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Some common fixed point theorems in metric spaces satisfying an implicit relation involving quadratic terms

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Abstract. In the present paper, we prove some new common fixed point theorems in metric spaces for weakly compatible mappings which satisfy an implicit relation involving quadratic terms. Our results generalize some earlier results of Imdad et al. [5]. Some illustrative examples are furnished to realize the improvements which are made in this paper.

1. Introduction

The study of nonlinear functional analysis, fixed point theorem is necessary due to its wide application to nonlinear science in many different fields of mathematics. In 1922, Banach [2] proved a theorem which ensures, under appropriate conditions, the existence and uniqueness of a fixed point. His result is known as Banach fixed point theorem or Banach contraction principle. This theorem provides a technique for solving variety of applied problems in mathematical sciences and engineering. Results of Banach [2] continue to be the source of inspiration for many researchers working in metric fixed point theory. Many authors have extended, generalized and improved Banach fixed point theorem in different ways.

In 1976 Jungck [6] generalized the Banach contraction principle by using the notion of commuting mappings. He also introduced idea of compatible mappings, which is more general than weakly commuting mappings due to Sessa [14]. In, 1998 Jungck and Rhoades [8] improved this results by coincidentally commuting (or weakly compatible mappings).

Recently, Popa [[12], [13]] introduced implicit functions which are proving fruitful due to their unifying power besides admitting new contraction conditions. Imdad et al. [5] proved some common fixed point theorems in metric spaces under a different set of conditions. Most recently, Savita et al. [3] proved the Generalization of a fixed point theorem of Suzuki type in complete convex space.

The object of this paper is to prove some common fixed point theorems in metric spaces for weakly compatible mapping satisfying an implicit relation involving quadratic terms. We also furnish some examples to justify our results. Our results generalize the result of Imdad et al. [5].

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2. Preliminaries

We recall some related definitions which will be needed in the sequel.
Definition 2.1. A pair (A, S) of self-mappings defined on a metric space (X, d) is said to be,

1. Compatible (see [7]) if $lim_{n\to\infty}d(ASx_n, SAx_n) = 0$ whenever x_n is a sequence in X such that $lim_{n\to\infty}Ax_n = lim_{n\to\infty}Sx_n = t$ for some $t \in$ X.

2. Non-compatible (see [10]) if there exists at least one sequence x_n in X, such that $lim_{n\to\infty}Ax_n = lim_{n\to\infty}Sx_n = t$ for some $t \in$ X, but $lim_{n\to\infty}d(ASx_n, SAx_n)$ is either non-zero or non-existent.

3. Weakly compatible (see [9]) if the mappings commute at their coincidence points, that is, $Ax = Sx$ for some $x \in$ X implies $ASx = SAx$.

4. Tangential (or satisfying the property (E.A)[1] if there exists a sequence x_n in X such that $lim_{n\to\infty}Ax_n = lim_{n\to\infty}Sx_n = t$ for some $t \in X$.

Definition 2.2. (see [4]) Two finite families of self-mappings $\{A_i\}_{i=1}^m$ and $\{S_k\}_{k=1}^n$ of a non-empty set X are said to be pairwise commuting if

1. $A_iA_j = A_jA_i; i, j \in \{1, 2, ...m\}$,

2. $S_kS_l = S_lS_k; k, l \in \{1, 2, ...p\}$,

3. $A_iS_k = S_kA_i; i \in \{1, 2, ...m\}$ *and* $k \in \{1, 2, ...p\}$.

3. Implicit relations

Imdad et al. [5] used the results of Popa ([11], [12]) to proved several fixed point theorems satisfying suitable implicit relations. For proving such results, he considered Ψ to be the set of all continuous functions $\psi(t_1, t_2, ..., t_6) : R_+^6 \to R$ satisfying the following conditions:
(ψ_1) is non-increasing in variables t_5 and t_6,
(ψ_2) *there exists* $k \in (0, 1)$ such that for $u, v \geq 0$ with
(ψ_{2a}) $\psi(u, v, v, u, u + v, 0) \leq 0$ or
(ψ_{2b}) $\psi(u, v, u, v; , 0, u + v) \leq 0$ implies $u \leq kv$,
(ψ_3) $\psi(u, u, 0, 0, u, u) > 0$, for all $u > 0$.

Now, we present some examples of such functions satisfying (ψ_1), (ψ_2) and (ψ_3).

Example 3.1. Define $\psi(t_1, t_2, ..., t_6) : R_+^6 \to R$ as:

(3.1) $\psi(t_1, t_2, ..., t_6) = t_1 - kmax\{t_2, t_3, t_4, \frac{1}{3}(t_5 + t_6)\}, k \in (0, 1)$.

(3.2) $\psi(t_1, t_2, ..., t_6) = t_1^2 - t_1(at_6 + bt_3 + ct_5) - d' t_2t_4$, where $a > 0, b, c, d \geq 0, a + b + c < 1$ *and* $a + d < 1$.

(3.3) $\psi(t_1, t_2, ..., t_6) = t_2^3 - at_2^2t_1 - bt_2t_3t_1 - ct_6^2t_5 - d' t_5t_6^2$, where $a, c, d \geq 0, b > 0, a + b < 1$ *and* $a + c + d' < 1$.

(3.4) $\psi(t_1, t_2, ..., t_6) = t_1^3 - c\frac{t_3^2t_5^2 + t_4^2t_6^2}{t_2 + t_3 + t_5 + 1}; c \in (0, 1)$.

(3.5) $\psi(t_1, t_2..., t_6) = t_2^2 - bt_1^2 - a\frac{t_5t_6}{t_3^2 + t_4^2 + 1}$; where $b > 0, a \geq 0$ and $a + b < 1$.

(3.6) $\psi(t_1, t_2, ..., t_6) = t_1^2 - a\, max\, \{t_2^2, t_3^2, t_4^2\} - b\, max\, \{t_3t_4, t_6t_5\} - ct_3t_4$, where $a, b > 0, c \geq 0, a + 2b < 1$ and $a + c < 1$.

We add some more examples to demonstrate how this implicit relation can cover several other known contractive conditions.

Example 3.2. Define $\psi(t_1, t_2, ..., t_6) : R^6_+ \to R$ as follows:

(3.7) $\psi(t_1, t_2, ..., t_6) = t_1 - a_1[a_2 max\{t_2, t_3, t_4, \frac{1}{3}(t_5+t_6)\} + (1-a_2)[max\{t_2^2;, t_3t_4, t_5t_6, \frac{t_3t_6}{3}, \frac{t_4t_5}{3}\}]^{1/2}]$, where $a_1 \in (0, 1)$ and $a_2 \in [0, 1]$.

Example 3.3. Define $\psi(t_1, t_2, ..., t_6) : R^6_+ \to R$ as follows:

(3.8) $\psi(t_1t_2, ..., t_6) = t_1^2 - a_1 max\{t_2^2, t_3^2, t_4^2\} - a_2 max\{\frac{t_3t_4}{3}, \frac{t_5t_6}{3}\} - a_3t_4t_6$, where $a_1, a_2, a_3 \geq 0$ and $a_1 + a_2 + a_3 < 1$.

Example 3.4. Define $\psi(t_1, t_2, ..., t_6) : R^6_+ \to R$ as:

(3.9) $\psi(t_1, t_2, ..., t_6) = t_2^2 - \phi(t_1^2, t_3t_5, t_4t_6, t_3t_4, t_4t_5)$; where $\phi : R^5_+ \to R^+$ is an upper semi-continuous and non-decreasing function in each coordinate variable such that $\phi(t, t, at, bt, ct) < t$ for each $t > 0$ and $a, b, c \geq 0$ with $a + b + c \leq 3$.

Imdad et al.[5] proved the following common fixed point theorem in metric spaces.

Theorem 3.5. Let A and S be two self-mappings of a metric space (X, d) such that
1. $\overline{A(X)} \subseteq S(X)$,
2. for all $x, y \in X$ and some $\psi \in \Psi$,
$\psi\Big(d(Ax, Ay), d(Sx, Sy), d(Sx, Ax), d(Sy, Ay), d(Sx, Ay), d(Sy, Ax)\Big) \leq 0$
3. $\overline{A(X)}$ is a complete subspace of X.
Then the pair (A, S) has a point of coincidence. Moreover, the mappings A and S have a unique common fixed point in X provided the pair (A, S) is weakly compatible.

4. Main Result

In this section, we prove a common fixed point theorem for quadruple of weakly compatible mappings satisfying an implicit relation involving quadratic terms.

Theorem 4.1. *Let A and S be two self-mappings of a metric space (X, d) such that*
1. $\overline{A(X)} \subseteq S(X)$,
2. For all $x, y \in X$ and some $\psi \in \Psi$,

$$(4.1)\ \psi\Big(d^2(Ax, Ay), d^2(Sx, Sy), d(Sx, Ax)d(Ax, Sy), d(Sy, Ay)d(Sy, Ax),$$
$$d(Sx, Ay)d(Sy, Ax), d^2(Sy, Ax)\Big) \leq 0,$$

3. $\overline{A(X)}$ is a complete subspace of X.
Then the pair (A, S) has a point of coincidence. Moreover, the mappings A and S have a unique common fixed point in X provided the pair (A, S) is weakly compatible.

Proof. Let x_0 be an arbitrary element in X. Then due to (1), $A(X) \subseteq \overline{A(X)} \subseteq S(X)$. Hence one can inductively define a sequence
(4.2) $\{Ax_0, Ax_1, Ax_2, ..., Ax_n, Ax_{n+1}, ...\}$,

such that $Ax_n = Sx_{n+1}$ for n = 0, 1, 2,..... Now, we show that the sequence defined by (4.2) is Cauchy. Using (4.1) with $x = x_n$ and $y = x_{n+1}$, we have

$$\psi\Big(d^2(Ax_n, Ax_{n+1}), d^2(Sx_n, Sx_{n+1}), d(Sx_n, Ax_n)d(Ax_n, Sx_{n+1}),$$
$$d(Sx_{n+1}, Ax_{n+1})d(Sx_{n+1}, Ax_n), d(Sx_n, Ax_{n+1})d(Sx_{n+1}, Ax_n), d^2(Sx_{n+1}, Ax_n)\Big) \leq 0.$$

As $Ax_n = Sx_{n+1}$ for $n = 0, 1, 2, ...$ we have

$$\psi\Big(d^2(Ax_n, Ax_{n+1}), d^2(Sx_n, Ax_n), d(Sx_n, Ax_n)d(Ax_n, Ax_n),$$
$$d(Ax_n, Ax_{n+1})d(Ax_n, Ax_n), d(Sx_n, Ax_{n+1})d(Ax_n, Ax_n), d^2(Ax_n, Ax_n)\Big) \leq 0.$$

Since ψ is non decreasing in variable t_5,
we have $\psi\Big(d^2(Ax_n, Ax_{n+1}), d^2(Sx_n, Ax_n), 0, 0, 0, 0\Big) \leq 0$.

Now, using the property (ψ_{2a}), we have $d(Ax_n, Ax_{n+1}) \leq kd(Sx_n, Ax_n) = kd(Ax_{n-1}, Ax_n)$
and so $d^2(Ax_n, Ax_{n+1}) \leq k^n d^2(Ax_0, Ax_1)$ for all $n \geq 0$. Hence by a simple calculation, it follows that $\{Ax_n\}$ is a Cauchy sequence. Since $\overline{A(X)}$ is a complete subspace of X, we have

$$lim_{n\to\infty}Ax_n = lim_{n\to\infty}Sx_{n+1} \in \overline{A(X)} \subseteq S(X) \subset X,$$

$$lim_{n\to\infty}Ax_n = lim_{n\to\infty}Sx_{n+1} = t \in S(X).$$

Hence there exists $u \in X$ such that $Su = t$. We claim that $Su = Au$. If not, then $d^2(Au, Su) > 0$. Using (4.1) with $x = u$ and $y = x_n$, we get

$$\psi\Big(d^2(Au, Ax_n), d^2(Su, Sx_n), d(Su, Au)d(Au, Sx_n),$$
$$d(Sx_n, Ax_n)d(Sx_n, Au), d(Su, Ax_n)d(Sx_n, Au), d^2(Sx_n, Au)\Big) \leq 0.$$

Taking limit as $n \to \infty$ we get
$\psi\Big(d^2(Au, t), d^2(Su, t), d(Su, Au)d(Au, t), d(t, t)d(t, Au), d(Su, t)d(t, Au), d^2(t, Au)\Big) \leq 0,$
or, $\psi\Big(d^2(Au, Su), 0, d(Su, Au)d(Au, Su), 0, 0, d^2(Su, Au)\Big) \leq 0,$
or, $\psi\Big(d^2(Au, Su), 0, d^2(Su, Au), 0, 0, d^2(Su, Au)\Big) \leq 0,$

yielding thereby (due to (ψ_{2b}))$d^2(Au, Su) \leq 0$ which is a contradiction. Then we have $Au = Su$, which shows that u is a coincidence point of A and S. Since the pair (A, S) is weakly compatible, we have $St = SAu = ASu = At$. Now, we show that t is a common fixed point of mappings A and S. We assert that $At = t$. If not then $d(At, t) > 0$. Again using (4.1) with $x = t$ and $y = u$, we have

$$\psi\Big(d^2(At, Au), d^2(St, Su), d(St, At)d(At, Su),$$
$$d(Su, Au)d(Su, At), d(St, Au)d(Su, At), d^2(Su, At)\Big) \leq 0,$$
or
$$\psi\Big(d^2(At, t), d^2(At, t), d(At, At)d(At, t), d(t, t)d(t, At), d(At, t)d(t, At), d^2(t, At)\Big) \leq 0,$$
or,
$$\psi\Big(d^2(At, t), d^2(At, t), 0, 0, d^2(t, At), d^2(t, At)\Big) \leq 0,$$
which contradicts (ψ_3). Hence $At = t$ or $St = At = t$. This shows that t is a common fixed point of A and S. The uniqueness of common fixed point is an easy consequence of implicit relation (4.1) in view of (ψ_3). This completes the proof.

Remark 4.2. Theorem 4.1 is a generalized and improved form of Theorem 3.5 due to Imdad et al. [5].
From Theorem 4.1, we can deduce the following corollaries:

Corollary 4.3. *The conclusions of Theorem 4.1 remain true if for all $x, y \in X$ ($x \neq y$) the implicit relation (4.1) is replaced by one of the following:*

(4.1) $$d^2(Ax,Ay) \leq k\, max\{d^2(Sx,Sy), d(Sx,Ax)d(Ax,Sy), d(Sy,Ay)d(Sy,Ax), \tfrac{1}{2}[d(Sx,Ay)d(Sy,Ax) + d^2(Sy,Ax)]\} \quad k \in [0,1].$$

(4.2) $$d^2(Ax,Ay) \leq k\, max\{d^2(Sx,Sy), d(Sx,Ax)d(Ax,Sy), d(Sy,Ay)d(Sy,Ax), \tfrac{1}{2}d(Sx,Ay)d(Sy,Ax), \tfrac{1}{2}d^2(Sy,Ax)\} \quad k \in [0,1].$$

(4.3) $$d^2(Ax,Ay) \leq k\, max\{d^2(Sx,Sy), \tfrac{1}{2}[d(Sx,Ax)d(Ax,Sy) + d(Sy,Ay)d(Sy,Ax)], \tfrac{1}{2}[d(Sx,Ay)d(Sy,Ax) + d^2(Sy,Ax)]\} \quad k \in [0,1].$$

(4.4) $$d^2(Ax,Ay) \leq a\, d^2(Sx,Sy) + b\, d(Sx,Ax)d(Ax,Sy) + c\, d(Sy,Ay)d(Sy,Ax) + d' d(Sx,Ay)d(Sy,Ax) + e\, d^2(Sy,Ax),$$
where $a + b + c + d' + e < 1$ and $d', e \geq 0$.

(4.5) $$d^2(Ax,Ay) \leq \frac{k}{2} max\{d^2(Sx,Sy), d(Sx,Ax)d(Ax,Sy), d(Sy,Ay)d(Sy,Ax), d(Sx,Ay)d(Sy,Ax), d^2(Sy,Ax)\} \quad k \in [0,1].$$

(4.6) $$d^2(Ax,Ay) \leq ad^2(Sx,Sy) + bd(Sx,Ax)d(Ax,Sy), cd(Sy,Ay)d(Sy,Ax) + d'[d(Sx,Ay)d(Sy,Ax) + d^2(Sy,Ax)]$$
where $a + b + c + 2d' < 1$ and $d' \geq 0$.

(4.7) $$d^2(Ax,Ay) \leq a_1 d^2(Sx,Sy) + \frac{a_2 d(Sx,Ax)d(Ax,Sy)d(Sy,Ay)d(Sy,Ax) + a_3 d(Sx,Ay)d(Sy,Ax)d^2(Sy,Ax)}{d(Sx,Ax)d(Ax,Sy) + d(Sy,Ay)d(Sy,Ax)}.$$

where $a_1, a_2, a_3 \geq 0$ such that $1 < 2a_1 + a_2 < 2$

(4.8) $$d^2(Ax,Ay) \leq \phi\Big(d^2(Sx,Sy), d(Sx,Ax)d(Ax,Sy), d(Sy,Ay)d(Sy,Ax), d(Sx,Ay)d(Sy,Ax), d^2(Sy,Ax)\Big),$$

where $\phi : R^5_+ \to R^+$ is an upper semi-continuous and non-decreasing function in each coordinate variable such that $\phi(t,t,at,bt,ct) < t$ for each $t > 0$ and $a, b, c \geq 0$ with $a + b + c \leq 3$.

Corollary 4.4. *Let A be a self-mappings of a metric space (X,d) such that*
1. for all $x, y \in X$ and some $\psi \in \Psi$,

(4.9) $$\psi\Big(d^2(Ax,Ay), d^2(x,y), d(x,Ax)d(Ax,y), d(y,Ay)d(y,Ax), d(x,Ay)d(y,Ax), d^2(y,Ax)\Big) \leq 0,$$

2. $\overline{A(X)}$ is a complete subspace of X.
Then A has a unique fixed point in X.

5. A common fixed point theorem for finite families of self-mappings

As an application of Theorem 4.1, we prove a common fixed point theorem for two finite families of mappings as follows:

Theorem 5.1. *Let $\{A_1, A_2, ..., A_m\}$ and $\{S_1, S_2, ..., S_p\}$ be two finite families of self-mappings of a metric space (X,d) with $A = A_1A_2....A_m$ and $S = S_1S_2....S_p$ satisfying condition (4.1) of Theorem 4.1. Suppose that $\overline{A(X)} \subseteq S(X)$, wherein $\overline{A(X)}$ is a complete subspace of X. Then (A,S) has a point of coincidence.*

Moreover, if $A_iA_j = A_jA_i$, $S_kS_l = S_lS_k$ and $A_iS_k = S_kA_i$ for all $i, j \in I_1 = \{1, 2, ...,m\}$ and $k, l \in I_2 = \{1, 2, ..., p\}$, then (for all $i \in I_1$ and $k \in I_2$) A_i and S_k have a common fixed point in X.

Proof. The conclusion is immediate as A and S satisfy all the conditions of Theorem 4.1. Now appealing to componentwise commutativity of various pairs, one can immediately assert that $AS = SA$ and hence, obviously the pair (A, S) is weakly compatible. Note that all the conditions of Theorem 4.1 (for mappings A and S) are satisfied ensuring the existence of unique common fixed point, say t. Now we show that t remains the fixed point of all the component mappings. For this consider

$$\begin{aligned} A(A_it) &= ((A_1A_2...A_m)A_i)t = (A_1A_2....A_{m-1})((A_mA_i)t) \\ &= (A_1...A_{m-1})(A_iA_mt) \\ &. \\ &. \\ &. \\ &= A_1A_i(A_2A_3A_4....A_mt) \\ &= A_iA_1(A_2A_3....A_mt) = Ai_(A_t) = A_it. \end{aligned}$$

Similarly, one can show that, $A(S_kt) = S_k(At) = S_kt, S(S_kt) = S_k(St) = S_kt$, and $S(A_it) = A_i(St) = A_it$, which shows that (for all i and k) A_it and S_kt are other fixed points of the pair (A, S). Now appealing to the uniqueness of common fixed points of the pair separately, we get

$t = A_it = S_kt$. Which shows that t is a common fixed point of A_i and S_k for all i and k.

By setting $A_1 = A_2 = ... = A_m = A$ and $S_1 = S_2 = ... = S_p = S$ in Theorem 5.1, we deduce the following corollary.

Corollary 5.2. *Let A and S be self-mappings of a metric space (X, d) satisfying inequality (4.1) of Theorem 4.1 for all distinct $x, y \in X$. If $\overline{A^m(X)} \subseteq S^p(X)$, then A and S have a unique common fixed point in X provided $AS = SA$.*

6. Illustrative examples

Now we furnish an example to demonstrate the validity of Theorem 4.1.

Example 6.1. Consider X = $[3, 13]$ with usual metric. Define self-mappings A and S on X as

$$Ax = \begin{cases} 3, & if \quad x \in \{3\} \bigcup (7, 13]; \\ 6, & if \quad 3 < x \le 7; \end{cases}$$

$$Sx = \begin{cases} 3, & if \quad x = 3; \\ 8, & if \quad 3 < x \le 7; \\ x - 4, & if \quad x > 7. \end{cases}$$

We can see that the mappings A and S commute at 3 which is their coincidence point. Also $A(X) = \{3, 6\}$ and $S(X) = [3, 9]$. Clearly, $\overline{A(X)} = \{3, 6\} \subset [3, 9] = S(X)$.

Now define $\psi(t_1, t_2, ..., t_6) : R^6_+ \to R$ as,

$$\psi(t_1, t_2, ..., t_6) = t_1 - at_2 - bt_3 - ct_4 - d' t_5 - et_6,$$

where $a + b + c + d' + e < 1$ and $d', e \ge 0$

By a simple calculation one can verify that contraction condition (4.1) is satisfied for $a = \frac{1}{5}$ and $b = c = \frac{1}{4}$ and $d' = e = \frac{1}{8}$. If $x, y \in \{3\} \bigcup [(7, 13]$, then $d(Ax, Ay) = 0$ and verification is trivial. If $x \in (3, 6]$ and $y > 6$, then from Corollary 4.4

$$
\begin{aligned}
& a\, d^2(Sx, Sy) + b\, d(Sx, Ax)d(Ax, Sy) + c\, d(Sy, Ay)d(Sy, Ax) \\
& + d' d(Sx, Ay)d(Sy, Ax) + e\, d^2(Sy, Ax) \\
= \;& \frac{1}{5}|y-12|^2 + \frac{1}{4}[2\,|y-10| + |y-7||y-10|] \\
& + \frac{1}{8}[5|y-10| + |y-10|^2] \\
& \geq \begin{cases} 3, & if \quad y \in (3,6]; \\ 3, & if \quad y > 6. \end{cases}
\end{aligned}
$$

Similarly, one can verify the other cases. Thus all the conditions of Theorem 4.1 are satisfied and 3 is the unique common fixed point of the mappings A and S, which is their coincidence point also.

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