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Diagonalizability of the product versus diagonalizability of the individual factors

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Abstract. The following question is attempted: Given two commuting matrices A and B with complex (real) entries such that AB is unitarily (orthogonally) diagonalizable, when will it happen that A and B are diagonalizable?

1. Introduction

We work with either the field of complex numbers $\mathbb{C}$ or the field of real numbers $\mathbb{R}$. The vector space of all square matrices with real and complex entries will be denoted, respectively, by $M_n(\mathbb{R})$ and $M_n(\mathbb{C})$. The transpose and adjoint of a matrix A will be denoted by A^t and A^*, respectively. We assume throughout that $\mathbb{C}^n$ (respectively, $\mathbb{R}^n$) is equipped with the Euclidean inner product $\langle x, y\rangle = \sum_{i=1}^{n} x_i\overline{y_i}$ for $x = (x_i), y = (y_i) \in \mathbb{C}^n$ (respectively, $\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i$) and $M_n(\mathbb{C})$ (respectively, $M_n(\mathbb{R})$) is equipped with the Hilbert-Schmidt trace inner product $\langle A, B\rangle = trace(B^*A)$ for $A, B \in M_n(\mathbb{C})$ (respectively, $\langle A, B\rangle = trace(B^tA)$). A complex matrix A is said to be unitarily diagonalizable if there exists a unitary matrix U such that $U^*AU = D$ for some diagonal matrix D. A real matrix A is said to be orthogonally diagonalizable, if there exists a (real) orthogonal matrix Q and a diagonal matrix D such that $Q^tAQ = D$. Notice that in this case, A becomes a symmetric matrix and so the matrix D must have only real entries on the diagonal. Let us recall the finite dimensional spectral theorem, which states that a square matrix is normal (real symmetric) if and only if it is unitarily (real orthogonally) similar to a diagonal matrix. Let us also recall that a linear map $T : M_n(\mathbb{C}) \longrightarrow M_n(\mathbb{C})$ is said to be unitarily diagonalizable if there exists an orthonormal basis of $M_n(\mathbb{C})$ with respect to which the matrix representation of T is diagonal. A similar definition holds for orthogonal diagonalizability.

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The question we are interested is the following:
Given two matrices A and B with complex (real) entries with the following properties:
(1) $AB \neq 0$.
(2) $AB = BA$.
(3) AB is unitarily (orthogonally) diagonalizable.
(4) Either A or B is unitarily (orthogonally) diagonalizable.
When does it happen that A and B are diagonalizable?

Let us first observe the following:

Theorem 1.1. *Suppose A and B are complex matrices such that $0 \neq AB = BA$ with A diagonalizable, and B non-diagonalizable. If A is invertible, then AB is not diagonalizable.*

Proof. Since $AB = BA$ and A is given to be diagonalizable, if AB were diagonalizable, then A and AB would be simultaneously diagonalizable. Let S be an invertible matrix such that $S^{-1}(AB)S = D$, and $S^{-1}AS = D'$, where D and D' are diagonal matrices with D' invertible. Then, $S^{-1}BS = (D')^{-1}D$, a diagonal matrix. This contradiction proves the result. □

Our motivation to study the problem stated above comes from maps of the form $X \mapsto AXB$ for fixed matrices A and B. Such maps typically appear in linear preserver problems (Refer [2] for some interesting problems and techniques). For fixed matrices A and B, consider the linear maps L_A and R_B on $M_n(\mathbb{C})$ or $M_n(\mathbb{R})$ defined by $L_A(X) = AX$, $R_B(X) = XB$. Note that $L_AR_B = R_BL_A$. For a field $\mathbb{F}$ and the set $M_{m,n}(\mathbb{F})$ of $m \times n$ matrices over $\mathbb{F}$, a linear preserver ϕ is a linear map $\phi : M_{m,n}(\mathbb{F}) \longrightarrow M_{m,n}(\mathbb{F})$ that preserves a certain property or a relation. Most such maps are of the form $\phi(X) = AXB$ for some invertible matrices A and B of orders $m \times m$ and $n \times n$, respectively, or $m = n$ and $\phi(X) = AX^tB$ for some invertible matrices A and B of order $n \times n$. Both these maps are called standard maps. The first such problem was perhaps studied by Frobenius, who proved that any determinant preserver is of the form AXB or AX^tB with $det(AB) = 1$. Other properties of matrices such as rank, inertia, trace, invertibility, functions of eigenvalues and so on, were investigated later on. For instance, rank preservers on the space of complex as well as real matrices are of the above form. There are two types of preserver problems one usually considers. Given a subset $\mathcal{S}$ of $M_{m,n}(\mathbb{F})$, what are the linear maps ϕ on $M_{m,n}(\mathbb{F})$ such that $\phi(\mathcal{S}) \subseteq \mathcal{S}$. The other one is to characterize those linear maps ϕ on $M_{m,n}(\mathbb{F})$ such that $\phi(\mathcal{S}) = \mathcal{S}$. For some general techniques on linear preserver problems, one may refer to [2] and the references cited therein. It is therefore natural to consider the properties of these maps L_A and R_B. More about these maps will be added later.

2. Results

We present the main results in this section. We prove our results for complex matrices. Proofs for real matrices follow with obvious modifications. It is well known that if A and B commute, then there exists an orthonormal vector that is an eigenvector for both A and B (Refer Observation 1.3.18 and Lemma 1.3.19, [1]). If, in addition, A and B are also unitarily diagonalizable, then A and B are simultaneously unitarily diagonalizable (Refer Theorem 1.3.21, [1]). Our first main result is the following.

Theorem 2.1. *Let $A, B \in M_n(\mathbb{C})$ be two commuting matrices such that $AB \neq 0$ and both A as well as AB are unitarily diagonalizable. Further, assume that the algebraic multiplicity of the eigenvalue 0 of A is at most 1. Then B is also unitarily diagonalizable.*

Proof. Note first that A and AB commute, as $AB = BA$. Therefore, A and AB are simultaneously unitarily diagonalizable. Thus, there is a common basis $\mathcal{B}$ of orthonormal eigenvectors for AB and A. We claim that $\mathcal{B}$ forms a basis of eigenvectors for B. Let us discuss two cases.

Case(i): Suppose 0 is not an eigenvalue of A. Then A is invertible. Let $x \in \mathcal{B}$ be an eigenvector of A corresponding to an eigenvalue λ. Then, $Bx = (1/\lambda)B(\lambda x) = (1/\lambda)BAx = (1/\lambda)ABx = (\lambda'/\lambda)x$, where λ' is the eigenvalue of AB for which x is an eigenvector. Thus, x is an eigenvector for B.

Case(ii): Suppose 0 is an eigenvalue of A of algebraic multiplicity 1. Then, the corresponding eigenspace is one dimensional. Let $u \in \mathcal{B}$ be an eigenvector of A corresponding to the eigenvalue 0. If $Bu = 0$, then u will be an eigenvector of B corresponding to the eigenvalue 0. Let us therefore assume that $Bu \neq 0$. Consider the subspace $span\{u\}$. Since $AB = BA$, we see that u is an eigenvector of B. □

Remark 2.2. *The proof of the above theorem can be adapted to two arbitrary linear maps T_1 and T_2 on $M_n(\mathbb{C})$, instead of matrices A and B, by suitably modifying the assumptions. This will be used later on in Theorem 2.10.*

We state below as a corollary, an analogous result for real matrices.

Corollary 2.3. *Let $A, B \in M_n(\mathbb{R})$ be two commuting matrices such that $AB \neq 0$ and both A as well as AB are orthogonally diagonalizable. Further, assume that the algebraic multiplicity of the eigenvalue 0 of A is at most 1. Then B is also orthogonally diagonalizable.*

The assumption on A in Corollary 2.3 cannot be dropped, as the following examples illustrate.

Example 2.4. *Consider the matrices $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then $AB = BA = I$, which is orthogonally diagonalizable. However, A and B have complex eigenvalues. Now consider the matrices $C = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}$. Then $CD = DC = -I$, which is again orthogonally diagonalizable. However, neither C nor D is diagonalizable.*

Example 2.5. *Consider the matrices $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Then, $AB = BA = A$, which is orthogonally diagonalizable, whereas B is not. Note that the geometric multiplicity of the eigenvalue 0 of A equals 2, which is also its algebraic multiplicity.*

The following is a contrapositive of Theorem 2.1. We present its proof for the sake of completeness. It is worth noting that the proof follows an entirely different approach, namely, simultaneous triangularization of two commuting matrices.

Theorem 2.6. *Suppose A and B are complex matrices such that $0 \neq AB = BA$ with A and AB both being unitarily diagonalizable, and B non-diagonalizable (and hence is not unitarily diagonalizable). Then 0 must be an eigenvalue of A of algebraic multiplicity at least 2.*

Proof. Since A and B commute, they are simultaneously unitarily triangularizable (by Theorem 2.3.3, [1]). Let T_1 and T_2 be the respective triangular matrices obtained as a result of triangularization. Since A is unitarily diagonalizable and B is non-diagonalizable, T_1 is a diagonal matrix and then there will exist distinct indices i, j such that $i < j$, with $(T_2)_{ij} \neq 0$. Since T_1 and T_2 commute and T_1T_2 is diagonal, we have $(T_1)_{ii} = (T_1)_{jj} = 0$. This proves the result. □

Let us now consider the linear maps L_A and R_B introduced in the introduction. Once again, we consider the complex case first. As noted before, $L_AR_B(X) = R_BL_A(X)$ for all $X \in M_n(\mathbb{C})$. Recall that a linear map $T : M_n(\mathbb{C}) \longrightarrow M_n(\mathbb{C})$ is said to be unitarily diagonalizable if there exists an orthonormal basis of $M_n(\mathbb{C})$ with respect to which the matrix representation of T is diagonal. The following result is an easy observation.

Theorem 2.7. *Consider a linear map L on $M_n(\mathbb{C})$ of the form $X \mapsto AXB$ for some unitarily diagonalizable matrices A and B. Then, L is unitarily diagonalizable.*

Proof. Let λ_i and μ_j be the eigenvalues of A and B, respectively. Let the corresponding linearly independent orthonormal eigenvectors of A and B^* be denoted by x_i and y_j. Define $X_{ij} := x_i y_j^*$. We then have $L(X_{ij}) = \lambda_i \overline{\mu_j} X_{ij}$. Thus, X_{ij} is an eigenvector of L corresponding to the eigenvalue $\lambda_i \overline{\mu_j}$. It is easy to verify that the collection of matrices $\{X_{ij} : 1 \leq i, j \leq n\}$ is an orthonormal set in $M_n(\mathbb{C})$, and so is a linearly independent set. It follows that L is unitarily diagonalizable. $\square$

The following corollary follows immediately from the above theorem.

Corollary 2.8. *If $A, B \in M_n(\mathbb{C})$ are unitarily diagonalizable, then the maps L_A and R_B on $M_n(\mathbb{C})$ defined by $L_A(X) = AX$, $R_B(X) = XB$ are both unitarily diagonalizable.*

It is the converse of Theorem 2.7 that is more interesting. Before proceeding further, we point out two useful facts that will be used in the following theorem.

Lemma 2.9. *Let $A \in M_n(\mathbb{C})$ and consider the map L_A on $M_n(\mathbb{C})$ as above. Then*
(i) A and L_A have the same set of eigenvalues.
(ii) The minimal polynomials of A and L_A are the same.

Proof. Suppose λ is an eigenvalue of L_A. Then, there exists a nonzero matrix X such that $L_A(X) = \lambda X = AX$. If x is a nonzero column of X, then, $Ax = \lambda x$. Thus, λ is an eigenvalue of A. On the otherhand, take any eigenvector x of A corresponding to an eigenvalue λ of A and form the matrix all of whose columns are the vector x. Then, $AX = \lambda X$. Thus, λ is an eigenvalue of L_A. This proves (i).
Let us denote the minimal polynomial of A by $m(x)$. Then, for any $X \in M_n(\mathbb{C})$, we have $m(A)X = m(L_A)(X) = 0$. Therefore, the polynomial $m(x)$ annihilates L_A and so, its minimal polynomial divides $m(x)$. On the otherhand, let $p(x)$ be any polynomial such that $p(L_A) = 0$. Then, we have $p(L_A)(I) = 0$. But, $p(L_A)(I) = p(A)I = p(A)$. Therefore, $p(x)$ is divisible by $m(x)$. This proves (ii). $\square$

It is easy to see that a similar conclusion holds for the map R_B. We now prove the converse of Theorem 2.7.

Theorem 2.10. *Let L be a unitarily diagonalizable linear transformation on $M_n(\mathbb{C})$ of the form $X \mapsto AXB$ for some A and B such that either A or B is unitarily diagonalizable and is invertible. Then B is also diagonalizable.*

Proof. Notice that $L = L_A R_B = R_B L_A$. The linear maps L_A and R_B are invertible if and only if A and B are invertible. Since A is unitarily diagonalizable, it follows from Theorem 2.7 that the map L_A is unitarily diagonalizable. It now follows from Theorem 2.1 and Remark 2.2, that R_B is unitarily diagonalizable. In fact, L_A and R_B are simultaneously unitarily diagonalizable. Therefore the minimal polynomial of R_B splits as a product of distinct linear factors. But we know from Lemma 2.9 that the minimal polynomials of R_B and B are the same. It follows that B is diagonalizable. $\square$

We end with a few remarks.
(1) Consider the maps L_A and L_B, where A and B are as in Example 2.5. Then, $L_A L_B = L_B L_A = L_A$, which is unitarily diagonalizable. However, L_B is not diagonalizable. Note that 0 is an eigenvalue of L_A of algebraic multiplicity at least 2. The matrices E_{11} and E_{12} are two linearly independent eigenvectors of L_A. (E_{ij} is the matrix with 1 at the ij^{th} entry and 0s elsewhere.)
(2) Theorems 1.1, 2.1 and 2.6 and Lemma 2.9 hold for matrices with entries from an arbitrary field by replacing unitary similarity with similarity.
(3) The proof of Theorem 2.1 can be extended to a family $\mathcal{F}$ of matrices with entries over any field that are pairwise commuting, one of the matrices in each pair being diagonalizable with 0 as an eigenvalue of algebraic multiplicity at most 1, and the product diagonalizable.

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