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L^2-boundedness of Fourier integral operators with weighted symbols

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Abstract. In this paper we study a class of Fourier integral operators defined by symbols with tempered weight. These operators are bounded (respectively compact) in L^2 if the weight of the symbol is bounded (respectively tends to 0).

1. Introduction

A Fourier integral operator or FIO for short has the following form

$$[I(a,\phi)f](x) = \iint\limits_{\mathbb{R}^n_y \times \mathbb{R}^N_\theta} e^{i\phi(x,y,\theta)} a(x,y,\theta) f(y)\, dy\, d\theta, \quad f \in \mathcal{S}(\mathbb{R}^n) \tag{1}$$

where ϕ is called the *phase function* and a is the *symbol* of the FIO $I(a,\phi)$. In particular when $\phi(x,y,\theta) = \langle x-y,\theta\rangle$, $I(a,\phi)$ is called a pseudodifferential operator.

The study of FIO was started by considering the well known class of symbols $S^m_{\rho,\delta}$ introduced by Hörmander which consists of functions $a(x,\theta) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^N)$ that satisfy

$$|\partial^\alpha_\theta \partial^\beta_x a(x,\theta)| \le C_{\alpha,\beta}(1+|\theta|)^{m-\rho|\alpha|+\delta|\beta|},$$

with $m \in \mathbb{R}$, $0 \le \rho, \delta \le 1$. For the phase function one usually assumes that $\phi(x,\theta) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^N \setminus 0)$ is positively homogeneous of degree 1 with respect to θ and ϕ does not have critical points for $\theta \neq 0$.

Later on, other classes of symbols and phase functions were studied. In ([2]) and [6], D. Robert and B. Helffer treated the symbol class $\Gamma^\mu_\rho(\Omega)$ that consists of elements $a \in C^\infty(\Omega)$ such that for any multi-indices $(\alpha,\beta,\gamma) \in \mathbb{N}^n \times \mathbb{N}^n \times \mathbb{N}^N$, there exists $C_{\alpha,\beta,\gamma} > 0$,

$$|\partial^\alpha_x \partial^\beta_y \partial^\gamma_\theta a(x,y,\theta)| \le C_{\alpha,\beta,\gamma} \lambda^{\mu-\rho(|\alpha|+|\beta|+|\gamma|)}(x,y,\theta),$$

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$$\lambda(x, y, \theta) = (1 + |x|^2 + |y|^2 + |\theta|^2)^{1/2},$$

where Ω is an open set of $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^N$, $\mu \in \mathbb{R}$ and $\rho \in [0, 1]$ and they considered phase functions satisfying certain properties. In ([5]), Messirdi and Senoussaoui treated the L^2 boundedness and L^2 compactness of FIO with symbol class just defined. These operators are continuous (respectively compact) in L^2 if the weight of the symbol is bounded (respectively tends to 0). Noted that in Hörmander's class this result is not true in general. In fact, in ([7]) the author gave an example of FIO with symbol belonging to $\bigcap_{0<\rho<1} S^0_{\rho,1}$ that cannot be extended as a bounded operator on $L^2(\mathbb{R}^n)$.

The aim of this work is to extend results obtained in ([5]), the same hypothesis on the phase function are kept but we consider symbols with weight (m, ρ) (see below).

So in the second section we define symbol and phase functions used in this paper and we give the sense of the integral (1) by using the known oscillatory integral method developed by Hörmander ([3]). In addition, we discuss a special case of phase functions of type $\phi(x, y, \theta) = S(x, \theta) - \langle y, \theta\rangle$. The last section is devoted to treat the L^2 boundedness and L^2 compactness of FIO with phase ϕ.

2. Preliminaries

Definition 2.1. *A continuous function $m : \mathbb{R}^n \to [0, +\infty[$ is called a tempered weight on $\mathbb{R}^n$ if*

$$\exists C_0 > 0,\ \exists l \in \mathbb{R}\ ;\ m(x) \leq C_0\, m(y) \left(1 + \left|y - x\right|\right)^l, \text{ for all } x, y \in \mathbb{R}^n.$$

The simplest example is given by $\lambda^k(x) = (1 + |x|)^k$, where k is a natural number.

Definition 2.2. *Let Ω be an open set in $\mathbb{R}^n$, $\rho \in [0, 1]$ and m a tempered weight. A function $a \in C^\infty(\Omega)$ is called symbol with weight (m, ρ) or (m, ρ)-weighted symbol on Ω if*

$$\forall \alpha \in \mathbb{N}^n, \exists C_\alpha > 0;\ \left|\partial_x^\alpha a(x)\right| \leq C_\alpha m(x)(1 + |x|)^{-\rho|\alpha|}, \text{ for all } x \in \Omega.$$

We note $S^m_\rho(\Omega)$ the space of (m, ρ)-weighted symbols.

Proposition 2.3. *Let m and l be two tempered weights.*

(i) If $a \in S^m_\rho(\Omega)$ then $\partial_x^\alpha a \in S^{m\lambda^{-\rho|\alpha|}}_\rho(\Omega)$;
(ii) If $a \in S^m_\rho(\Omega)$ and $b \in S^l_\rho(\Omega)$ then $ab \in S^{ml}_\rho(\Omega)$;
(iii) If $\rho \leq \delta$, $S^m_\delta(\Omega) \subset S^m_\rho(\Omega)$;
(iv) Let $a \in S^m_\rho(\Omega)$. If there exists $C_0 > 0$ and $\mu \in \mathbb{R}$ such that $|a| \geq C_0\lambda^\mu$ uniformly on Ω then $\frac{1}{a} \in S^{m\lambda^{-2\mu}}_\rho(\Omega)$.

Proof. The proof is based on Leibniz's formula. □

Now, we consider the class of Fourier integral operators

$$[I(a, \phi)f](x) = \iint_{\mathbb{R}^n_y \times \mathbb{R}^N_\theta} e^{i\phi(x,y,\theta)} a(x, y, \theta) f(y)\, dy\, \widehat{d\theta}, \quad f \in \mathcal{S}(\mathbb{R}^n) \tag{2}$$

where $\widehat{d\theta} = (2\pi)^{-n}\, d\theta$, $a \in S^m_\rho(\mathbb{R}^n)$ and ϕ be a phase function which satisfies the following assumptions

(H1) $\phi \in C^\infty(\mathbb{R}^n_x \times \mathbb{R}^n_y \times \mathbb{R}^N_\theta, \mathbb{R})$;
(H2) For all $(\alpha, \beta, \gamma) \in \mathbb{N}^n \times \mathbb{N}^n \times \mathbb{N}^N$, there exists $C_{\alpha,\beta,\gamma} > 0$ such that

$$|\partial_x^\alpha \partial_y^\beta \partial_\theta^\gamma \phi(x, y, \theta)| \leq C_{\alpha,\beta,\gamma}\lambda^{(2-|\alpha|-|\beta|-|\gamma|)}(x, y, \theta);$$

(H3) There exists $C_1, C_2 > 0$ such that

$$C_1\lambda(x,y,\theta) \le \lambda(\partial_y\phi, \partial_\theta\phi, y) \le C_2\lambda(x,y,\theta), \text{ for all } (x,y,\theta) \in \mathbb{R}^n_x \times \mathbb{R}^n_y \times \mathbb{R}^N_\theta;$$

(H3*) There exists $C_1^*, C_2^* > 0$ such that

$$C_1^*\lambda(x,y,\theta) \le \lambda(x, \partial_\theta\phi, \partial_x\phi) \le C_2^*\lambda(x,y,\theta), \text{ for all } (x,y,\theta) \in \mathbb{R}^n_x \times \mathbb{R}^n_y \times \mathbb{R}^N_\theta.$$

It is clear that the phase function $\phi(x,y,\theta) = \langle x-y, \theta\rangle$ satisfies the hypothesis (H1)-(H3*).

To give a meaning to the right hand side of (2), we use the oscillatory integral method. So we consider $g \in \mathcal{S}(\mathbb{R}^n_x \times \mathbb{R}^n_y \times \mathbb{R}^N_\theta)$ such that $g(0) = 1$. Let $a \in S_0^m(\mathbb{R}^n)$ and define

$$a_r(x,y,\theta) = g\left(\frac{x}{r}, \frac{y}{r}, \frac{\theta}{r}\right) a(x,y,\theta), \quad r > 0.$$

Theorem 2.4. *Let ϕ be a phase function satisfying (H1)-(H3*). Then*

1. *For all $f \in \mathcal{S}(\mathbb{R}^n)$, $\lim_{r\to+\infty}[I(a_r,\phi)f](x)$ exists for every point $x \in \mathbb{R}^n$ and is independent of the choice of the function g. We define*

$$[I(a,\phi)f](x) := \lim_{r\to+\infty}[I(a_r,\phi)f](x);$$

2. *$I(a,\phi)$ defines a linear continuous operator on $\mathcal{S}(\mathbb{R}^n)$ and on $\mathcal{S}'(\mathbb{R}^n)$ respectively.*

Proof. (1) Let $\chi \in C_0^\infty(\mathbb{R})$, supp $\chi \subset [-2,2]$ such that $\chi \equiv 1$ in $[-1,1]$. For $\varepsilon > 0$, put

$$\omega_\varepsilon(x,y,\theta) = \chi\left(\frac{|\nabla_y\phi|^2 + |\nabla_\theta\phi|^2}{\varepsilon\,\lambda^2(x,y,\theta)}\right).$$

In supp ω_ε, $|\nabla_y\phi|^2 + |\nabla_\theta\phi|^2 \le 2\varepsilon\,\lambda^2(x,y,\theta)$. Using $(H3)$ we have

$$C_1^2\,\lambda^2(x,y,\theta) \le 2\varepsilon\lambda^2(x,y,\theta) + |y|^2.$$

So for $\varepsilon = \varepsilon_0$ small enough and $f \in \mathcal{S}(\mathbb{R}^n)$, we obtain

$$|I(\omega_{\varepsilon_0}a_r,\phi)f(x)| \le \iint_{\mathbb{R}^n_y\times\mathbb{R}^N_\theta} m(x,y,\theta)\,|f(y)|\,\mathrm{d}y\,\widehat{\mathrm{d}\theta}.$$

Using the definition of the tempered weight, there exist $C > 0$ and $l \in \mathbb{R}$ such that

$$|I(\omega_{\varepsilon_0}a_r,\phi)f(x)| \le C\,m(0,0,0)\iint_{\mathbb{R}^n_y\times\mathbb{R}^N_\theta} \lambda^l(x,y,\theta)\,|f(y)|\mathrm{d}y\,\widehat{\mathrm{d}\theta},$$

thus $I(\omega_{\varepsilon_0}a_r,\phi)f$ is absolutely convergent on supp ω_ε. By Lebesgue's dominated convergence Theorem we can see that

$$I(\omega_{\varepsilon_0}a,\phi)f = \lim_{r\to+\infty} I(\omega_{\varepsilon_0}a_r,\phi)f.$$

Moreover, it remains to prove this convergence on supp $(1-\omega_{\varepsilon_0})$. First note that

$$\operatorname{supp}(1-\omega_{\varepsilon_0}) \subset \Omega_0 = \{(x,y,\theta) : |\nabla_y\phi|^2 + |\nabla_\theta\phi|^2 \ge \varepsilon_0\,\lambda^2(x,y,\theta)\}.$$

Consider the differential operator

$$L = \frac{1}{i(|\nabla_y\phi|^2 + |\nabla_\theta\phi|^2)}\left(\sum_{j=1}^n \frac{\partial\phi}{\partial y_j}\frac{\partial}{\partial y_j} + \sum_{k=1}^N \frac{\partial\phi}{\partial\theta_k}\frac{\partial}{\partial\theta_k}\right).$$

A basic calculus shows that $L\,\mathrm{e}^{i\phi} = \mathrm{e}^{i\phi}$ and

$$({}^tL)^k[(1-\omega_{\varepsilon_0})b] = \sum_{|\alpha|+|\beta|\le k} g^{(k)}_{\alpha\beta}\partial^\alpha_y\partial^\beta_\theta((1-\omega_{\varepsilon_0})b), \tag{3}$$

where $b \in C^\infty(\mathbb{R}^n_y \times \mathbb{R}^N_\theta)$, $k \in \mathbb{N}$, ${}^tL$ is the transpose operator of L and $g^{(k)}_{\alpha,\beta} \in S_0^{\lambda^{-k}}(\Omega_0)$. For more details on equality (3) see [6, Lemma II.3].

Thus,

$$\begin{aligned}
\lim_{r\to\infty} I((1-\omega_{\varepsilon_0})a_r,\phi)f(x) &= \lim_{r\to\infty} \iint_{\mathbb{R}^n_y\times\mathbb{R}^N_\theta} \mathrm{e}^{i\phi(x,y,\theta)}({}^tL)^k[(1-\omega_{\varepsilon_0})\,a_r f(y)]\,\mathrm{d}y\,\widehat{\mathrm{d}\theta}\\
&= \iint_{\mathbb{R}^n_y\times\mathbb{R}^N_\theta} \mathrm{e}^{i\phi(x,y,\theta)}({}^tL)^k[(1-\omega_{\varepsilon_0})\,a f(y)]\,\mathrm{d}y\,\widehat{\mathrm{d}\theta}.
\end{aligned}$$

(2) is an immediate consequence of formula (3). The continuity of $I(a,\phi)$ on the space $\mathcal{S}'(\mathbb{R}^n)$ is obtained by the same way via the assumption (H3*). □

In the sequel we study the special phase function

$$\phi(x,y,\theta) = S(x,\theta) - \langle y,\theta\rangle, \tag{4}$$

where $S \in C^\infty(\mathbb{R}^n_x \times \mathbb{R}^n_\theta, \mathbb{R})$ satisfying

(G1) There exists $\delta_0 > 0$ such that

$$\inf_{x,\theta\in\mathbb{R}^n} \left|\det \frac{\partial^2 S}{\partial x\partial\theta}(x,\theta)\right| \ge \delta_0;$$

(G2) For all $(\alpha,\beta) \in \mathbb{N}^n \times \mathbb{N}^n$, there exists $C_{\alpha,\beta} > 0$, such that

$$|\partial^\alpha_x\partial^\beta_\theta S(x,\theta)| \le C_{\alpha,\beta}\lambda(x,\theta)^{(2-|\alpha|-|\beta|)}.$$

Proposition 2.5 ([5]).
(1) If S satisfies (G1) and (G2), then S satisfies (H1), (H2) and (H3). In addition there exists $C > 0$ such that for all $x, x', \theta \in \mathbb{R}^n$,

$$|x - x'| \le C|(\partial_\theta S)(x,\theta) - (\partial_\theta S)(x',\theta)|. \tag{5}$$

(2) If S satisfies (G2), then there exists a constant $\epsilon_0 > 0$ such that the phase function ϕ given in (4) belongs to $S_1^{\lambda^2}(\Omega_{\phi,\epsilon_0})$ where

$$\Omega_{\phi,\epsilon_0} = \left\{(x,\theta,y) \in \mathbb{R}^{3n};\ \ |\partial_\theta S(x,\theta) - y|^2 < \epsilon_0\,(|x|^2 + |y|^2 + |\theta|^2)\right\}.$$

Example 2.6. *Consider the function given by*

$$S(x,\theta) = \sum_{|\alpha|+|\beta|=2,\,\alpha,\beta\in\mathbb{N}^n} C_{\alpha,\beta}x^\alpha\theta^\beta, \quad \textit{for } (x,\theta)\in\mathbb{R}^{2n}$$

where $C_{\alpha,\beta}$ are real constants. $S(x,\theta)$ verifies (G1) and (G2).

Using Proposition 2.5, we deduce that

$$\lambda(x,y,\theta) \sim \lambda(x,\theta) \text{ in } \Omega_{\phi,\varepsilon_0} \tag{6}$$

and we establish the following result:

Proposition 2.7. *If $(x,\theta) \to a(x,\theta)$ belongs to $S_k^m(\mathbb{R}_x^n \times \mathbb{R}_\theta^n)$, then $\Omega_{\phi,\epsilon_0} \ni (x,y,\theta) \to a(x,\theta)$ belongs to $S_k^{\tilde{m}}(\mathbb{R}_x^n \times \mathbb{R}_y^n \times \mathbb{R}_\theta^n) \cap S_k^{\tilde{m}}(\Omega_{\phi,\epsilon_0})$, $k \in \{0,1\}$, where $\tilde{m}(x,y,\theta) = m(x,\theta)$.*

Proof. Denote $b(x,y,\theta) = a(x,\theta)$. We have to prove that $b(x,y,\theta) \in S_k^{\tilde{m}}(\mathbb{R}_x^n \times \mathbb{R}_y^n \times \mathbb{R}_\theta^n) \cap S_k^{\tilde{m}}(\Omega_{\phi,\epsilon_0})$, $k \in \{0,1\}$.
Let $\alpha, \beta, \gamma \in \mathbb{N}^n \times \mathbb{N}^n \times \mathbb{N}^n$.
If $|\gamma| \geq 1$, $|\partial_x^\alpha \partial_y^\gamma \partial_\theta^\beta b(x,y,\theta)| = 0$.
Else, if $|\gamma| = 0$,

$$\begin{aligned} |\partial_x^\alpha \partial_\theta^\beta b(x,y,\theta)| &= |\partial_x^\alpha \partial_\theta^\beta a(x,\theta)| \\ &\leq C_{\alpha,\beta}\, m(x,\theta)\lambda^{-(|\alpha|+|\beta|)}(x,\theta). \end{aligned}$$

By virtue of (6), we deduce that

$$|\partial_x^\alpha \partial_y^\gamma \partial_\theta^\beta b(x,y,\theta)| \leq C_{\alpha,\beta,\gamma}\, \tilde{m}\,(x,y,\theta)\lambda^{-(|\alpha|+|\beta|+|\gamma|)}(x,y,\theta).$$

□

3. L^2-boundedness and L^2-compactness of FIO

We prove now our main results.

Proposition 3.1. *Let F be the integral operator with the distribution kernel*

$$K(x,y) = \int_{\mathbb{R}^n} e^{i(S(x,\theta)-\langle y,\theta\rangle)} a(x,\theta)\, \widehat{d\theta}, \tag{7}$$

$a \in S_k^m(\mathbb{R}_{x,\theta}^{2n})$, $k = 0,1$ and S satisfies (G1) and (G2). Then FF^* and F^*F are pseudodifferential operators with symbol in $S_k^{m^2}(\mathbb{R}^{2n})$, $k = 0,1$, given by

$$\begin{aligned} \sigma(FF^*)(x, \partial_x S(x,\theta)) &\equiv |a(x,\theta)|^2 \left| \left(\det \frac{\partial^2 S}{\partial\theta\partial x} \right)^{-1} (x,\theta) \right|, \\ \sigma(F^*F)(\partial_\theta S(x,\theta), \theta) &\equiv |a(x,\theta)|^2 \left| \left(\det \frac{\partial^2 S}{\partial\theta\partial x} \right)^{-1} (x,\theta) \right|. \end{aligned} \tag{8}$$

We denote here $a \equiv b$ for $a, b \in S_k^{m^2}(\mathbb{R}^{2n})$ if $(a-b) \in S_k^{m^2\lambda^{-2}}(\mathbb{R}^{2n})$ and σ stands for the symbol.

The proof of Proposition 3.1 is similar to that of [5, Theorem 4.1].

Theorem 3.2. *Let F be the integral operator with the distribution kernel*

$$K(x,y) = \int_{\mathbb{R}^n} e^{i(S(x,\theta)-\langle y,\theta\rangle)} a(x,\theta)\, \widehat{d\theta},$$

where $a \in S_0^m(\mathbb{R}_{x,\theta}^{2n})$ and S satisfies (G1) and (G2). Then,

1. *F is bounded on $L^2(\mathbb{R}^n)$ if the weight $m(x,\theta)$ is bounded on $\mathbb{R}^{2n}$;*

2. *F is compact on* $L^2(\mathbb{R}^n)$ *if* $\lim\limits_{|x|+|\theta|\to\infty} m(x,\theta) = 0$.

Proof. Let $u \in \mathcal{S}(\mathbb{R}^n_x)$ and $v \in \mathcal{S}(\mathbb{R}^n_\theta)$.

$$\begin{aligned} Fu(x) &= \int_{\mathbb{R}^n} K(x,y)u(y)\,\mathrm{d}y \\ &= \int_{\mathbb{R}^n} \mathrm{e}^{iS(x,\theta)} a(x,\theta)\mathcal{F}u(\theta)\,\widehat{\mathrm{d}\theta}, \end{aligned}$$

$$(FF^*v)(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathrm{e}^{i(S(x,\theta)-S(\widetilde{x},\theta))} a(x,\theta)\,\overline{a}(\widetilde{x},\theta)v(\theta)\,\mathrm{d}\widetilde{x}\,\widehat{\mathrm{d}\theta},$$

and

$$(\mathcal{F}(F^*F)\mathcal{F}^{-1}v)(\theta) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathrm{e}^{-i(S(x,\theta)-S(x,\tilde{\theta}))}\,\overline{a}(x,\theta)\,a(x,\widetilde{\theta})v(\tilde{\theta})\,\widehat{\mathrm{d}\widetilde{\theta}}\,\mathrm{d}x,$$

where $\mathcal{F}$ is the Fourier transform.
(1) By virtue of Proposition 3.1, F^*F is a pseudodifferential operator with symbol in $S_0^{m^2}(\mathbb{R}^{2n})$ and by using Caldéron-Vaillancourt Theorem (see [1]), we deduce that there exists a positive constant $\gamma(n)$ and an integer $k(n)$ such that for all $u \in \mathcal{S}(\mathbb{R}^n)$,

$$\|(F^*F)\; u\|_{L^2(\mathbb{R}^n)} \le \gamma(n)\; Q_{k(n)}(\sigma(F^*F))\,\|u\|_{L^2(\mathbb{R}^n)},$$

where

$$Q_{k(n)}(\sigma(F^*F)) = \sum_{|\alpha|+|\beta|\le k(n)} \sup_{(x,\theta)\in\mathbb{R}^{2n}} \left|\partial_x^\alpha \partial_\theta^\beta \sigma(F^*F)(\partial_\theta S(x,\theta),\theta)\right|.$$

$Q_{k(n)}(\sigma(F^*)F)$ is a finite positive constant because $m(x,\theta)$ is bounded on $\mathbb{R}^n_x \times \mathbb{R}^n_\theta$. Hence F^*F is a bounded operator on $L^2(\mathbb{R}^n)$, therefore

$$\begin{aligned} \|Fu\|_{L^2(\mathbb{R}^n)} &\le \|F^*F\|^{1/2}_{\mathcal{L}(L^2(\mathbb{R}^n))}\|u\|_{L^2(\mathbb{R}^n)} \\ &\le (\gamma(n)\; Q_{k(n)}(\sigma(FF^*)))^{1/2}\|u\|_{L^2(\mathbb{R}^n)}. \end{aligned} \tag{9}$$

Thus, F is bounded on $\mathcal{S}(\mathbb{R}^n)$ and it can be extended by density to a bounded operator on $L^2(\mathbb{R}^n)$.

(2) If $\lim\limits_{|x|+|\theta|\to\infty} m(x,\theta) = 0$, it follows from [6, Theorem II.40] that F^*F and FF^* are compact operators on $L^2(\mathbb{R}^n)$. Show that F (and F^*) is compact on $L^2(\mathbb{R}^n)$.

Let $(u_p)_{p\in\mathbb{N}}$ be a bounded sequence in $L^2(\mathbb{R}^n)$; $\sup\limits_{p\in\mathbb{N}}\|u_p\|_{L^2(\mathbb{R}^n)} \le C,\ C > 0$; such that $(F^*Fu_p)_p$ converges on a suitable subsequence, which, for convenience, we will again denote by $(u_n)_n$. The following calculation shows that $(Fu_n)_n$ is also convergent. Indeed,

$$\begin{aligned} \|F(u_p - u_q)\|^2_{L^2(\mathbb{R}^n)} &= \langle F^*F(u_p-u_q),(u_p-u_q)\rangle_{L^2(\mathbb{R}^n)} \\ &\le \|F^*F(u_p-u_q)\|_{L^2(\mathbb{R}^n)}\|u_p-u_q\|_{L^2(\mathbb{R}^n)} \\ &\le 2C\|F^*F(u_p-u_q)\|_{L^2(\mathbb{R}^n)} \underset{p,q\to\infty}{\longrightarrow} 0. \end{aligned}$$

Finally, F is compact on $L^2(\mathbb{R}^n)$. □

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