal{F} = F(\mathcal{R}) \cap F(\mathcal{T}) \cap F(I)$ is nonempty and $q \in \mathcal{F}$. Let $\rho = \sup_n a_n$, $\theta = \sup_n k_n \geq 1$, $\xi = \sup_n \sigma_n \geq 1$ and $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$ are four real sequences in $[0, 1]$ which satisfy the following conditions:*

(1) $\sum_{n=1}^{\infty}(k_n\sigma_n - 1)a_n < \infty$,
(2) $\sum_{n=1}^{\infty} c_n < \infty$, $\sum_{n=1}^{\infty} d_n < \infty$,
(3) $\rho < \frac{1}{\theta^2\xi^2}$.

For some $x_1 \in D$, let $\{x_n\}$ be $\mathcal{R}$- generated Ishikawa iterative with errors defined by (3), then converges strongly to a common fixed point in $\mathcal{F}$ if and only if

$$\lim_{n\to\infty} \inf d(x_n, \mathcal{F}) = 0. \tag{11}$$

Proof. The necessity of restriction (11) is understandable. Let us prove the sufficiency part of Theorem. As $\mathcal{T}$, $I : D \to D$ are uniformly L_1, L_2-Lipschitzian mappings respectively, $\mathcal{R}$, $\mathcal{T}$ and I are continuous mappings. So the sets $F(\mathcal{R})$, $F(\mathcal{T})$ and $F(I)$ are closed. Therefore $\mathcal{F} = F(\mathcal{R}) \cap F(\mathcal{T}) \cap F(I)$ is nonempty closed set. For given $q \in \mathcal{F}$, we have from (9)

$$\|x_{n+1} - q\| \leq (1 + \lambda_n)\|x_n - q\| + \beta_n, \tag{12}$$

as before where $\lambda_n = a_n(k_n^2\sigma_n^2 - 1)$ with $\sum_{n=1}^{\infty} \lambda_n < \infty$ and $\beta_n = c_n\|u_n - q\| + a_n k_n \sigma_n d_n \|v_n - q\|$ with $\sum_{n=1}^{\infty} \beta_n < \infty$. Therefore, we get

$$d(x_{n+1}, \mathcal{F}) \leq (1 + \lambda_n)d(x_n, \mathcal{F}) + \beta_n. \tag{13}$$

So the inequality(13) with Lemma(3.1) implies the existence of the limit $\lim_{n\to\infty} d(x_n, \mathcal{F})$. Then by condition (11), one gets

$$\lim_{n\to\infty} d(x_n, \mathcal{F}) = \lim_{n\to\infty} \inf d(x_n, \mathcal{F}) = 0. \tag{14}$$

Let us prove that the sequence $\{x_n\}$ converges to a common fixed point of $\mathcal{R}$, $\mathcal{T}$ and I. Indeed, due to $1 + p \le \exp(p)$ for all $p > 0$, and from (12) we get,

$$\|x_{n+1} - q\| \le \exp(\lambda_n)\|x_n - q\| + \beta_n. \tag{15}$$

Therefore, for any positive integers m, n, from (15) we obtain,

$$\begin{aligned}
\|x_{n+m} - q\| &\le \exp(\lambda_{n+m-1})\|x_{n+m-1} - q\| + \beta_{n+m-1}\\
&\le \exp(\lambda_{n+m-1} + \lambda_{n+m-2})\|x_{n+m-2} - q\| + \beta_{n+m-1} + \beta_{n+m-2}\exp(\lambda_{n+m-1})\\
&\le \exp(\lambda_{n+m-1} + \lambda_{n+m-2} + \lambda_{n+m-3})\|x_{n+m-3} - q\|\\
&\quad + \beta_{n+m-1} + \beta_{n+m-2}\exp(\lambda_{n+m-1}) + \beta_{n+m-3}\exp(\lambda_{n+m-1} + \beta_{n+m-2})\\
&\le \exp\Big(\sum_{i=n}^{n+m-1}\lambda_i\Big)\|x_n - q\| + \beta_{n+m-1} + \sum_{j=n+2}^{n+m-2}\beta_j\exp\Big(\sum_{i=j+2}^{n+m-1}\lambda_i\Big)\exp\Big(\sum_{i=n+2}^{n+m-2}\lambda_i\Big)\\
&\le \exp\Big(\sum_{i=n}^{n+m-1}\lambda_i\Big)\|x_n - q\| + \beta_{n+m-1}\exp\Big(\sum_{i=n}^{n+m-1}\lambda_i\Big) + \sum_{j=n+2}^{n+m-2}\beta_j\exp\Big(\sum_{i=n}^{n+m-1}\lambda_i\Big)\\
&\le \exp\Big(\sum_{i=n}^{n+m-1}\lambda_i\Big)\Big(\|x_n - q\| + \sum_{j=n+2}^{n+m-2}\beta_j\Big)\\
&\le \exp\Big(\sum_{i=n}^{\infty}\lambda_i\Big)\Big(\|x_n - q\| + \sum_{j=n+2}^{\infty}\beta_j\Big)\\
&\le \mathcal{Z}\Big(\|x_n - q\| + \sum_{j=n+2}^{\infty}\beta_j\Big),
\end{aligned} \tag{16}$$

for all $q \in \mathcal{F}$, where $\mathcal{Z} = \exp\left(\sum_{i=1}^{\infty}\lambda_i\right) < \infty$. Since $\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0$ and $\sum_{i=1}^{\infty}\beta_i < \infty$, for any given $\epsilon > 0$ there exists a positive integer r_0 such that

$$d(x_{r_0}, \mathcal{F}) < \frac{\epsilon}{2\mathcal{Z}}, \quad \sum_{j=r_0+2}^{\infty}\beta_j < \frac{\epsilon}{2\mathcal{Z}}$$

Therefore there exists $q_1 \in \mathcal{F}$ such that

$$\|x_{r_0} - q_1\| < \frac{\epsilon}{2\mathcal{Z}}, \quad \sum_{j=r_0+2}^{\infty}\beta_j < \frac{\epsilon}{2\mathcal{Z}}$$

Thus, for all $n \ge r_0$ from (16), we get

$$\begin{aligned}
\|x_n - q_1\| &\le \mathcal{Z}\Big(\|x_{r_0} - q_1\| + \sum_{j=r_0+2}^{\infty}\Big)\\
&< \mathcal{Z}\Big(\frac{\epsilon}{2\mathcal{Z}} + \frac{\epsilon}{2\mathcal{Z}}\Big) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,
\end{aligned} \tag{17}$$

this means that the sequence $\{x_n\}$ converges strongly to a common fixed point q_1 of $\mathcal{R}$, $\mathcal{T}$ and I. This completes the proof. □

If $\mathcal{R} = \mathrm{Id}$(Id is the identity mapping) to prove main results we need one more an auxiliary results.

Proposition 3.3. *Let E be a real uniformly convex Banach space and let D be a nonempty closed convex subset of E. Let $\mathcal{T} : D \to D$ be a uniformly L_1- Lipschitzian asymptotically quasi-I-nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$ and $I : D \to D$ be a uniformly L_2- Lipschitzian asymptotically quasi-nonexpansive mapping with a sequence $\{\sigma_n\} \subset [1, \infty)$. Suppose $\mathcal{F} = F(\mathcal{T}) \cap F(I)$ is nonempty and $q \in \mathcal{F}$. Let $\rho_* = \inf_n a_n$, $\rho^* = \sup_n a_n$, $\theta = \sup_n k_n \geq 1$, $\xi = \sup_n \sigma_n \geq 1$ and $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$ are four real sequences in $[0, 1]$ which satisfy the following conditions:*

(1) $\sum_{n=1}^{\infty}(k_n\sigma_n - 1)a_n < \infty$,
(2) $\sum_{n=1}^{\infty} c_n < \infty$, $\quad \sum_{n=1}^{\infty} d_n < \infty$,
(3) $0 < \rho_* \leq \rho^* \frac{1}{\theta^2\xi^2}$,
(4) $0 < \rho_* = \inf_n a_n \leq \sup_n a_n = \rho^* < 1$.

For some $x_1 \in D$, let $\{x_n\}$ be an Ishikawa iterative with errors defined by (3), satisfies the following

$$\lim_{n\to\infty} \|x_n - \mathcal{T} x_n\| = 0, \quad \lim_{n\to\infty} \|x_n - I x_n\| = 0.$$

Proof. First, we will prove that

$$\lim_{n\to\infty} \|x_n - \mathcal{T}^n x_n\| = 0, \quad \lim_{n\to\infty} \|x_n - I^n x_n\| = 0. \tag{18}$$

According to Lemma (3.1) for any $q \in \mathcal{F}$ we have from (10) $\lim_{n\to\infty} \|x_n - q\| = d$. It follows from (3) that

$$\begin{aligned} \|x_{n+1} - q\| &= \|(1 - a_n - c_n)x_n + a_n\mathcal{T}^n y_n + c_n u_n - q\| \\ &= \|(1 - a_n)(x_n - q) + a_n(\mathcal{T}^n y_n - q) + c_n(u_n - x_n)\| \\ &= \|(1 - a_n)[(x_n - q) + c_n(u_n - x_n)] + a_n[(\mathcal{T}^n y_n - q) + c_n(u_n - x_n)]\|. \end{aligned} \tag{19}$$

Owing to condition (2) and boundedness of the sequences $\{u_n\}$ and $\{x_n\}$ we have

$$\limsup_{n\to\infty} \|(x_n - q) + c_n(u_n - x_n)\| \leq \limsup_{n\to\infty} \|x_n - q\| + \limsup_{n\to\infty} c_n\|u_n - x_n\| = d. \tag{20}$$

By means of asymptotically quasi-I-nonexpansivity of $\mathcal{T}$ and asymptotically quasi-nonexpansivity of I and boundedness of sequences $\{u_n\}$, $\{v_n\}$, $\{x_n\}$ with condition (2) we take

$$\begin{aligned} \limsup_{n\to\infty} [(\mathcal{T}^n y_n - q) + c_n(u_n - x_n)]\| &\leq \limsup_{n\to\infty} k_n\sigma_n\|y_n - q\| + \limsup_{n\to\infty} c_n\|u_n - x_n\| \\ &\leq \limsup_{n\to\infty} \|y_n - q\| \\ &\leq \limsup_{n\to\infty} k_n\sigma_n\|x_n - q\| + \limsup_{n\to\infty} d_n\|v_n - q\| = d. \end{aligned} \tag{21}$$

Now using (20), (21) and applying to Lemma (2.2) to (19) one finds

$$\lim_{n\to\infty} \|x_n - \mathcal{T}^n y_n\| = 0. \tag{22}$$

With the help of (3), (22) and condition (2) we infer that

$$\begin{aligned} \lim_{n\to\infty} \|x_{n+1} - x_n\| &= \lim_{n\to\infty} \|(1 - a_n - c_n)x_n + a_n\mathcal{T}^n y_n + c_n u_n - x_n\| \\ &= \lim_{n\to\infty} \|a_n(\mathcal{T}^n y_n - x_n) + c_n(u_n - x_n)\| \\ &= \lim_{n\to\infty} \|a_n(\mathcal{T}^n y_n - x_n)\| = 0. \end{aligned} \tag{23}$$

By (22) and (23) we get

$$\lim_{n\to\infty} \|x_{n+1} - \mathcal{T}^n y_n\| \leq \lim_{n\to\infty} \|x_{n+1} - x_n\| + \lim_{n\to\infty} \|x_n - \mathcal{T}^n y_n\| = 0. \tag{24}$$

On the other hand, we have

$$\begin{aligned}\|x_n - q\| &\leq \|x_n - \mathcal{T}^n y_n\| + \|\mathcal{T}^n y_n - q\| \\ &\leq \|x_n - \mathcal{T}^n y_n\| + k_n\sigma_n\|y_n - q\|, \end{aligned} \tag{25}$$

which means

$$\|x_n - q\| - \|x_n - \mathcal{T}^n y_n\| \leq k_n\sigma_n\|y_n - q\|. \tag{26}$$

The last inequality with (8) implies that

$$\|x_n - q\| - \|x_n - \mathcal{T}^n y_n\| \leq k_n^2\sigma_n^2\|x_n - q\| + k_n\sigma_n d_n\|v_n - q\|. \tag{27}$$

Then condition (2) and (22), (10) with squeeze Theorem yield

$$\lim_{n\to\infty} \|y_n - q\| = d. \tag{28}$$

Again from (3) we can see that

$$\begin{aligned}\|y_n - q\| &= \|(1 - b_n - d_n)x_n + b_n I^n x_n + d_n v_n - q\| \\ &= \|(1 - b_n)(x_n - q) + d_n(v_n - x_n)(1 - b_n) + b_n(I^n x_n - q) + b_n d_n(v_n - x_n)\| \\ &= \|(1 - b_n)[x_n - q + d_n(v_n - x_n)] + b_n[I^n x_n - q + d_n(v_n - x_n)]\| \end{aligned} \tag{29}$$

From (10) and condition (2) one finds

$$\limsup_{n\to\infty} \|x_n - q + d_n(v_n - x_n)\| \leq \limsup_{n\to\infty} \|x_n - q\| + \limsup_{n\to\infty} d_n\|v - n - x_n\| = d, \tag{30}$$

and

$$\begin{aligned}\limsup_{n\to\infty} \|I^n x_n - q + d_n(v_n - x_n)\| &\leq \limsup_{n\to\infty} \|I^n x_n - q\| + \leq \limsup_{n\to\infty} d_n\|v_n - x_n\| \\ &\leq \limsup_{n\to\infty} \sigma_n\|x_n - q\| + \limsup_{n\to\infty} d_n\|v_n - x_n\| \\ &\leq \limsup_{n\to\infty} \|x_n - q\| = d. \end{aligned} \tag{31}$$

Now apply Lemma (2.2) to (29), we obtain

$$\lim_{n\to\infty} \|x_n - I^n x_n\| = 0. \tag{32}$$

From (23) and (32) we get

$$\lim_{n\to\infty} \|x_{n+1} - I^n x_n\| \leq \lim_{n\to\infty} \|x_{n+1} - x_n\| + \lim_{n\to\infty} \|x_n - I^n x_n\| = 0. \tag{33}$$

Again from (3) we can see that

$$\begin{aligned}\|y_n - x_n\| &= \|(1 - b_n - d_n)x_n + b_n I^n x_n + d_n v_n - x_n\| \\ &= \|b_n(I^n x_n - x_n) + d_n(v_n - x_n)\| \\ &\leq b_n\|I^n x_n - x_n\| + d_n\|v_n - x_n\| \end{aligned}$$

Taking limit to both side to above inequality and from (32) to condition (2) we wet

$$\lim_{n\to\infty} \|y_n - x_n\| \leq b_n\|I^n x_n - x_n\| + \lim_{n\to\infty} d_n\|v_n - x_n\| = 0. \tag{34}$$

Consider

$$\begin{aligned}\|x_n - \mathcal{T}^n x_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - \mathcal{T}^n y_n\| + \|\mathcal{T}^n y_n - \mathcal{T}^n x_n\| \\ &\le \|x_n - x_{n+1}\| + \|x_{n+1} - \mathcal{T}^n y_n\| + L_1\|y_n - x_n\| \\ &\le \|x_n - x_{n+1}\| + \|x_{n+1} - \mathcal{T}^n y_n\| + L_1\|b_n(I^n x_n - x_n) + d_n(v_n - x_n)\| \\ &\le \|x_n - x_{n+1}\| + \|x_{n+1} - \mathcal{T}^n y_n\| + L_1 b_n\|I^n x_n - x_n\| + L_1 d_n\|v_n - x_n\|\end{aligned}$$

From (23), (24), (32) and condition (2) we get

$$\lim_{n\to\infty} \|x_n - \mathcal{T}^n x_n\| = 0. \tag{35}$$

Next, from (23) and (34) we take

$$\lim_{n\to\infty} \|x_{n+1} - y_n\| \le \lim_{n\to\infty} \|x_{n+1} - x_n\| + \lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{36}$$

Finally, we get

$$\begin{aligned}\|x_n - \mathcal{T} x_n\| &\le \|x_n - \mathcal{T}^n x_n\| + \|\mathcal{T}^n x_n - \mathcal{T}^n y_{n-1}\| + \|\mathcal{T}^n y_{n-1} - \mathcal{T} x_n\| \\ &\le \|x_n - \mathcal{T}^n x_n\| + L_1\|x_n - y_{n-1}\| + L_1\|\mathcal{T}^{n-1} y_{n-1} - x_n\|\end{aligned}$$

which with (24), (35) and (36),we get

$$\lim_{n\to\infty} \|x_n - \mathcal{T} x_n\| = 0. \tag{37}$$

Similarly, one has

$$\begin{aligned}\|x_n - I x_n\| &\le \|x_n - I^n x_n\| + \|I^n x_n - \mathcal{T}^n x_{n-1}\| + \|I^n x_{n-1} - I x_n\| \\ &\le \|x_n - I^n x_n\| + L_2\|x_n - x_{n-1}\| + L_2\|I^{n-1} x_{n-1} - x_n\|\end{aligned}$$

which with (23), (32) and (33) implies

$$\lim_{n\to\infty} \|x_n - I x_n\| = 0. \tag{38}$$

□

Next we are ready to formulate one of main result concerning weak convergence of the sequence $\{x_n\}$.

Theorem 3.4. *Let E be a real uniformly convex Banach space satisfying Opial condition and let D be a nonempty closed convex subset of E. Let $\mathcal{T} : D \to D$ be a uniformly L_1- Lipschitzian asymptotically quasi-I-nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$ and $I : D \to D$ be a uniformly L_2- Lipschitzian asymptotically quasi-nonexpansive mapping with a sequence $\{\sigma_n\} \subset [1, \infty)$. Suppose $\mathcal{F} = F(\mathcal{T}) \cap F(I)$ is nonempty and $q \in \mathcal{F}$. Let $\rho_* = \inf_n a_n$, $\rho^* = \sup_n a_n$, $\rho = \sup_n a_n$, $\theta = \sup_n k_n \ge 1$, $\xi = \sup_n \sigma_n \ge 1$ and $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$ are four real sequences in $[0, 1]$ which satisfy the following conditions:*

(1) $\sum_{n=1}^{\infty}(k_n\sigma_n - 1)a_n < \infty$,
(2) $\sum_{n=1}^{\infty} c_n < \infty$, $\sum_{n=1}^{\infty} d_n < \infty$,
(3) $0 < \rho_* \le \rho^* \frac{1}{\theta^2\xi^2}$,
(4) $0 < \rho_* = \inf_n a_n \le \sup_n a_n = \rho^* < 1$.

If the mapping $X - \mathcal{T}$ and $X - I$ are semiclosed at zero and let $\{x_n\}$ be Ishikawa iterative sequence with errors defied by (3) converges weakly to a common fixed point in $\mathcal{F}$.

Proof. Let $q \in \mathcal{F}$, then according to Lemma 3.1 the sequence $\{\|x_n - q\|\}$ converges. This provides that $\{x_n\}$ is bounded sequence. Since E be a uniformly convex, then every bounded subset of E is weakly compact. From boundedness of $\{x_n\}$ in D, we find a subsequence $\{x_{n_r}\} \subset \{x_n\}$ such that $\{x_{n_r}\}$ converges weakly to $p \in D$. Therefore, from (37) and (38) it follows that

$$\lim_{n_r\to\infty} \|x_{n_r} - \mathcal{T}x_{n_r}\| = 0, \quad \lim_{n_r\to\infty} \|x_{n_r} - Ix_{n_r}\| = 0. \tag{39}$$

Since the mapping $X - \mathcal{T}$ and $X - I$ are semiclosed at zero, therefore, we find $\mathcal{T}p = p$ and $Ip = p$, which means $p \in \mathcal{F} = F(\mathcal{T}) \cap F(I)$.

Finally, let us prove that $\{x_n\}$ converges weakly to p. In fact, suppose the contrary, that is there exists some subsequence $\{x_{n_r}\} \subset \{x_n\}$ such that $\{x_{n_r}\}$ converges weakly to $p_1 \in D$ and $p_1 \neq p$. Then by the same method as given above, we can also prove that $p_1 \in \mathcal{F} = F(\mathcal{T}) \cap F(I)$.

Taking $q = p$ and $q = p_1$ and using the similar case in the proof of (10), we can prove that the limits

$$\lim_{n\to\infty} \|x_n - p\| = d, \quad \lim_{n\to\infty} \|x_n - p_1\| = d_1, \tag{40}$$

where d and d_1 are two nonnegative numbers. By asset of the Opial condition of E, we obtain

$$\begin{aligned} d &= \lim_{n_r\to\infty} \sup \|x_{n_r} - p\| < \lim_{n_r\to\infty} \sup \|x_{n_r} - p_1\| = d_1 \\ &= \lim_{n_k\to\infty} \sup \|x_{n_k} - p_1\| < \lim_{n_k\to\infty} \sup \|x_{n_k} - p\| = d. \end{aligned} \tag{41}$$

This is a negation. Therefore $p_1 = p$. This implies that $\{x_n\}$ converges weakly to p. This completes the proof of Theorem 3.4. □

Now we formulate next results concerning strong convergence of the sequence $\{x_n\}$.

Theorem 3.5. *Let E be a real uniformly convex Banach space and let D be a nonempty closed convex subset of E. Let $\mathcal{T} : D \to D$ be a uniformly L_1- Lipschitzian asymptotically quasi-I-nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$ and $I : D \to D$ be a uniformly L_2- Lipschitzian asymptotically quasi-nonexpansive mapping with a sequence $\{\sigma_n\} \subset [1, \infty)$. Suppose $\mathcal{F} = F(\mathcal{T}) \cap F(I)$ is nonempty and $q \in \mathcal{F}$. Let $\rho_* = \inf_n a_n$, $\rho^* = \sup_n a_n$, $\rho = \sup_n a_n$, $\theta = \sup_n k_n \geq 1$, $\xi = \sup_n \sigma_n \geq 1$ and $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$ are four real sequences in $[0, 1]$ which satisfy the following conditions:*

(1) $\sum_{n=1}^{\infty}(k_n\sigma_n - 1)a_n < \infty$,
(2) $\sum_{n=1}^{\infty} c_n < \infty$, $\quad \sum_{n=1}^{\infty} d_n < \infty$,
(3) $0 < \rho_* \leq \rho^* \frac{1}{\theta^2\xi^2}$,
(4) $0 < \rho_* = \inf_n a_n \leq \sup_n a_n = \rho^* < 1$.

If at least one mappings of $\mathcal{T}$ and I is semicompact and let $\{x_n\}$ be Ishikawa iterative sequence with errors defied by (3) converges weakly to a common fixed point in $\mathcal{F}$.

Proof. Without any loss of generality, we may assume the $\mathcal{T}$ is semicompact. This with (37) means that there exists a subsequence $\{x_{n_r}\} \subset \{x_n\}$ such that $x_{n_r} \to q^*$ strongly and $q^* \in D$. Since $\mathcal{T}$, I are continuous, them by (37) and (38), we obtain

$$\|q^* - \mathcal{T}q^*\| = \lim_{n_r\to\infty} \|x_{n_r} - \mathcal{T}x_{n_r}\| = 0, \ \ \|q^* - Iq^*\| = \lim_{n_r\to\infty} -Ix_{n_r}\| = 0. \tag{42}$$

This shows that $q^* \in \mathcal{F}$. According to Lemma 3.1 the limit $\lim_{n\to\infty} \|x_n - q^*\|$ exists. Then

$$\lim_{n\to\infty} \|x_n - q^*\| = \lim_{n_r\to\infty} \|x_{n_r} - q^*\| = 0,$$

which means that $\{x_n\}$ converges to $q^* \in \mathcal{F}$. This completes the proof. □

4. A comparison table of iterative algorithms

In this section, using Example 4.1 below, we numerically demonstrate the convergence of the algorithm defined in this paper and compare its behavior with the iterative algorithms of Purtas and Kiziltunc, (1), Deng et al.(2).

Example 4.1 Let $X = \mathbb{R}$ be the set of real numbers equipped with the norm $\|\cdot\| = |\cdot|$, $D = [0, 1]$, and let $a_n,\ b_n,\ c_n,\ d_n$ be four sequences in [0, 1] define as follows :

$$a_n = \frac{1}{2n},\ c_n = \frac{1}{2n+1},\ b_n = \frac{1}{3n^2} \text{ and } d_n = \frac{1}{n^2+1} \text{ for all } n \geq 1.$$

Let $\mathcal{R},\ \mathcal{T},\ I : D \to D$ are three operators defined, respectively, by

$$\mathcal{R}x = \frac{x^2}{1+x},\ \mathcal{T}x = \frac{x}{1+x}, \text{ and } Ix = \frac{2x}{2+x} \text{ for all } x \in D.$$

It can be easily verified that $\mathcal{R}$ is quasi-nonexpansive $\mathcal{T}$ is asymptotically quasi -I-nonexpansive and I is asymptotically quasi-nonexpansive mapping.

Let $\{u_n\}$, $\{v_n\}$ be two bounded sequences in D. Notice that $F = F(\mathcal{R}) \cap F(\mathcal{T}) \cap F(I) = \{0\}$. Suppose $u_n = 0.5 = v_n$ for all n with initial value $x_1 = 0.5$. The comparison given in the following table illustrate the proposed iteration scheme (3) and compared it with the well known iteration schemes proposed by Purtas and Kiziltunc (1), Deng et al.(2) [scheme up to the accuracy of thirteen decimal places.]

Table 1: A comparison table of proposed iterative algorithm with known iterative algorithms

Steps	items	proposed iteration	Purtas and Kiziltunc iteration	Deng et al. iteration
1	x_1	0.5000000000000	0.5000000000000	0.5000000000000
2	x_2	0.0753061175857	0.4090909090909	0.4127906976744
3	x_3	0.0003026515882	0.3696568060177	0.2491793046970
4	x_4	0.0000000005209	0.3370761147569	0.1416379304178
5	x_5	0.0000000000000	0.3128197174358	0.0791007541585
6	x_6	0.0000000000000	0.2937097933159	0.0527988434229
7	x_7	0.0000000000000	0.2780808238022	0.0372944032745
8	x_8	0.0000000000000	0.2649512540392	0.0275894816122
9	x_9	0.0000000000000	0.2533695413141	0.0221913236398
10	x_{10}	0.0000000000000	0.2435833272750	0.0173721633682

The iterative sequence generated by the proposed method converges to $q = 0.0000000000000$. The above comparison table also shows that the iterative sequence generated by the proposed iterative method (3) converges faster than some well known iterative methods (see Figure 1 below).

5. Conclusion

In the foregoing discussion, a new $\mathcal{R}$-generated Ishikawa iterative algorithm is proposed which enable us to prove some weak and strong convergence results, related to quasi-nonexpansive, asymptotically quasi I-nonexpansive and asymptotically quasi-nonexpansive mappings in real Banach space. A comparison of the proposed iterative scheme (3) to some known iterative schemes such as (1) and (2) reveals the fact that the iterative sequence generated by the proposed iterative algorithm converges to common fixed point faster than to iterative sequence generated by some known iterative algorithms as shown in Example 4.1 above.

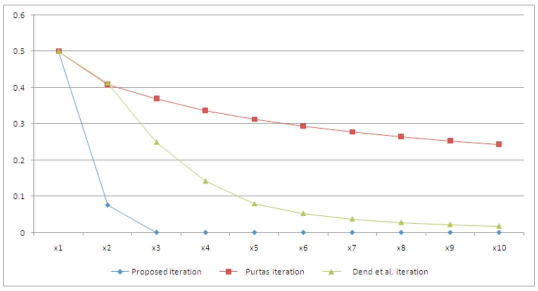

Figure 1: Convergence analysis of iteration schemes

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