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Principal pivot transforms of range symmetric matrices in indefinite inner product space

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Abstract. It is shown that the property of a matrix being range symmetric in an indefinite inner product space is preserved under the principal pivot transformation.

1. Introduction

An indefinite inner product in C^n is a conjugate symmetric **sesquilinear** form $[x, y]$ together with the regularity condition that $[x, y] = 0$ for all $y \in C^n$ only when $x = 0$. Any indefinite inner product is associated with a unique invertible complex matrix J (called a weight) such that $[x, y] =< x, Jy >$ where $<,>$ denotes the Euclidean inner product on C^n. We also make an additional assumption on J, that is, $J^2 = I$, to present the results with much algebraic ease. There are two different values for dot product of vectors in indefinite inner product spaces and to overcome these difficulties, a new matrix product, called indefinite matrix multiplication is introduced and some of its properties are investigated in [7]. For $A \in C^{m\times n}, B \in C^{n\times k}$, the indefinite matrix product of A and B (relative to J) is defined as $A \circ B = AJB$. When J is the identity matrix the product reduces to the usual product of matrices. It can be verified that with respect to the indefinite matrix product, $rank(A \circ A^{[*]}) = rank(A^{[*]} \circ A) = rank(A)$, where as this rank property fails under the usual matrix multiplication, where $A \circ A^{[*]} = J_n A^* J_m$ is the adjoint of A relative to J_n and J_m, the weights in the appropriate spaces. Thus the Moore Penrose inverse of a complex matrix over an indefinite inner product space, with respect to the indefinite matrix product always exists and this is one of its main advantages. Recently, in [6] we have extended the concept of range symmetric matrix to an indefinite inner product space and presented some interesting characterizations of range symmetric matrices similar to EP matrices in the setting of indefinite matrix product. Further, we have exhibited that the class of range symmetric matrices in an indefinite inner product space coincides with the class of J-EP matrices studied in [3]. A set of necessary and sufficient conditions for a Schur complement in an EP matrix to be EP are determined in [4]. The aim of this manuscript is to discuss the range symmetry of a block matrix and the structure of the principal pivot transform of a range symmetric block matrix in an indefinite inner product space. We

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recall the definitions and preliminary results on complex matrices over an indefinite inner product space in Section 2. In Section 3, we deal with the main results. We present equivalent conditions for a block matrix to be range symmetric in terms of Schur complements of the principal sub-matrices. We begin with the principal pivot transform of a complex matrix and exhibit that it carries over as such for a matrix over an indefinite inner product space, under the indefinite matrix multiplication. We prove that the principal pivot transform of a range symmetric matrix is range symmetric. In general, rank of a matrix and that of its principal pivot are not the same. Here, we have determined certain conditions under which they are equal. Wherever possible, we provide examples to illustrate our results.

2. Preliminaries

First we shall state a well known lemma concerning the invariance of the usual matrix product involving generalized inverses.

Lemma 2.1: ([8],p.21) If X and Y are generalized inverses of A, then $CXB = CYB \Leftrightarrow N(A) \subseteq N(C)$ and $N(A^*) \subseteq N(B^*) \Leftrightarrow C = CA^-A$ and $B = AA^-B$ for every A^-.

Definition 2.1: For $A \in C^{n\times n}$, in an indefinite inner product space $\wp$,with weight J, a matrix $X \in C^{nn}$ satisfying $A \circ X \circ A = A$ is called a generalized inverse of A relative to the weight J. $A_J\{1\}$ is the set of all generalized inverses of A relative to the weight J.

Remark 2.1: It can be easily verified that X is a generalized inverse of A under the indefinite matrix product if and only if J_nXJ_m is a generalized inverse of A under the usual product of matrices. Hence $A_J\{1\} = \{X/J_nXJ_m$ is a generalized inverse of $A\}$.

Definition 2.2: For $A \in C^{n\times n}$ a matrix $X \in C^{n\times n}$ is called the Moore-Penrose inverse of A relative to the weight J if it satisfies the following equations:

$$A \circ X \circ A = A, X \circ A \circ X = X, (A \circ X)^{[*]} = A \circ X, (X \circ A)^{[*]} = X \circ A.$$

Such an X is denoted by $A^{[\dagger]}$ and represented as $A^{[\dagger]} = J_nA^{\dagger}J_n$.

The indefinite matrix product $C \circ X^- \circ B$ is said to be invariant for all choice of $X^- \in A_J\{1\}$ if $C \circ X^- \circ B = C \circ Y^- \circ B$ for $X^-, Y^- \in A_J\{1\}$.

Lemma 2.2: The indefinite matrix product $C \circ X^- \circ B$ is invariant for all choice of $X^- \circ A_J\{1\}$ if and only if the usual matrix product CXB is invariant for all choice of $X \in A\{1\}$.

Proof: From Definition 2.1 and Remark 2.1,

$$\begin{aligned} CXB = CYB for X, Y \in A\{1\} \quad &\Leftrightarrow \quad C \circ J_nXJ_m \circ B = C \circ J_nYJ_m \circ B \ for\ X, Y \in A\{1\} \\ &\Leftrightarrow \quad C \circ X^- \circ B = C \circ Y^- \circ B \ for\ X^-, Y^- \in A_J\{1\}. \end{aligned}$$

Hence the Lemma holds.

Definition 2.3: The Range space of $A \in C^{m\times n}$ is defined by $R(A) = \{y = A \circ x \in C^n/x \in C^n\}$.The Null space of A is defined by $Nu(A) = \{x \in C^n/A \circ x = 0\}$. It is clear that $Nu(A^{[*]}) = N(A^*)$.

Lemma 2.3: For $A, B, C \in C^{n\times n}$,

(i) $N(A) \subseteq N(C) \Leftrightarrow Nu(A) \subseteq Nu(C)$.
(ii) $N(A^*) \subseteq N(B^*) \Leftrightarrow Nu(A^{[*]}) \subseteq Nu(B^{[*]})$

Property 2.1: For $A \in C^{n\times n}$ the following hold:

(i) $(A^{[*]})^{[*]} = A$.
(ii) $(A^{[\dagger]})^{[\dagger]} = A$

Definition 2.2: $A \in C^{n\times n}$ is said to be range symmetric in an indefinite inner product space $\wp$ with weight J, that is, A is range symmetric relative to J if $R(A) = R(A^{[*]})$.

In particular if J is the identity matrix, it reduces to EP matrix [1].

In the sequel we shall use the following result found in [6].

Theorem 2.1: For $A \in C^{n\times n}$ the following are equivalent:

(i) $A$is range symmetric in $\wp$
(ii) AJ is EP
(iii) JA is EP
(iv) A is J-EP
(v) $N(A) = N(A^{[*]})$
(vi) $[AA^{\dagger}]^{[*]} = J(AA^{\dagger})J = A^{\dagger}A$

Let us consider $M \in C^{(m+n)\times(m+n)}$, a *block* matrix of the form

(2.1) $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$, where A and D are square matrices of orders m and n respectively. With respect to this partitioning, a Schur complement of A in M is a matrix of the form $M/A = D - CA^{-}B$ [2], where A^{-}, a generalized inverse of A is a solution of $AXA = A$. On account of Remark (2.1), Lemma (2.1) and Lemma (2.2), it is obvious that under certain conditions M/A is independent of the choice of generalized inverse of A and it turns out that the definition of a Schur complement carries over as such to indefinite inner product spaces. However, in the sequel we shall always assume that M/A is given in terms of specific choice of A^{-}. Let J, J_m and J_n be the weights associated with the indefinite inner products in C^{m+n}, C^m and C^n respectively. Since $J_m = J_m{}^* = J_m{}^{-1}$ and $J_n = J_n{}^* = J_n{}^{-1}$, it can be verified that J is of the form

(2.2) $\begin{bmatrix} J_m & 0 \\ 0 & J_n \end{bmatrix}$.

Theorem 2.2(Theorem 1 of [4]): Let M be a matrix of the form (2.1) with $N(A) \subseteq N(C)$ and $N(M/A) \subseteq N(B)$, then the following are equivalent:

(i) M is an EP matrix.
(ii) A and M/A are EP, $N(A^*) \subseteq N(B*)$ and $N(M/A)^* \subseteq N(C^*)$.
(iii) Both the matrices

$\begin{bmatrix} A & 0 \\ C & M/A \end{bmatrix}$ and $\begin{bmatrix} A & B \\ 0 & M/A \end{bmatrix}$ are EP.

3. Principal pivot on a matrix

In [5], we have introduced the concept of principal pivot transform for a block complex matrix and proved that $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ satisfying $N(A) \subseteq N(C)$ and $N(A^*) \subseteq N(B^*)$ can be transformed into the matrix

(3.1) $\widehat{M} = \begin{bmatrix} A^{\dagger} & -A^{\dagger}B \\ CA^{\dagger} & S \end{bmatrix}$ Where $S = D - CA^{\dagger}B$ is the Schur complement of A in M. $\widehat{M}$ is called a principal pivot transform of M. The operation that transforms $M \to \widehat{M}$ is called a principal pivot by pivoting the block A. If A is non-singular it reduces to the principal pivot by pivoting the block A [9]. Properties and applications of the principal pivot transforms are well recognized in Mathematical programming [9 and 10].

A system of linear equations $M \circ z = t$ in an indefinite inner product space $\wp$ with weight J is identical with the system $MJz = t$, where $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ and $J = \begin{bmatrix} J_m & 0 \\ 0 & J_n \end{bmatrix}$. Further, by Lemma (2.3), the conditions on the matrix M to be transformed into its principal pivot are equivalent under the indefinite matrix multiplication with respect to the weight J. Hence it turns out that a principal pivot transform of a complex matrix in $\wp$ is the same as that of a matrix in Euclidean space. In this section, we are concerned with complex matrices over an indefinite inner product space $\wp$ with weight J. We shall discuss the relation between the principal transforms of the block matrices M, JM, MJ and PMP^T for some permutation matrix P. For

$A \in C^{m\times m}$, A is J-EP and A is range symmetric in $\wp$ are equivalent by Theorem (2.1), hence forth we use, A is $J - EP_r$ if A is range symmetric in $\wp$ and of rank r.

Lemma 3.1: Let $A \in C^{m\times m}$ and $U \in C^{m\times m}$ be any nonsingular matrix. Then, A is EP $\Leftrightarrow$ UAU^* is EP.

Proof:

$$\begin{aligned} A \text{ is } EP &\Leftrightarrow R(A) = R(A^*). \\ &\Leftrightarrow R(UAU^*) = R(UA^*U^*) \quad \text{(By Lemma 3 of [1])}. \\ &\Leftrightarrow R(UAU^*) = R(UAU^*)^*. \\ &\Leftrightarrow UAU^* is EP. \end{aligned}$$

Theorem 3.1: Let M of the form (2.1) with $N(A) \subseteq N(C)$ and $N(A^{[*]}) \subseteq N(B^{[*]})$ be a J-EP matrix, then, $\widehat{M}$ the principal pivot transform of M is J-EP.

Proof: For M of the form (2.1) and J of the form (2.2), under the usual matrix product, $JM = \begin{bmatrix} J_mA & J_mB \\ J_nC & J_nD \end{bmatrix}$ with $N(J_mA) = N(A) \subseteq N(C) = N(J_nC)$ and $N(J_mA)^* = N(A^*J_m) = N(A^{[*]}) \subseteq N(B^{[*]}) = N(B^*J_m) = N(J_mB)^*$. By using $(J_mA)^\dagger = A^\dagger J_m$, the principal pivot transform of JM reduces to the form

$\widehat{JM} = \begin{bmatrix} A^\dagger J_m & -A^\dagger B \\ J_nA^\dagger J_m & J_nC \end{bmatrix}$.

$\widehat{M}J = \begin{bmatrix} A^\dagger & -A^\dagger B \\ CA^\dagger & S \end{bmatrix}\begin{bmatrix} J_m & 0 \\ 0 & J_n \end{bmatrix} = \begin{bmatrix} A^\dagger J_m & -A^\dagger BJ_n \\ CA^\dagger J_m & SJ_n \end{bmatrix}$. Let us define

(3.2) $P_m = \begin{bmatrix} J_m & 0 \\ 0 & I_n \end{bmatrix}$ and $P_n = \begin{bmatrix} I_m & 0 \\ 0 & J_n \end{bmatrix}$. Since J_m and J_n are nonsingular P_m and P_n are nonsingular. Further,

(3.3) $JP_n = P_nJ = P_m$, $P_m{}^* = P_m$ and $JP_m = P_mJ = P_n$.

$$P_n\widehat{M}P_m = \begin{bmatrix} J_m & 0 \\ 0 & J_n \end{bmatrix}\begin{bmatrix} A^\dagger & -A^\dagger B \\ CA^\dagger J_m & S \end{bmatrix}\begin{bmatrix} J_m & 0 \\ 0 & I_n \end{bmatrix} = \begin{bmatrix} A^\dagger J_m & -A^\dagger B \\ J_nCA^\dagger J_m & J_nS \end{bmatrix} = \widehat{JM}.$$

Then by using $JP_n = P_nJ = P_m, P_m{}^* = P_m$ and $J^2 = I_{m+n}$,

(3.4) $\widehat{JM} = P_n\widehat{M}P_m = (P_nJ)(\widehat{JM})P_m = P_m(\widehat{JM})P_m{}^*$. Since M is J-EP, by Theorem (2.1) (iii), JM is EP and by Theorem 1 of [5], $\widehat{JM}$ is EP. Since P_m is invertible, by Lemma 3.1, $P_m\widehat{JM}P_m{}^* = \widehat{JM}$ is EP. Again, by Theorem (2.1) (iv) it follows that $\widehat{M}$ is J-EP. Hence the Theorem holds.

Corollary 3.1: Let M be of the form (2.1) with $N(A) \subseteq N(C)$ and $N(A^{[*]}) \subseteq N(B^{[*]})$ and J of the form (2.2) . Let P_m and P_n be defined as in (3.2). Then the following hold:

(i) $\widehat{JM} = P_n\widehat{M}P_m = P_m(\widehat{JM})P_m{}^* = P_n\widehat{MJ}P_n{}^*$.

(ii) $\widehat{MJ} = P_m\widehat{M}P_n = P_n(\widehat{JM})P_n{}^* = P_m(\widehat{MJ})P_m{}^*$.

(iii) $\widehat{JM} = J(\widehat{MJ})J$.

Proof: (i) directly follows from (3.4) and using (3.3). (ii) can be proved in the same manner as that of (i) for the matrix MJ and hence omitted. (iii)$\widehat{JM} = P_n.\widehat{M}.P_m = P_nP_m\widehat{MJ}P_nP_m = J\widehat{MJ}J$, by using $P_nP_m = P_mP_n = J$. Thus (iii) holds.

Remark 3.1: For M of the form (2.1), if $N(A) \subseteq N(C)$ and $N(A^*) \subseteq N(B^*)$, then by Lemma (2.1) $C = CA^\dagger A$ and $B = AA^\dagger B$. Hence, $M = \begin{bmatrix} I & 0 \\ CA^\dagger & I \end{bmatrix}\cdot\begin{bmatrix} A & 0 \\ 0 & M/A \end{bmatrix}\cdot\begin{bmatrix} I & A^\dagger B \\ 0 & I \end{bmatrix}$ and $rank(M) = rank(A) + rank(M/A)$. In the same manner for the matrix $\widehat{M}$ in (3.1), $rank(\widehat{M}) = rank(A) + rank(D)$.Thus in general, $rank(M) \neq rank(\widehat{M})$. Here, we shall determine conditions for the equality of the ranks of a matrix and its principal transform. First , we discuss the range symmetry of a block matrix.

Theorem3.2: Let M be a matrix of the form (2.1) with $N(A) \subseteq N(C)$ and $N(S) \subseteq N(B)$, S is the Schur complement of A in M. Then the following are equivalent:

(i) M is a J-EP matrix.

(ii) A is $J_m - EP$ and S is $J_n - EP$, $N(A^{[*]}) \subseteq N(B^{[*]})$ and $N(S^{[*]}) \subseteq N(C^{[*]})$.

Proof: $(i) \Rightarrow (ii)$. Since M is a matrix of the form (2.1), the weight J in conformity with that of M is of the form(2.2), $JM = \begin{bmatrix} J_mA & J_mB \\ J_nC & J_nD \end{bmatrix}$. Since M is a J-EP matrix by Theorem(2.1) (iii), JM is an EP matrix. In the EP matrix JM, the Schur complement of J_mA in JM , that is, $JM/J_mA = J_nD-(J_nC)(J_mA)^\dagger J_mB = J_n(D-CA^\dagger B) = J_nS$. The matrix JM satisfies the conditions $N(J_mA) = N(A) \subseteq N(C) = N(J_nC)$ and $N(J_nS) = N(S) \subseteq N(B) = N(J_mB)$. Now by applying Theorem(2.2) for the EP matrix JM and by using Theorem(2.1) (iv),we conclude that A is Jm-EP and S is Jn-EP, $N(A^{[*]}) = N(J_mA^*J_m) = N(J_mA)^* \subseteq N(J_mB)^* = N(J_mB^*J_m) = N(B^{[*]})$ and $N(S^{[*]}) = N(J_nS)^* \subseteq N(J_nC)^* = N(C^{[*]})$. Thus (ii) holds.
$(ii) \Rightarrow (i)$: Since A is J_m-EP and S is J_n-EP, by Theorem(2.1) (iii) J_mA is EP and the Schur complement of J_mA in $JM = JM/J_mA = J_nS$ is EP. $N(J_mA)^* = N(A^{[*]}) \subseteq N(B^{[*]}) = N(J_mB)^*$ and $N(J_nS)^* = N(S^{[*]}) \subseteq N(C^{[*]}) = N(J_nC)^*$. Then by Theorem (2.2) JM is EP and finally, M is J-EP follows from Theorem (2.1) (iv). Hence the Theorem holds.

Lemma 3.2: Let M be of the form (2.1) and J of the form (2.2). Let $S = D - CA^\dagger B$ be the Schur complement of A in M and $S_1 = A - BD^\dagger C$ be the Schur complement of D in M. Let $N(A) \subseteq N(C)$ and $N(D) \subseteq N(B)$. Then the following are equivalent:

(i) M is J-EP with $N(S) \subseteq N(B)$ and $N(S_1) \subseteq N(C)$.

(ii) A and S_1 are J_m-EP matrices, D and S are J_n-EP matrices with $N(A) = N(S_1) \subseteq N(B^{[*]})$ and $N(D) = N(S) \subseteq N(C^{[*]})$.

Proof : $(i) \Rightarrow (ii)$. Since M is J-EP with $N(A) \subseteq N(C)$ and $N(S) \subseteq N(B)$, by Theorem (3.2), it follows that A is Jm-EP and S is Jn-EP, $N(A) = N(A^{[*]} \subseteq N(B^{[*]})$ and $N(S) = N(S^{[*]}) \subseteq N(C^{[*]})$. Since M is J-EP, by Theorem (2.1) (iii) $JM = \begin{bmatrix} J_mA & J_mB \\ J_nC & J_nD \end{bmatrix}$ is EP. For some permutation matrix P, let $M_1 = PMP^T = \begin{bmatrix} D & C \\ B & A \end{bmatrix}$ and $J_1 = PJP^T$. Then,$J_1M_1 = PJMP^T = \begin{bmatrix} J_nD & J_nC \\ J_mB & J_mA \end{bmatrix}$. Since JM is EP,by Lemma (3.1),J_1M_1 is EP. By Theorem (2.1) (iv), M_1 is J_1-EP,with $N(D) \subseteq N(B)$ and $N(S_1) \subseteq N(C)$. Therefore by Theorem (3.2) it follows that D is Jn-EP and S_1 is Jm-EP, $N(D) = N(D^{[*]}) \subseteq N(C^{[*]})$ and $N(S_1) = N(S_1^{[*]}) \subseteq N(B^{[*]})$. Finally, to prove that $N(A) = N(S_1)$ and $N(D) = N(S)$, it is enough to prove that $A^\dagger A = {S_1}^\dagger S_1$ and $D^\dagger D = S^\dagger S$. Since we have $N(A) \subseteq N(C), N(A^*) \subseteq N(B^*), N(S) \subseteq N(B)$ and $N(S^*) \subseteq N(C^*)$, according to the assumptions of Theorem 1 (v) of [2], we have

(3.5) $M^\dagger = \begin{bmatrix} A^\dagger + A^\dagger BS^\dagger CA^\dagger & -A^\dagger BS^\dagger \\ -S^\dagger CA^\dagger & S^\dagger \end{bmatrix}$ By using Lemma (2.1), for the conditions $N(A^*) \subseteq N(B^*)$ and $N(S^*) \subseteq N(C^*)$, we have $AA^\dagger B = B$ and $C = SS^\dagger C$. Hence, on computation $MM^\dagger$ reduces to the form $MM^\dagger = \begin{bmatrix} AA^\dagger & 0 \\ 0 & SS^\dagger \end{bmatrix}$. Beside $N(D) \subseteq N(B), N(D^*) \subseteq N(C^*), N(S_1) \subseteq N(C)$ and $N({S_1}^*) \subseteq N(B^*)$ and by corollary 1 of [2], $M^\dagger$ is also given by

(3.6) $M^\dagger = \begin{bmatrix} {S_1}^\dagger & -A^\dagger BS^\dagger \\ -D^\dagger C{S_1}^\dagger & S^\dagger \end{bmatrix}$. Again by using Lemma (2.1), on computation we have $MM^\dagger = \begin{bmatrix} S_1{S_1}^\dagger & 0 \\ 0 & SS^\dagger \end{bmatrix}$. On comparing the corresponding blocks of $MM^\dagger$ we get $AA^\dagger = S_1S_1^\dagger$. Since both A and S_1 are J_m-EP, by Theorem (2.1) (vi) $J_mAA^\dagger J_m = J_mS_1{S_1}^\dagger J_m$. Implies $A^\dagger A = {S_1}^\dagger S_1$ and hence, $N(A) = N(S_1)$. In the same manner, by using the formulae (3.5) and (3.6), we obtain two more expressions for $M^\dagger M$ and comparing the corresponding block yields $D^\dagger D = S^\dagger S$, hence $N(D) = N(S)$. Thus (ii) holds.
$(ii) \Rightarrow (i)$: $N(S) \subseteq N(B)$ follows directly from $N(S) = N(D) \subseteq N(B)$. Similarly, $N(S_1) \subseteq N(C)$ follows from $N(S_1) = N(A) \subseteq N(C)$. Since, A and S_1 are J_m-EP matrices, A is J_m-EP and S is J_n-EP, $N(A^{[*]}) \subseteq N(B^{[*]})$ and $N(S^{[*]}) \subseteq N(C^{[*]})$ with $N(A) \subseteq N(C)$, $N(S) \subseteq N(B)$ by Theorem (3.2) M is J-EP. Hence the Lemma holds.

Theorem 3.3: Let M be of the form (2.1) and J of the form (2.2). Let $S = D - CA^\dagger B$ be the Schur complement of A in M and $S_1 = A - BD^{\dagger}C$ be the Schur complement of D in M. If M is $J - EP_r$ with $N(A) \subseteq N(C)$, $N(D) \subseteq N(B), N(S) \subseteq N(B)$ and $N(S_1) \subseteq N(C)$, then the following hold:

(i) Principal sub-matrices A is J_m-EP and D is J_n-EP.
(ii) The Schur complements S is J_n-EP and S_1 is J_m-EP.
(iii) The principal pivot transform $\widehat{M}$ of M by pivoting the block A is J-EP and rank $\widehat{M} = r$.
(iv) The principal pivot transform $\widehat{M_1} of M_1$ by pivoting the block D is J_1EP where $J_1 = PJP^T$ and $M_1 = PMP^T$ for some permutation matrix P and rank $\widehat{M_1} = r$.

Proof: (i) and (ii) are consequences of Lemma 3.2.
(iii):The principal pivot transform $\widehat{M}$ is J-EP, whenever M is J-EP has been proved in Theorem(3.1). Here we prove the equality of ranks of M and its principal pivot $\widehat{M}$. The proof runs as follows:

$$\begin{aligned} rank(\widehat{M}) &= rank(A^{\dagger}) + rank(D).(By\ Remark\ (3.1)) \\ &= rank(A) + rank(S)(By\ using\ N(D) = N(S)) \\ &= rank(M).(By\ Remark\ (3.1)). \end{aligned}$$

(iv): Let $M_1 = \begin{bmatrix} D & C \\ B & A \end{bmatrix}$ and $J_1 = \begin{bmatrix} J_n & 0 \\ 0 & J_m \end{bmatrix}$. By the assumption $N(D) \subseteq N(B)$ holds . Since D is J_n-EP and M is J-EP, $N(D^*) = N(D^{[*]}) \subseteq N(C^{[*]}) = N(C^*)$ holds. Hence M_1 can be transformed into its principal pivot transform by pivoting the block D and $\widehat{M_1} = \begin{bmatrix} D^{\dagger} & -D^{\dagger}C \\ BD^{\dagger} & S_1 \end{bmatrix}$. where $S_1 = A - BD^{\dagger}C$ is the Schur complement of D in M. We claim that $\widehat{M_1}$ is J_1-EP. By proceeding as in the proof of Theorem (3.1) for the matrix J_1M_1, corresponding to(3.4)we get

(3.7) $\widehat{J_1M_1} = U_n\widehat{M_1}U_m = U_n\widehat{M_1}J_1U_n = U_mJ_1\widehat{M_1}U_m$, where $U_m = \begin{bmatrix} I_n & 0 \\ 0 & J_m \end{bmatrix}$. and $\begin{bmatrix} J_n & 0 \\ 0 & I_m \end{bmatrix}$. The rest of the proof runs as follows.

$$\begin{aligned} M is J - EP &\Leftrightarrow JM \text{ is EP (By Theorem (2.1)).} \\ &\Leftrightarrow PJMP^T \text{ is EP (By Lemma 3.1)} \\ &\Leftrightarrow (PJP^T)(PMP^T) \text{ is EP.} \\ &\Leftrightarrow J_1M_1 \text{ is EP.} \end{aligned}$$

Then, by Theorem 1 of [5], $\widehat{J_1M_1}$, is EP. Since U_m in (3.7) is invertible, by Lemma 3 .1, $J_1\widehat{M_1}$ is EP. Then $\widehat{M_1}$ is J_1-EP follows from Theorem(2.1) (ii). $rank(\widehat{M_1}) = rank(D^{\dagger}) + rank(\widehat{M_1}/D^{\dagger}) = rank(D) + rank(A) = rank(S) + rank(A) = rank(M)$, by using $N(D) = N(S)$.Thus (iv) holds. Hence the Theorem holds.

Remark 3.2: In particular if M is non-singular with A and D are non-singular, then the conditions $N(A) \subseteq N(C)$, $N(D) \subseteq N(B)$ automatically hold and by Remark (3.1), S and S_1 are non-singular, further $rank(\widehat{M}) = rank(A) + rank(D)$. Hence it follows that the principal pivot transform of a non-singular matrix is nonsingular. However , we note that the non-singularity of $\widehat{M}$ need not imply that M is non-singular. This is illustrated in the following Example.

Let us consider $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ with $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = C^* = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ Here, $S = D - CA^{\dagger}B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. $rank(M) = rank(A) + rank(S) = 3$. For $J = \begin{bmatrix} J_2 & 0 \\ 0 & J_2 \end{bmatrix}$, where $J_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. On computation we can see that the vector $[oxxx]^t \in N(MJ)$ and $[oxxx]^t \notin N(JM) = N(MJ)^*$. Hence MJ is not EP and by Theorem (2.1) M is not J-EP. Here A and D are non-singular. By Remark (3.1), $rank(\widehat{M}) = rank(D) + rank(A) = 2 + 2 = 4$. $\widehat{M}$ is non-singular. Hence, $\widehat{JM}$ is EP being nonsingular and by Theorem (2.1) (iv) $\widehat{M}$ is J-EP.

Conclusion: We have discussed the range symmetry of the principal pivot transform of a block matrix M,

in an indefinite inner product space with weight J. We have determined the relation between the principal pivot transforms of M, JM, MJ and P^TMP for some permutation matrix P.
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References

[1] T.S.Baskett and I.J.Katz, Theorems on product of EP_r matrices,Lin.Alg.Appln.2 (1969), 87-103.
[2] F.Burns, D.Carlson, Emilie Haynsworth and Thomas Markham,Generalized inverse formulas using the Schur complement, SIAM.J.Appl.Math. 26 (1974) 254-259.
[3] S.Jayaraman, EP matrices in indefinite inner product spaces, Funct.Anal. Approx. Comput. 4 (2) (2012), 23-31.
[4] AR.Meenakshi, On Schur complement in an EP matrix, Periodica Math. Hung.16(1985) 193-200.
[5] AR.Meenakshi, Principal pivot transform of an EP matrix,C.R.Math.Rep.Acad.Sci.2 (1986) 121-126.
[6] AR.Meenakshi,Range symmetric matrices in indefinite inner product space, Int.J.Fuzzy Math.Archive 5 # 2 (2015) 49-56.
[7] K.Ramanathan,K.Kamaraj and K.C.Sivakumar,Indefinite product of matrices and applications To indefinite inner product spaces,J.Anal.,12(2004)135-142.
[8] C.R.Rao and S.K.Mitra,Generalized Inverses of Matrices and its Applications, Wiley New York,1971.
[9] A.W.Tucker,Combinatorial Analysis (Bellman and Hall Eds.),Amer.Math.Soc. Providence RI.(1960),129-140.
[10] A.W.Tucker,Principal pivot transforms of square matrices,SIAM.Rev.5(1963)305.