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Spectral picture, perturbed Browder and Weyl theorems, and their variations

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Abstract. The holes (i.e., the union of the bounded components of the complement in the complex plane), alongwith the isolated points, of the Weyl and the approximate Weyl spectrum (and their B-Fredholm avatars) play a decisive role in determining Browder and Weyl theorems type properties for Banach space operators and their perturbations.

Dedicated to Marigold Maisie Duggal on her birthday

1. Introduction

Let $B(\mathcal{X})$ (resp., $B(\mathcal{H})$) denote the algebra of operators, equivalently bounded linear transformations, on a complex infinite dimensional Banach space $\mathcal{X}$ (resp., Hilbert space $\mathcal{H}$) into itself. Given $A \in B(\mathcal{X})$, let $\sigma(A)$, $\sigma_a(A)$, $\sigma_w(A)$ and $\sigma_{aw}(A)$ and $\sigma_{ab}(A)$ denote, respectively, the spectrum, the approximate point spectrum, the Weyl spectrum and the approximate Weyl spectrum of A; let $\Pi_0(A)$, $\Pi_0^a(A)$, $E_0(A)$ and $E_0^a(A)$ denote, respectively, the set of finite rank poles (of the resolvent) of A, the set of finite rank left poles of A, the set of finite multiplicity eigenvalues which are isolated points of $\sigma(A)$ and the set of finite multiplicity eigenvalues which are isolated points of $\sigma_a(A)$. Following current terminology [1], we say that $A \in B(\mathcal{X})$ satisfies *Browder's theorem* (*a-Browder's theorem*) , $A \in (Bt)$ (resp., $A \in (a-Bt)$), if $\sigma(A) \setminus \sigma_w(A) = \Pi_0(A)$ (resp., if $\sigma_a(A) \setminus \sigma_{aw}(A) = \Pi_0^a(A)$), and A satisfies *Weyl's theorem* (*a-Weyl's theorem*), $A \in (Wt)$ (resp.,$A \in (a-Wt)$), if $\sigma(A) \setminus \sigma_w(A) = E_0(A)$ (resp., $\sigma_a(A) \setminus \sigma_{aw}(A) = E_0^a(A)$). Browder and Weyl type theorems have been considered in the recent past by a number of authors and there exists in extant literature a large body of information on Browder and Weyl thoerems, their generalized extensions and their variations (see [1, 2, 5, 6, 9–22, 24] for further references).

If we let $\eta\sigma_w(A)$ denote the connected hull of $\sigma_w(A)$, $\eta\sigma_w(A)^C$ the unbounded component of the complement of $\eta\sigma_w(A)$ in the complex plane $\mathbb{C}$ and $\eta'\sigma_w(A)$ the union of the holes of $\sigma_w(A)$, then $A \in (Bt)$ if and only if A has SVEP, *the single-valued extension property*, on $\eta\sigma_w(A)^C \cup \eta'\sigma_w(A)$. Similarly, $A \in (a-Bt)$ if and only if

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A has SVEP on $\eta\sigma_{aw}(A)^C \cup \eta'\sigma_{aw}(A)$, $A \in (Wt)$ if and only if $A \in (Bt)$ and $E_0(A) \cap \{\eta\sigma_w(A)^C \cup \eta'\sigma_w(A)\} = \Pi_0(A)$ and $A \in (a - Wt)$ if and only if $A \in (a - Bt)$ and $E_0^a(A) \cap \{\eta\sigma_{aw}(A)^C \cup \eta'\sigma_{aw}(A)\} = \Pi_0^a(A)$.

We show in the following that, when proving Browder and Weyl theorem type results, the action takes place on the holes and the isolated points of the Weyl spectrum. Thus, $A \in (Bt)$ if and only if A has SVEP on $\sigma(A) \cap \eta'\sigma_w(A)$ (resp., $A \in (a - Bt)$ if and only if A has SVEP on $\sigma_a(A) \cap \eta'\sigma_{aw}(A)$), and $A \in (Wt)$ if and only if $A \in (Bt)$ and $E_0(A) \cap \sigma_w(A) = \emptyset$ (resp., $A \in (a - Wt)$ if and only if $A \in (a - Bt)$ and $E_0^a(A) \cap \sigma_{aw}(A) = \emptyset$). Similar assertions hold for the generalized versions, i.e. the B-Browder and B-Weyl versions [12], of these results. It is seen that A satisfies property (b) [7, 9], $\sigma_a(A) \setminus \sigma_{aw}(A) = \Pi_0(A)$, if and only if A^* has SVEP on $\sigma_a(A) \cap \eta'\sigma_{aw}(A)$ (resp., A satisfies property (w) [2, 5], $\sigma_a(A) \setminus \sigma_{aw}(A) = E_0(A)$, if and only if A satisfies property (b) and $E_0(A) \cap \sigma_{aw}(A) = \emptyset$); A satisfies property (ab) [9], $\sigma(A) \setminus \sigma_w(A) = \Pi_0^a(A)$, if and only if A^* has SVEP on $\mathrm{iso}\sigma_a(A) \cap \eta'\sigma_{aw}(A)$. Perturbation by commuting Riesz operators preserves SVEP at points [8]. This implies that Browder's theorem type results, including property (ab), survive perturbation by commuting Riesz operators, but this does not extend to Weyl's theorem type results. A typical result here goes as follows: If R is a Riesz operator which commutes with A, then A satisfies property (w) (A satisfies property (b) implies $A + R$ satisfies property (b)) if and only if A^* has SVEP on $\sigma_a(A) \cap \eta'\sigma_{aw}(A)$ and $E_0(A) \cap \sigma_{aw}(A) = \emptyset$ (resp., A^* has SVEP on $(\sigma_a(A + R) \setminus \sigma_a(A)) \cap \eta'\sigma_{aw}(A)$ and $E_0(A + R) \cap \sigma_{aw}(A + R) = \emptyset$). We apply these results, and their generalized versions, to perturbation by commuting (Riesz operators which are) nilpotent, quasinilpotent and finite rank operators. SVEP does not survive perturbation by non-commuting compact operators. Given a compact operator K, it is seen that: A satisfies property (b) implies $A + K$ satisfies property (b) (A satisfies property (w) implies $A + K$ satisfies property (w)) if and only if $\sigma_a(A + K) \cap \eta'\sigma_{aw}(A) \subseteq \mathrm{iso}\sigma(A + K)$ and $\mathrm{iso}\sigma_a(A + K) \setminus \sigma_a(A) \subseteq \mathrm{iso}\sigma(A + K)$ (resp., if and only if $A + K$ satisfies property (b) and $\mathrm{iso}\sigma_w(A) = \emptyset$); if the complement of $\sigma_{aw}(A)$ in $\mathbb{C}$ is connected, then $A \in (ab)$ implies $A + K \in (ab)$ for all compact operators K.

2. Notation and terminology

In keeping with standard terminology, we shall denote the spectrum, the approximate point spectrum, the surjectivity spectrum and the isolated points of the spectrum of an operator $A \in B(\mathcal{X})$ by $\sigma(A)$, $\sigma_a(A)$, $\sigma_s(A)$ and isoσ(A), respectively. The boundary of a subset S of the set $\mathbb{C}$ of complex numbers will be denoted by ∂S, the interior of S will denoted by int(S) and we shall write S^C for the complement of S in $\mathbb{C}$. We shall denote the open unit disc by $\mathcal{D}$. An operator $A \in B(\mathcal{X})$ has SVEP, *the single-valued extension property*, at a point $\lambda_0 \in \mathbb{C}$ if for every open disc $\mathcal{D}_{\lambda_0}$ centered at λ_0 the only analytic function $f : \mathcal{D}_{\lambda_0} \longrightarrow \mathcal{X}$ satisfying $(A - \lambda)f(\lambda) = 0$ is the function $f \equiv 0$. (Here, and in the sequel, we have shortened $A - \lambda I$ to $A - \lambda$.) Evidently, every A has SVEP at points in the resolvent $\rho(A) = \mathbb{C} \setminus \sigma(A)$ and the boundary $\partial\sigma(A)$ of the spectrum $\sigma(A)$. *We say that T has SVEP if it has SVEP at every $\lambda \in \mathbb{C}$.* The *ascent of A*, asc(A) (resp. *descent of A*, dsc(A)), is the least non-negative integer n such that $A^{-n}(0) = A^{-(n+1)}(0)$ (resp., $A^n(\mathcal{X}) = A^{n+1}(\mathcal{X})$): If no such integer exists, then asc(A) (resp. dsc(A))$= \infty$. It is well known, see [1, 25, 27], that asc$(A) < \infty$ implies A has SVEP at 0, dsc$(A) < \infty$ implies A^* (= the dual operator) has SVEP at 0, finite ascent and descent for an operator implies their equality, and that a point $\lambda \in \sigma(A)$ is a pole (of the resolvent) of A if and only if $\mathrm{asc}(A - \lambda) = \mathrm{dsc}(A - \lambda) < \infty$.

An operator $A \in B(\mathcal{X})$ is: *upper semi–Fredholm at $\lambda \in \mathbb{C}$, $\lambda \in \Phi_{usf}(A)$ or $A - \lambda \in \Phi_+(\mathcal{X})$*, if $A(\mathcal{X})$ is closed and the deficiency index $\alpha(A - \lambda) = \dim(A - \lambda)^{-1}(0) < \infty$; *lower semi–Fredholm at $\lambda \in \mathbb{C}$, $\lambda \in \Phi_{lsf}(A)$ or $A - \lambda \in \Phi_-(\mathcal{X})$*, if $\beta(A - \lambda) = \dim(\mathcal{X}/(A - \lambda)(\mathcal{X})) < \infty$; A is semi–Fredholm, $\lambda \in \Phi_\pm(A)$ or $A - \lambda \in \Phi_\pm(\mathcal{X})$, if $A - \lambda$ is either upper or lower semi–Fredholm, and A is Fredholm, $\lambda \in \Phi(A)$ or $A - \lambda \in \Phi(\mathcal{X})$, if $A - \lambda$ is both upper and lower semi–Fredholm. The index of a semi–Fredholm operator is the integer $\mathrm{ind}(A) = \alpha(A) - \beta(A)$. Corresponding to these classes of one sided Fredholm operators, we have the following spectra: The *upper semi Fredholm spectrum* $\sigma_{usf}(A) = \{\lambda \in \sigma(A) : A - \lambda \notin \Phi_+(\mathcal{X})\}$, the *lower semi Fredholm spectrum* $\sigma_{lsf}(A) = \{\lambda \in \sigma(A) : A - \lambda \notin \Phi_-(\mathcal{X})\}$ and the *Fredholm spectrum* $\sigma_e(A) = \sigma_{usf}(A) \cup \sigma_{lsf}(A)$. $A \in B(\mathcal{X})$ is upper Weyl (resp., lower Weyl, (simply) Weyl) at 0 if it is upper semi Fredholm with $\mathrm{ind}(A) \leq 0$ (resp., lower semi Fredholm with $\mathrm{ind}(A) \geq 0$, Fredholm with $\mathrm{ind}(A) = 0$). The upper (or, approximate) Weyl spectrum, the lower (or, surjectivity) Weyl spectrum and the Weyl spectrum of A are respectively the sets $\sigma_{aw}(A) = \{\lambda \in \sigma_a(A) : \lambda \notin \Phi_+(A) \text{ or } \operatorname{ind}(A-\lambda) \nleq 0\}$, $\sigma_{sw}(A) = \{\lambda \in \sigma_s(A) : \lambda \notin \Phi_-(A) \text{ or } \operatorname{ind}(A-\lambda) \ngeq 0\}$ and $\sigma_w(A) = \sigma_{aw}(A) \cup \sigma_{sw}(A)$. It is well known, [1, Theorems 3.16, 3.17], that a semi- Fredholm operator A (resp., its conjugate operator A^*) has SVEP at a point λ if and only if $\operatorname{asc}(A-\lambda) < \infty$ (resp., $\operatorname{dsc}(A-\lambda) < \infty$) ; furthermore, if $A-\lambda$ is Weyl (resp., upper Weyl), i.e. if $\lambda \in \Phi(A)$ and $\operatorname{ind}(A-\lambda) = 0$ (resp., $\lambda \in \Phi_+(A)$ and $\operatorname{ind}(A-\lambda) \leq 0$), then A has SVEP at λ implies $\lambda \in \operatorname{iso}\sigma(\mathrm{A})$ with $\operatorname{asc}(A-\lambda) = \operatorname{dsc}(A-\lambda) < \infty$ (resp., $\lambda \in \operatorname{iso}\sigma_{\mathrm{a}}(\mathrm{A})$ with $\operatorname{asc}(A-\lambda) < \infty$). If we let $\sigma_{ab}(A) = \{\lambda \in \sigma_a(A) : \lambda \notin \Phi_+(A) \text{ or } \operatorname{asc}(\mathrm{A}-\lambda) \nless \infty\}$ and $\sigma_{sb}(A) = \{\lambda \in \sigma_s(A) : \lambda \notin \Phi_-(A) \text{ or } \operatorname{des}(\mathrm{A}-\lambda) \nless \infty\}$ denote, respectively, the upper (or approximate) and the lower (or surjectivity) Browder spectrum of A, then $\sigma_{sb}(A) = \sigma_{ab}(A^*)$ and $\sigma_b(A) = \sigma_{ab}(A) \cup \sigma_{sb}(A)$ is the *Browder spectrum* of A. (See [1, 25–27, 32] for further information on Fredholm theory, SVEP, and isolated points.)

A generalization of semi Fredholm and Fredholm operators is obtained as follows. We say that the operator $A \in B(\mathcal{X})$ is semi B-Fredholm, $A \in \Phi_{sBf}(\mathcal{X})$, if there exists an integer $n \geq 1$ such that $A^n(\mathcal{X})$ is closed and the induced operator $A_{[n]} = A|_{A^n(\mathcal{X})}$, $A_{[0]} = A$, is semi Fredholm (in the usual sense). It is seen that if $A_{[n]} \in \Phi_\pm(\mathcal{X})$ for an integer $n \geq 1$, then $A_{[m]} \in \Phi_\pm(\mathcal{X})$ for all integers $m \geq n$, and one may (unambiguously) define the index of A by $\operatorname{ind}(A) = \alpha(A) - \beta(A)$ $(= \operatorname{ind}(A_{[n]}))$ [12]. Upper semi B-Fredholm, lower semi B-Fredholm and B-Fredholm spectra of A are then the sets

$\sigma_{uBf}(A) = \{\lambda \in \sigma(A) : A-\lambda$ is not upper semi B-Fredholm$\}$,

$\sigma_{lBf}(A) = \{\lambda \in \sigma(A) : A-\lambda$ is not lower semi B-Fredholm$\}$, and

$\sigma_{Be}(A) = \sigma_{uBf}(A) \cup \sigma_{lBf}(A)$.

If we let

$$\begin{aligned}
&\sigma_{Bw}(A) = \{\lambda \in \sigma(A) : \lambda \in \sigma_{Be}(A) \text{ or } \operatorname{ind}(A-\lambda) \neq 0\},\\
&\sigma_{aBw}(A) = \{\lambda \in \sigma_a(A) : \lambda \in \sigma_{uBf}(A) \text{ or } \operatorname{ind}(A-\lambda) \nleq 0\},\\
&\sigma_{sBw}(A) = \{\lambda \in \sigma_s(A) : \lambda \in \sigma_{lBf}(A) \text{ or } \operatorname{ind}(A-\lambda) \ngeq 0\},\\
&\sigma_{Bb}(A) = \{\lambda \in \sigma(A) : \lambda \in \sigma_{Be}(A) \text{ or } \operatorname{asc}(A-\lambda) \neq \operatorname{dsc}(A-\lambda)\},\\
&\sigma_{aBb}(A) = \{\lambda \in \sigma_a(A) : \lambda \in \sigma_{uBf}(A) \text{ or } \operatorname{asc}(A-\lambda) = \infty\}, \text{ and}\\
&\sigma_{sBb}(A) = \{\lambda \in \sigma_s(A) : \lambda \in \sigma_{lBf}(A) \text{ or } \operatorname{dsc}(A-\lambda) = \infty\}
\end{aligned}$$

denote, respectively, the *the B-Weyl, the upper B-Weyl, the lower B-Weyl, the B-Browder, the upper B-Browder* and *the lower B-Browder spectrum* of A, then $\sigma_{Bw}(A) = \sigma_{aBw}(A) \cup \sigma_{sBw}(A)$, $\sigma_{Bb}(A) = \sigma_{aBb}(A) \cup \sigma_{sBb}(A)$, $\sigma_{aBw}(A) = \sigma_{sBw}(A^*)$ and $\sigma_{aBb}(A) = \sigma_{sBb}(A^*)$. The following implications are well known [10, Theorems 2.1 and 2.2]:

$\sigma_w(A) = \sigma_b(A) \Longleftrightarrow \sigma_{Bw}(A) = \sigma_{Bb}(A) \Longleftrightarrow \sigma(A) \setminus \sigma_{Bw}(A) = \Pi(A) \Longleftrightarrow A$ has SVEP at points in $\sigma(A) \setminus \sigma_{Bw}(A)$, and

$\sigma_{aw}(A) = \sigma_{ab}(A) \Longleftrightarrow \sigma_{aBw}(A) = \sigma_{aBb}(A) \Longleftrightarrow \sigma_a(A) \setminus \sigma_{aBw}(A) = \Pi^a(A) \Longleftrightarrow A$ has SVEP at points in $\sigma_a(A) \setminus \sigma_{aBw}(A)$.

Evidently, $\sigma_{aw}(A) \subseteq \sigma_w(A)$ and $\sigma_{aBw}(A) \subseteq \sigma_{Bw}(A)$; hence

$\sigma_{aBw}(A) = \sigma_{aBb}(A) \Longleftrightarrow \sigma_{aw}(A) = \sigma_{ab}(A) \Longrightarrow \sigma_w(A) = \sigma_b(A) \Longleftrightarrow \sigma_{Bw}(A) = \sigma_{Bb}(A)$

(where the one way implications are strict). In keeping with current terminology [10, 12, 17], we say in the following that an operator

A satisfies generalized Browder's theorem, or $A \in (gBt)$, if $\sigma_{Bw}(A) = \sigma_{Bb}(A)$;

A satisfies generalized a-Browder's theorem, or $A \in (a-gBt)$, if $\sigma_{aBw}(A) = \sigma_{aBb}(A)$.

A *hole* of a compact subset of $\mathbb{C}$ (more generally, of a subset of a topological space) is any bounded component of its complement [26]. (Thus, a hole of $\sigma_w(A)$, respectively $\sigma_{aw}(A)$, for an operator A is a bounded maximal connected subset of $\mathbb{C} \setminus \sigma_w(A)$, respectively $\mathbb{C} \setminus \sigma_{aw}(A)$).The *connected hull* ηS of a compact subset S of $\mathbb{C}$ is the complement of the unique unbounded component of the complement S^C of S in $\mathbb{C}$. It is clear that for every compact subset S of $\mathbb{C}$,

$$\eta' S = \eta S \setminus S = \cup Hole(S)$$

(i.e., $\eta' S$ is the union of the holes, equivalently bounded components of $\mathbb{C} \setminus S$, of S). If E, F are compact subsets of $\mathbb{C}$, then $\partial E \subset F \subset E$ implies $\eta E = \eta F$ (and E can be obtained from F by filling in some of its holes). If F has no holes, then ($F = \eta F$, and hence) $\partial E \subset F \subset E \Longrightarrow E \supset F = \eta F = \eta E \supset E \Longrightarrow E = F$. If either of A and A^* has SVEP at a point $\lambda \notin \sigma_w(A)$ for an operator $A \in B(\mathcal{X})$, then $\lambda \notin \sigma_b(A)$; equivalently, $\lambda \in \Pi_0(A)$ [1, Corollary 3.53]. It is evident that if $\Pi_0(A) = \sigma(A) \cap \sigma_w(A)^C$, then $\mathrm{int}\{\sigma(\mathrm{A}) \cap \sigma_{\mathrm{w}}(\mathrm{A})^C\} = \emptyset$ (i.e., $\sigma(A) \cap \sigma_w(A)^C$ has empty interior), A has SVEP on $\sigma_w(A)^C$ and $\sigma_w(A) = \sigma_b(A)$. Since $\partial\sigma_b(A) \subseteq \sigma_w(A) \subseteq \sigma_b(A)$ for every operator $A \in B(\mathcal{X})$, $\eta\sigma_w(A) = \eta\sigma_b(A)$ for every operator $A \in B(\mathcal{X})$. If $K \in B(\mathcal{X})$ is a compact operator and if $\eta'\sigma_w(A) = \emptyset$, then (the argument above implies)

$$\begin{aligned} \sigma_w(A+K) = \sigma_w(A) \quad &\Longrightarrow \quad \sigma_w(A+K) = \eta\sigma_w(A+K) = \eta\sigma_b(A+K) \\ &\Longrightarrow \quad \sigma_w(A+K) = \sigma_b(A+K), \end{aligned}$$

i.e., *given an operator $A \in B(\mathcal{X})$, a sufficient condition for $A + K$ to have SVEP on $\sigma_w(A)^C$ ($= \mathbb{C} \setminus \sigma_w(A)$) for every compact operator $K \in B(\mathcal{X})$ is that $\eta'\sigma_w(A) = \emptyset$.*

3. Some complementary results: Polaroid operators

An operator $A \in B(\mathcal{X})$ is said to be *polaroid* (*finitely polaroid*) if the isolated points of the spectrum of A are poles (of the resolvent) of A (resp., finite rank poles of A) ; A is *left polaroid* (*finitely left polaroid*) if isolated points of the approximate point spectrum of A are left poles of A (resp., finite rank left poles of A). Given $A \in B(\mathcal{X})$, it is clear that

$$\Pi_0(A) \subseteq \Pi_0^a(A) \subseteq \Pi^a(A), \Pi_0(A) \subseteq \Pi(A) \subseteq \Pi^a(A),$$

where the reverse inclusions generally fail. $A \in B(\mathcal{X})$ is *isoloid* (*finitely isoloid*) if points $\lambda \in \mathrm{iso}\sigma(A)$ are eigenvalues (resp., finite multiplicity eigenevalues) of A; A is *a-isoloid* (*finitely a-isoloid*) if points $\lambda \in \mathrm{iso}\sigma_a(A)$ are eigenvalues (resp., finite multiplicity eigenvalues) of A. It is clear that A is polaroid implies A is isoloid and A is left polaroid implies A is a-isoloid (where the reverse implications are, in general, false).

The left polaroid and polaroid properties do not survive perturbation by commuting Riesz operators: The 0 operator is polaroid but its perturbation $A = 0 + R$ by the non-nilpotent quasinilpotent operator $R(x_1, x_2, x_3, ...) = (\frac{x_2}{2}, \frac{x_3}{3}, ...)$ is neither left polaroid nor polaroid. However:

Proposition 3.1. *If a Riesz operator $R \in B(\mathcal{X})$ is such that $[A, R] = 0$ and $\mathrm{iso}\sigma_a(A + R) \subset \mathrm{iso}\sigma_a(A)$ for an operator $A \in B(\mathcal{X})$, then A is finitely left polaroid implies $A + R$ is finitely left polaroid.*

Proof. Since perturbation by Riesz operators preserves $\sigma_{aw}(.)$ [20, 31], $\sigma_{aw}(A + R) = \sigma_{aw}(A)$. Hence, if A is finitely left polaroid, then $\lambda \in \mathrm{iso}\sigma_a(A + R)$ implies

$$\begin{aligned} &\lambda \in \mathrm{iso}\sigma_a(A) \Longleftrightarrow \lambda \in \Pi_0^a(A) = \{\lambda : \lambda \in \mathrm{iso}\sigma_a(A) \cap \sigma_{aw}(A)^C\} \\ \Longleftrightarrow \quad &\lambda \in \mathrm{iso}\sigma_a(A + R) \cap \sigma_{aw}(A + R)^C \Longleftrightarrow \lambda \in \Pi_0^a(A + R), \end{aligned}$$

i.e., A is finitely left polaroid if and only if $A + R$ is finitely left polaroid. □

It is clear from the (argument) above that if $\mathrm{iso}\sigma_a(A + R) = \mathrm{iso}\sigma_a(A)$, then $A + R$ is left polaroid if and only if A is left polaroid, and if $\mathrm{iso}\sigma(A + R) = \mathrm{iso}\sigma(A)$, then $A + R$ is polaroid if and only if A is polaroid. The following proposition provides examples of Riesz operators satisfying this property.

Proposition 3.2. *Given operators $A, R \in B(\mathcal{X})$ such that $[A, R] = 0$, if either R is nilpotent (even, quasinilpotent), or R^n is finite rank for some integer $n > 0$ and $\mathrm{iso}\sigma_a(A) = \mathrm{iso}\sigma_a(A + R)$, then A is finitely left polaroid (finitely polaroid) if and only if $A + R$ is finitely left polaroid (resp., finitely polaroid).*

Proof. The hypotheses imply $\sigma_a(A) = \sigma_a(A+R)$, $\sigma(A) = \sigma(A+R)$, $\sigma_{aw}(A) = \sigma_{aw}(A+R)$ and $\sigma_w(A) = \sigma_w(A+R)$. □

Does Proposition 3.2 extend to: "A is left polaroid if and only if $A + R$ is left polaroid?" We have a partial answer.

Proposition 3.3. *Let $A, R \in B(\mathcal{X})$, where $[A, R] = 0$.*
(A) *If R^n is finite rank for some integer $n > 0$ and $iso\sigma_a(A) = iso\sigma_a(A+R)$, then A is left polaroid (polaroid) if and only if $A+R$ is left polaroid (resp., polaroid).*
(B) *If R is nilpotent, then A is polaroid if and only if $A+R$ is polaroid.*

Proof. **(A).** Recall from [11] that the semi B-Fredholm spectrum of an operator is stable under finite rank perturbations. Since $\sigma_a(A) = \sigma_a(A+R)$ and

$$\begin{aligned} & \lambda \in \mathrm{iso}\sigma_a(A+R) \cap \sigma_{aBw}(A+R)^C \Longleftrightarrow \lambda \in \mathrm{iso}\sigma_a(A+R) \cap \sigma_{aBe}(A+R)^C \\ \Longleftrightarrow \quad & \lambda \in \mathrm{iso}\sigma_a(A) \cap \sigma_{aBe}(A)^C \Longleftrightarrow \lambda \in \mathrm{iso}\sigma_a(A) \cap \sigma_{aBw}(A)^C, \end{aligned}$$

if A is left polaroid, then

$$\begin{aligned} & \lambda \in \mathrm{iso}\sigma_a(A+R) \Longleftrightarrow \lambda \in \mathrm{iso}\sigma_a(A) \Longleftrightarrow \lambda \in \Pi^a(A) \\ \Longleftrightarrow \quad & \lambda \in \mathrm{iso}\sigma_a(A) \cap \sigma_{aBw}(A)^C \Longleftrightarrow \lambda \in \mathrm{iso}\sigma_a(A+R) \cap \sigma_{aBw}(A+R)^C \Longleftrightarrow \lambda \in \Pi^a(A+R), \end{aligned}$$

i.e., A is left polaroid if and only if $A+R$ is left polaroid. Since $\sigma(A+R) = \sigma(A)$, and $\lambda \in \mathrm{iso}\sigma(T) \cap \sigma_{Bw}(T) \Longleftrightarrow \lambda \in \mathrm{iso}\sigma(T) \cap \sigma_{Be}(T)$ for every $T \in B(\mathcal{X})$, the argument above proves also that A is polaroid if and only if $A+R$ is polaroid.

(B). The hypotheses imply $\sigma(A+R) = \sigma(A)$. Since $\sigma_{Bb}(A+R) = \sigma_{Bb}(A)$ for nilpotent R [19, Theorem 2.6], if A is polaroid, then

$$\begin{aligned} & \lambda \in \mathrm{iso}\sigma(A+R) \Longleftrightarrow \lambda \in \mathrm{iso}\sigma(A) \Longleftrightarrow \lambda \in \Pi(A) \Longleftrightarrow \lambda \in \mathrm{iso}\sigma(A) \cap \sigma_{Bw}(A)^C \\ \Longleftrightarrow \quad & \lambda \in \mathrm{iso}\sigma(A) \cap \sigma_{Bb}(A)^C \Longleftrightarrow \lambda \in \mathrm{iso}\sigma(A+R) \cap \sigma_{Bb}(A+R)^C \Longleftrightarrow \lambda \in \Pi(A+R), \end{aligned}$$

i.e., A is polaroid if and only if $A+R$ is polaroid. □

4. Browder, Weyl theorems: Perturbations

It is well known that if either of A and A^* has SVEP, then A satisfies (all four versions of) Browder's theorem. A necessary and sufficient condition for $A \in (Bt)$ and $A \in (gBt)$ (resp., $A \in (a-Bt)$ and $A \in (a-gBt)$) is that A has SVEP on $\sigma_w(A)^C$ (resp., $\sigma_{aw}(A)^C$) [1, 10, 18]. We start in the following by proving that it is the activity on the bounded components $\eta'\sigma_w(A) = \bigcup \mathrm{Hole}\sigma_w(A)$ of A and the isolated points of $\sigma_w(A)$ (resp., $\eta'\sigma_{aw}(A) = \bigcup \mathrm{Hole}\sigma_{aw}(A)$ and isolated points of $\sigma_{aw}(A)$) which determines if the operator satisfies Browder's (resp., a-Browder's) and Weyl's (resp., a-Weyl's) theorem: $A \in (Bt)$ (resp., $A \in (a-Bt)$) if and only if A has SVEP on $\eta'\sigma_w(A)$ (resp., $\eta'\sigma_{aw}(A)$); $A \in (Wt)$ (resp., $A \in (a-Wt)$) if and only if $A \in (Bt)$ and $E_0(A) \cap \sigma_w(A) = \emptyset$ (resp., if and only if $A \in (a-Bt)$ and $E_0^a(A) \cap \sigma_{aw}(A) = \emptyset$). Given an operator $A \in B(\mathcal{X})$, let (for convenience) $\Xi(A)$ *and* $\Xi_a(A)$ denote, respectively, the sets $\Xi(A) = \{\lambda \in \sigma(A) : A \text{ does not have SVEP at } \lambda\}$, and $\Xi_a(A) = \{\lambda \in \sigma_a(A) : A \text{ does not have SVEP at } \lambda\}$.

Theorem 4.1. *If $A \in B(\mathcal{X})$, then:*
(i) $A \in (Bt) \Longleftrightarrow \eta'\sigma_w(A) \cap \Xi(A) = \emptyset$.
(ii) $A \in (a-Bt) \Longleftrightarrow \eta'\sigma_{aw}(A) \cap \Xi_a(A) = \emptyset$.
(iii) $A \in (gBt) \Longleftrightarrow \eta'\sigma_{Bw}(A) \cap \Xi(A) = \emptyset$.
(iv) $A \in (a-gBt) \Longleftrightarrow \eta'\sigma_{aBw}(A) \cap \Xi_a(A) = \emptyset$.

The proof of the theorem proceeds through a few steps, which we state below as lemmas, starting with the result that A has SVEP on $\eta\sigma_w(A)^C$ and $\eta\sigma_{aw}(A)^C$ for every $A \in B(\mathcal{X})$.

Lemma 4.2. *For operators $A \in B(\mathcal{X})$, both A and A^* have SVEP on $\eta\sigma_w(A)^C$ and $\eta\sigma_{aw}(A)^C$.*

Proof. The component $\eta\sigma_w(A)^C$ (resp., $\eta\sigma_{aw}(A)^C$) being unbounded, intersects the resolvent set $\rho(A) = \mathbb{C} \setminus \sigma(A)$. Consequently, $\mathrm{asc}(A-\lambda) = \mathrm{dsc}(A-\lambda) < \infty, \alpha(A-\lambda) = 0$ for all but a countable set of λ which are isolated poles of A. Hence A, also A^*, has SVEP everywhere on $\eta\sigma_w(A)^C$ (resp., $\eta\sigma_{aw}(A)^C$). □

Lemma 4.2 extends to $\eta\sigma_{Bw}(A)^C$ and $\eta\sigma_{aBw}(A)^C$.

Lemma 4.3. *For operators $A \in B(\mathcal{X})$, both A and A^* have SVEP on $\eta\sigma_{Bw}(A)^C$ and $\eta\sigma_{aBw}(A)^C$.*

Proof. The proof in both the cases is similar; we consider points $\lambda \in \eta\sigma_{aBw}(A)^C$. Take a point $\lambda \in \eta\sigma_{aBw}(A)^C$. There exists a large enough positive integer $n_0 = n_0(\lambda)$ such that $\lambda + \frac{1}{n} \in \Phi_+(A), \text{ind}(A - \lambda - \frac{1}{n}) \leq 0$ for all $n \geq n_0$ [23, Theorem 4.7]. The operator $A - \lambda$ and $(A - \lambda) - \frac{1}{n}$ commute and satisfy the property that $\lim_{n\to\infty} \|(A - \lambda) - \frac{1}{n}\| = \|A - \lambda\|$. Hence, since ($\lambda + \frac{1}{n} \in \eta\sigma_{aw}(A)^C$ implies) A has SVEP at $\lambda + \frac{1}{n}$, A has SVEP at λ. Since $\eta\sigma_{sBw}(A) = \eta\sigma_{aBw}(A)$, A^* also has SVEP on $\eta\sigma_{aBw}(A)$. □

Remark 4.4. An (instructive) alternative argument proving Lemmas 4.2 and 4.3 goes as follows. The inclusions $\partial\sigma_w(A) \subseteq \sigma_{aw}(A) \subseteq \sigma_w(A)$ and $\partial\sigma_{Bw}(A) \subseteq \sigma_{aBw}(A) \subseteq \sigma_{Bw}(A)$ imply $\eta\sigma_{aw}(A) = \eta\sigma_w(A)$ and $\eta\sigma_{aBw}(A) = \eta\sigma_{Bw}(A)$. Since the unbounded components $\eta\sigma_w(A)$ and $\eta\sigma_{Bw}(A)$ intersect the resolvent $\rho(A)$, $A - \lambda$ is invertible except perhaps for a countable set consisting of the poles of A.

Lemma 4.5. *For operators $A \in B(\mathcal{X})$:*
(i) $\eta'\sigma_{Bw}(A) \cap \Xi(A) = \emptyset \Longleftrightarrow \eta'\sigma_w(A) \cap \Xi(A) = \emptyset$.
(ii) $\eta'\sigma_{aBw}(A) \cap \Xi_a(A) = \emptyset \Longleftrightarrow \eta'\sigma_{aw}(A) \cap \Xi_a(A) = \emptyset$.

Proof. The proof in both the cases is similar; we prove (ii). Since $\sigma_{aBw}(A) \subseteq \sigma_{aw}(A)$ if and only if $\eta\sigma_{aw}(A)^C \cup \eta'\sigma_{aw}(A) \subseteq \eta\sigma_{aBw}(A) \cup \eta'\sigma_{aBw}(A)$ and since $\eta\sigma_{aw}(A)^C \cap \Xi_a(A) = \emptyset = \eta\sigma_{aBw}(A)^C \cap \Xi_a(A)$, we have

$$\eta'\sigma_{aBw}(A) \cap \Xi_a(A) = \emptyset \Longrightarrow \eta'\sigma_{aw}(A) \cap \Xi_a(A) = \emptyset.$$

Conversely, assume that $\eta'\sigma_{aw}(A) \cap \Xi_a(A) = \emptyset$. (Thus A has SVEP on $\sigma_{aw}(A)^C$.) Take a $\lambda \in \eta'\sigma_{aBw}(A)$, and apply the srgument of the proof of Lemma 4.3 to conclude that A has SVEP at λ. The choice of the point λ having been arbitrary, it follows that A has SVEP on $\eta'\sigma_{aBw}(A)$. □

Corollary 4.6. *For operators $A \in B(\mathcal{X})$:*
(i) $\sigma_w(A)^C \cap \Xi(A) = \emptyset \Longleftrightarrow \sigma_{Bw}(A)^C \cap \Xi(A) = \emptyset$.
(ii) $\sigma_{aw}(A)^C \cap \Xi_a(A) = \emptyset \Longleftrightarrow \sigma_{aBw}(A)^C \cap \Xi_a(A) = \emptyset$.

Proof. Immediate from the lemmas above since $\sigma_{xw}(A)^C = \eta\sigma_{xw}(A)^C \cup \eta'\sigma_{xw}(A)$, $\sigma_{xw} = \sigma_w$ or σ_{aw}, and $\sigma_{xBw}(A)^C = \eta\sigma_{xBw}(A)^C \cup \eta'\sigma_{xBw}(A)$, $\sigma_{xBw} = \sigma_{Bw}$ or σ_{aBw}. □

Proof of Theorem 4.1. We prove statements (*iii*) and (*iv*); statements (*i*) and (*ii*) are similarly proved (else, apply Lemma 4.5). Since A has SVEP on $\eta\sigma_{Bw}(A)^C$ and $\eta\sigma_{aBw}(A)^C$ for every $A \in B(\mathcal{X})$,

$$\begin{aligned} & \sigma(A) \cap \sigma_{Bw}(A)^C = \{\sigma(A) \cap \eta\sigma_{Bw}(A)^C\} \cup \{\sigma(A) \cap \eta'\sigma_{Bw}(A)\} \\ = \; & \{\Pi(A) \cap \eta\sigma_{Bw}(A)^C\} \cup \{\sigma(A) \cap \eta'\sigma_{Bw}(A)\}, \\ & \text{and} \\ & \sigma_a(A) \cap \sigma_{aBw}(A)^C = \{\sigma_a(A) \cap \eta\sigma_{aBw}(A)^C\} \cup \{\sigma_a(A) \cap \eta'\sigma_{aBw}(A)\} \\ = \; & \{\Pi^a(A) \cap \eta\sigma_{aBw}(A)^C\} \cup \{\sigma_a(A) \cap \eta'\sigma_{aBw}(A)\}. \end{aligned}$$

Hence $A \in (gBt)$ (resp., $A \in (a-gBt)$) if and only if $\sigma(A)\cap\eta'\sigma_{Bw}(A) = \Pi(A)\cap\eta'\sigma_{Bw}(A)$ (resp., $\sigma_a(A)\cap\eta'\sigma_{aBw}(A) = \Pi^a(A) \cap \eta'\sigma_{aBw}(A)$). Equivalently, $A \in (gBt)$ (resp., $A \in (a - gBt)$) if and only if $\eta'\sigma_{Bw}(A) \cap \Xi(A) = \emptyset$ (resp., $\eta'\sigma_{aBw}(A) \cap \Xi_a(A) = \emptyset$). □

If $R \in B(\mathcal{X})$ is a Riesz operator which commutes with an operator $A \in B(\mathcal{X})$, then:
(I) $\sigma_{xw}(A + R) = \sigma_{xw}(A)$, where $\sigma_{xw} = \sigma_w$ or σ_{aw} [20, 31];
(II) $A + R$ ($(A + R)^*$) has SVEP at a point λ if and only if A (resp., A^*) has SVEP at λ [8].
Hence:

Corollary 4.7. *Given operators $A, R \in B(\mathcal{X})$, if R is a Riesz operator which commutes with A, then $\eta'\sigma_{xw}(A+R) \cap \Xi_x(A+R) = \emptyset$ if and only if $\eta'\sigma_{xw}(A) \cap \Xi_x(A) = \emptyset$, where $\sigma_{xw} = \sigma_w$ or σ_{aw} and (correspondingly) $\Xi_x = \Xi$ or Ξ_a.*

It is immediate from Corollary 4.7 that

$A + R \in (x - Bt) \Longleftrightarrow A \in (x - Bt), A + R \in (x - gBt) \Longleftrightarrow A \in (x - gBt)$,

where $(x - Bt) = (Bt)$ or $(a - Bt)$ and $(x - gBt) = (gBt)$ or $(a - gBt)$. Corollary 4.7 applies in particular to commuting (with A) finite rank, quasinilpotent and compact operators R.

Weyl's Theorems. Given an operator $A \in B(\mathcal{X})$, let $E(A) = \{\lambda \in \sigma(A) : 0 < \alpha(A - \lambda)\}$ (denote the set of eigenvalues of A) and $E^a(A) = \{\lambda \in \sigma_a(A) : 0 < \alpha(A - \lambda)\}$. We say that the operator $A \in B(\mathcal{X})$ satisfies

generalized Weyl' theorem, $A \in (gWt)$, if $\sigma(A) \cap \sigma_{Bw}(A)^C = E(A)$;

generalized a-Weyl' theorem, $A \in (a - gWt)$, if $\sigma_a(A) \cap \sigma_{aBw}(A)^C = E^a(A)$.

The following implications

$$(a - gWt) \Longrightarrow (gWt) \Longrightarrow (Wt), \;\; (a - gWt) \Longrightarrow (a - Wt) \Longrightarrow (Wt)$$

hold, but the reverse implications are in general false [1, 12, 16–18]. It is evident that $(xWt) \Longrightarrow (xBt)$, where $(x..) = (..)$ or $(g..)$ or $(a - ..)$ or $(a - g..)$. Also, if $A \in (Bt)$ (resp., $A \in (gBt)$), then $\sigma(A) \cap \sigma_w(A)^C = \Pi_0(A) \subseteq E_0(A)$ (resp., $\sigma(A) \cap \sigma_{Bw}(A)^C = \Pi(A) \subseteq E(A)$); similarly, if if $A \in (a - Bt)$ (resp., $A \in (a - gBt)$), then $\sigma_a(A) \cap \sigma_{aw}(A)^C = \Pi^a_0(A) \subseteq E^a_0(A)$ (resp., $\sigma_a(A) \cap \sigma_{aBw}(A)^C = \Pi^a(A) \subseteq E^a(A)$). Thus, a necessary and sufficient condition for an $A \in (Bt)$ to satisfy $A \in (Wt)$ is that $E_0(A) \subseteq \Pi_0(A)$ (resp., $A \in (gBt)$ to satisfy $A \in (gWt)$ is that $E(A) \subseteq \Pi(A)$); similarly, a necessary and sufficient condition for an $A \in (a - Bt)$ to satisfy $A \in (a - Wt)$ is that $E^a_0(A) \subseteq \Pi^a_0(A)$ (resp., $A \in (a - gBt)$ to satisfy $A \in a - (gWt)$ is that $E^a(A) \subseteq \Pi^a(A)$). Since

$$E^a_0(A) = \{E^a_0(A) \cap \sigma_{aw}(A)^C\} \cup \{E^a_0(A) \cap \sigma_{aw}(A)\}$$

and since $E^a_0(A) \cap \sigma_{aw}(A)^C \subseteq \Pi^a_0(A)$, a sufficient condition for $E^a_0(A) \subseteq \Pi^a_0(A)$ is

$$E^a_0(A) \cap \sigma_{aw}(A) = E^a_0(A) \cap \mathrm{iso}\sigma_{aw}(A) = \emptyset.$$

Similarly, a sufficient condition for $E_0(A) \subseteq \Pi_0(A)$ is $E_0(A) \cap \mathrm{iso}\sigma_w(A) = \emptyset$. These conditions are necessary too.

Theorem 4.8. *A necessary and sufficient conditon for an operator $A \in B(\mathcal{X})$ to satisfy:*

(i) $A \in (Wt)$ is that $\eta'\sigma_w(A) \cap \Xi(A) = \emptyset$ and $E_0(A) \cap iso\sigma_w(A) = \emptyset$.

(ii) $A \in a - (Wt)$ is that $\eta'\sigma_{aw}(A) \cap \Xi_a(A) = \emptyset$ and $E^a_0(A) \cap iso\sigma_{aw}(A) = \emptyset$.

(iii) $A \in (gWt)$ is that $\eta'\sigma_{Bw}(A) \cap \Xi(A) = \emptyset$ and $E(A) \cap iso\sigma_{Bw}(A) = \emptyset$.

(iv) $A \in (a - gWt)$ is that $\eta'\sigma_{aBw}(A) \cap \Xi_a(A) = \emptyset$ and $E^a(A) \cap iso\sigma_{aBw}(A) = \emptyset$.

Proof. The proof in all cases is similar: We prove (iv). Since $(a - gWt)$ implies $(a - gBt)$, the necessity of the condition $\eta'\sigma_{aBw}(A) \cap \Xi_a(A) = \emptyset$ is clear (see Theorem 4.1). To see the necessity of the condition $E^a(A) \cap \mathrm{iso}\sigma_{aBw}(A) = \emptyset$, assume that $A \in (a - gBt)$ and that there exists a $\lambda \in E^a(A) \cap \mathrm{iso}\sigma_{aBw}(A)$. Then, since $0 < \alpha(A - \lambda)$, there does not exist an integer $d > 0$ such that $(A - \lambda)^d(\mathcal{X})$ is closed. Consequently, $\lambda \notin \Pi^a(A)$, and hence $\lambda \notin \mathrm{iso}\sigma_a(A) \cap \sigma_{aBw}(A)^C$ (implies $\lambda \notin E^a(A)$, since $A \in (a - gWt)$ implies $\Pi^a(A) = E^a(A)$.) This being a contradiction, we must have $E^a(A) \cap \mathrm{iso}\sigma_{aBw}(A) = \emptyset$. To prove the sufficiency of the conditions, we start by observing that the condition $\eta'\sigma_{aBw}(A) \cap \Xi_a(A) = \emptyset$ implies $A \in (a - gBt)$, hence $\sigma_a(A) \cap \sigma_{aBw}(A)^C = \Pi^a(A) \subseteq E^a(A)$. Let $\lambda \in E^a(A)$. Then A has SVEP at λ, and hence $\lambda \in \sigma_{aBw}(A)^C$ implies $\lambda \in \Pi^a(A)$. This, if $(E^a(A) \cap \sigma_{aBw}(A) =)\ E^a(A) \cap \mathrm{iso}\sigma_{aBw}(A) = \emptyset$, implies $E^a(A) \subseteq \Pi^a(A)$. Since the reverse inclusion is true for every operator A, $E^a(A) = \Pi^a(A)$, i.e., the conditions are sufficient. □

Remark 4.9. Examples of classes of operators satisfying the hypotheses of Theorems 4.1 and 4.8 abound. Let $(\mathcal{THN})$ denote the class of operators $A \in B(\mathcal{X})$ such that every part of A (A part of A is its restriction to a closed invariant subspace), and the inverse of every invertible part of A, is normaloid (i.e., its spectral radius equals its norm). Operators $A \in (\mathcal{THN})$ have have SVEP on $\eta'\sigma_{aw}(A)$ (equiavelntly, SVEP on $\sigma_{aw}(A)^C$) and are polaroid [16]. Hence operators $A \in (\mathcal{THN})$ satisfy Theorem 4.1 and Theorem 4.8 (i) and (iii). Prominent examples of $(\mathcal{THN})$ operators include hyponormal (Hilbert space) operators and paranormal operators [1, 16–18, 25]. Banach space operators $A \in H_0(p)$ for which the quasinilpotent part $H_0(T-\lambda) = \{x \in \mathcal{X} : \lim_{n\to\infty} \|(A-\lambda)^n x\|^{\frac{1}{n}} = 0\} = (A-\lambda)^{-p}(0)$ for some integer $p > 0$ and all complex λ [1, Page 172], in particular subscalar operators, are another important example of polaroid operators with SVEP. Operator $A \in B(\mathcal{X})$ satisfying $\sigma_a(A) = \sigma_{aw}(A)$ satisfy Theorem 4.1; furthermore, if also $\sigma_a(A)$ is connected (as, for example, is the case when $A \in B(\ell^p)$, $1 \leq p < \infty$, is a weighted right shift [1, 27]), then A satisfies Theorem 4.8. Analytic Toeplitz operators A_f (with symbol f) satisfy $\sigma(T_f)$ is connected and $\sigma(T_f) = \sigma_w(T_f)$: Hence analytic Toeplitz operators satisfy Theorem 4.8 (i) and (iii).

Unlike the situation for Browder theorem type results, results of the type of Weyl's theorem do not survive perturbation by commuting Riesz operators. Consider, for example, the operator $A = I \oplus 0 \in B(\ell^2 \oplus \ell^2)$, I the identity operator and 0 the zero operator. Then $\sigma(A) = \sigma_a(A) = \{0, 1\} = E(A) = E^a(A), \sigma_{Bw} = \sigma_{aBw}(A) = \emptyset$, and A satisfies both generalized Weyl's theorem and a-generalized Weyl's theorem. Let F be the finite rank operator defined by $F(x_1, x_2, x_3, ...) = (-x_1, 0, 0, ...)$ and let $R = F \oplus Q$, where $Q \in B(\ell^2)$ is a non-nilpotent injective quasinilpotent operator. Then the operator $A+R$ satisfies $\sigma(A) = \sigma_a(A+R) = \{0, 1\} = E(A+R) = E^a(A+R), \sigma_{Bw}(A+R) = \sigma_{aBw}(A+R) = \{0\}$. Clearly, $A + R \notin (gWt)$ and $A + R \notin (a - gWt)$. Observe here that the operator A of the example is isoloid (indeed, a-isoloid), but not finitely isoloid (or finitely a-isoloid). The following theorem, which generalizes a number of extant results [2, 5, 12, 17, 18, 29, 30], says that the (necessary) condition A is finitely isoloid (resp., finitely a-isoloid) is sufficient for the transfer of (Wt) and (gWt) (resp., $(a - Wt)$ and $(a - gWt)$) from A to $A + R$ for commuting Riesz operators R.

Theorem 4.10. *Let $A, R \in B(\mathcal{X})$, where R is a Riesz operator which commutes with A. If A is finitely isoloid, then $A \in (Wt)$ implies $A + R \in (Wt)$, $A \in (gWt)$ implies $A + R \in (gWt)$ and if A is finitely a-isoloid, then $A \in (a - Wt)$ implies $A + R \in (a - Wt)$, $A \in (a - gWt)$ implies $A + R \in (a - gWt)$.*

Proof. The proof in all the cases is similar: We prove $A \in (a - gWt) \Longrightarrow A + R \in (a - gWt)$, leaving it to the reader to make the minor changes in the argument required to prove the remaining cases. The hypothesis

$$\begin{aligned} A \in (a - gWt) \quad &\Longrightarrow \quad A \in (a - gBt) \Longrightarrow A + R \in (a - gBt) \\ &\Longleftrightarrow \quad \sigma_a(A+R) \cap \sigma_{aBw}(A+R)^C = \Pi^a(A+R) \subseteq E^a(A+R). \end{aligned}$$

We prove $E^a(A+R) \subseteq \Pi^a(A+R)$. Let $\lambda \in E^a(A+R)$. Then there exists a neighbourhood $\mathcal{N}_\epsilon(\lambda)$, $\epsilon > 0$, of λ such that $\mu \notin \sigma_a(A+R)$ for all $\mu \in \mathcal{N}_\epsilon(\lambda) \setminus \{\lambda\}$. Since $\mu \notin \sigma_a(A+R)$ implies $\mu \in \sigma_{aw}(A+R)^C$ (equivalently, $\mu \in \sigma_{aw}(A)^C$), it follows from ($A \in (a - gBt)$ implies) $A \in (a - Bt)$ that $\mu \in \Pi_0^a(A)$ implies $\mu \in \Pi^a(A)$ for every $\mu \in \mathcal{N}_\epsilon(\lambda) \setminus \{\lambda\}$. Observe that for every $\lambda \in E^a(A+R)$ either $\lambda \notin \sigma_a(A)$, or $\lambda \in \text{iso}\sigma_a(A)$, or $\lambda \in \text{acc}\sigma_\text{a}(\text{A})$. If $\lambda \notin \sigma_a(A)$, then

$$\lambda \in \sigma_{aw}(A)^C \Longrightarrow \lambda \in \sigma_{aw}(A+R)^C \Longrightarrow \lambda \in \Pi_0^a(A+R)$$

(since $\lambda \in E^a(A+R)$ implies $A + R$ has SVEP at λ). Again, if $\lambda \in \text{iso}\sigma_a(A)$, then ($A$ being finitely a-isoloid)

$$\begin{aligned} \lambda \in E_0^a(A) \quad &\Longrightarrow \quad \lambda \in \sigma_{aw}(A)^C, A \text{ has SVEP at } \lambda \\ &\Longleftrightarrow \quad \lambda \in \sigma_{aw}(A+R)^C, A+R \text{ has SVEP at } \lambda \\ &\Longleftrightarrow \quad \lambda \in \Pi_0^a(A+R) \Longrightarrow \lambda \in \Pi^a(A+R). \end{aligned}$$

Consider now $\lambda \in \text{acc}\sigma_\text{a}(\text{A})$. There exists an infinite sequence $\{\mu_n\} \subseteq \mathcal{N}_\epsilon(\lambda) \setminus \{\lambda\}$ such that (μ_n converges to μ and) $\mu_n \in \Pi_0^a(A)$ for all n. Since this implies $\mu_n \in \sigma_{aw}(A+R)^C$, and $A + R$ has SVEP at μ_n,

$$\mu_n \in \Pi_0^a(A+R) \Longrightarrow \mu_n \in \Pi^a(A+R) \subseteq E^a(A+R)$$

for all n. Since $\lambda \in E^a(A+R)$ is isolated, this is a contradiction. Hence ($\text{acc}\sigma_\text{a}(\text{A}) \cap \text{E}^\text{a}(\text{A}+\text{R}) = \emptyset$ and) every $\lambda \in E^a(A+R)$ is an element of $\Pi^a(A+R)$, i.e., $E^a(A+R) \subseteq \Pi^a(A+R)$. □

It is clear from the proof above that the hypotheses of Theorem 4.10 ensure $E_0(A+R) \cap \mathrm{iso}\sigma_w(A+R) = \emptyset$ and $E(A+R) \cap \mathrm{iso}\sigma_{Bw}(A+R) = \emptyset$ in part (*i*) of the theorem; similarly, $E_0^a(A+R) \cap \mathrm{iso}\sigma_{aw}(A+R) = \emptyset$ and $E^a(A+R) \cap \mathrm{iso}\sigma_{aBw}(A+R) = \emptyset$ in part (*ii*) of the theorem. More generally:

Proposition 4.11. *Given an operator $A \in B(\mathcal{X})$, a necessary and sufficient condition for:*

(i) $E_0(A) \subseteq \Pi_0(A)$, equivalently $E_0(A) = \Pi_0(A)$, is that $E_0(A) \cap iso\sigma_w(A) = \emptyset$.

(ii) $E_0^a(A) \subseteq \Pi_0^a(A)$, equivalently $E_0^a(A) = \Pi_0^a(A)$, is that $E_0^a(A) \cap iso\sigma_{aw}(A) = \emptyset$.

(iii) $E(A) \subseteq \Pi(A)$, equivalently $E(A) = \Pi(A)$, is that $E(A) \cap iso\sigma_{Bw}(A) = \emptyset$.

(iv) $E^a(A) \subseteq \Pi^a(A)$, equivalently $E^a(A) = \Pi^a(A)$, is that $E^a(A) \cap iso\sigma_{aBw}(A) = \emptyset$.

Proof. The proof in all cases being similar, we prove (*iv*). If $E^a(A) \subseteq \Pi^a(A)$, and if there exists a $\lambda \in E^a(A) \cap \mathrm{iso}\sigma_{aBw}(A)$, then (being a left pole of the resolvent of A) $\lambda \notin \sigma_{aBw}(A)$ – a contradiction. Conversely, if we assume that $E^a(A) \cap \mathrm{iso}\sigma_{aBw}(A) = \emptyset$ and that there exists a $\lambda \in E^a(A)$ such that $\lambda \notin \Pi^a(A)$, then $(A-\lambda)^d(\mathcal{X})$ is not closed for any integer $d > 0$. Reason: If $(A-\lambda)^d(\mathcal{X})$ is closed for an integer $d > 0$, then, since $\lambda \in E^a(A)$ implies A has SVEP at λ, $\lambda \in \Pi^a(A)$. Consequently, $\lambda \in \sigma_{aBw}(A) \cap E^a(A)$ – again a contradiction. □

5. Variations on Browder, Weyl Theorems

The a-Browder and a-Weyl theorems are obtained from (their classical counterparts) Browder and Weyl theorems $\sigma(A) \cap \sigma_w(A)^C = \Pi_0(A)$ and $\sigma(A) \cap \sigma_w(A)^C = E_0(A)$ by replacing $\sigma(A)$ by $\sigma_a(A)$, $\sigma_w(A)$ by $\sigma_{aw}(A)$, $\Pi_0(A)$ by $\Pi_0^a(A)$ and $E_0(A)$ by $E_0^a(A)$; similarly, the generalized versions of the Browder and Weyl theorems (resp., the a-generalized versions of the Browder and Weyl theorems) are obtained upon replacing $\sigma_w(A)$, $\Pi_0(A)$ and $E_0(A)$ by $\sigma_{Bw}(A)$, $\Pi(A)$ and $E(A)$ (resp., $\sigma_{aw}(A)$, $\Pi_0^a(A)$ and $E_0^a(A)$ by $\sigma_{aBw}(A)$, $\Pi^a(A)$ and $E^a(A)$). A number of further variations, obtained by making other suitably meaningful choices, have been considered in the recent past (see [2, 5, 9, 13–15] for a flavour of the type of variations considered). Prominenet amongst the variations to have attracted some attention are the properties (*a*), (*ab*) (*w*) and their generalized versions. We say that an operator $A \in B(\mathcal{X})$ satisfies property:

(*b*) if $\sigma_a(A) \cap \sigma_{aw}(A)^C = \Pi_0(A)$;

(*gb*) if $\sigma_a(A) \cap \sigma_{aBw}(A)^C = \Pi(A)$;

(*ab*) if $\sigma(A) \cap \sigma_w(A)^C = \Pi_0^a(A)$;

(*gab*) if $\sigma(A) \cap \sigma_{Bw}(A)^C = \Pi^a(A)$;

(*w*) if $\sigma_a(A) \cap \sigma_{aw}(A)^C = E_0(A)$;

(*gw*) if $\sigma_a(A) \cap \sigma_{aBw}(A)^C = E(A)$.

A number of the properties of operators satisfying the above defined properties lie on the surface and are easily adduced. Thus, if $A \in B(\mathcal{X})$ satisfies property (*b*), $A \in (b)$, then A has SVEP on $\sigma_{aw}(A)^C$, A^* has SVEP on $\sigma_a(A) \cap \sigma_{aw}(A)^C = \sigma_a(A) \cap \sigma_w(A)^C$, $A \in (a-Bt)$ and $\Pi_0^a(A) = \Pi_0(A)$; A may however fail to satisfy a-Weyl's (even, Weyl's) theorem. In this section we relate these properties to their spectral picture, study relations between these properties and consider the permanence of these properties under perturbation by commuting Riesz operators. In the process, we generalize a number of known results. We start with a characterization of properties (*b*) and (*w*).

Theorem 5.1. *Given an operator $A \in B(\mathcal{X})$:*

(i). *$A \in (w) \Longrightarrow A \in (b) \Longleftrightarrow A^*$ has SVEP on $\sigma_a(A) \cap \eta'\sigma_{aw}(A)$ (equivalently, $\sigma_a(A) \cap \eta'\sigma_{aw}(A) \subseteq iso\sigma(A)$).*

(ii). *$A \in (w) \Longleftrightarrow A \in (b)$ and $E_0(A) \cap iso\sigma_{aw}(A) = \emptyset$.*

Proof. **(i).** The defintion of property (w) implies that if $A \in (w)$, then both A and A^* have SVEP on $\sigma_a(A) \cap \sigma_{aw}(A)^C$. Since A has SVEP on $\sigma_a(A) \cap \sigma_{aw}(A)^C$ implies $\sigma_a(A) \cap \sigma_{aw}(A)^C = \Pi_0^a(A)(\subseteq E_0^a(A))$, if also A^* has SVEP on $\sigma_a(A) \cap \sigma_{aw}(A)^C$, then $\sigma_a(A) \cap \sigma_{aw}(A)^C = \Pi_0^a(A) = \Pi_0(A)$. Hence $A \in (w)$ implies $\sigma_a(A) \cap \sigma_{aw}(A)^C = \Pi_0^a(A) = \Pi_0(A) = E_0(A)$. In particular, $A \in (w)$ implies $A \in (b)$. It is clear that if A^* has SVEP at a $\lambda \in \sigma_{aw}(A)^C$, then $\lambda \in \sigma_w(A)^C$. Since an operator T and its adjoint T^* have SVEP on the unbounded component $\eta\sigma_w(T)^C$, A^* has SVEP at $\lambda \in \sigma_a(A) \cap \eta\sigma_{aw}(A)^C$ if and only if $\lambda \in \sigma_a(A) \cap \eta\sigma_w(A)^C$. (Indeed, since $\partial\sigma_w(A) \subseteq \sigma_{aw}(A) \subseteq \sigma_w(A)$ implies $\eta\sigma_{aw}(A) = \eta\sigma_w(A)$, $\sigma_a(A) \cap \eta\sigma_{aw}(A)^C = \sigma_a(A) \cap \eta\sigma_w(A)^C$.) Since $A \in (b) \iff \sigma_a(A) \cap \sigma_{aw}(A)^C = \Pi_0(A)$, we have that A, A^* have SVEP on $\sigma_a(A) \cap \sigma_{aw}(A)^C = \{\sigma_a(A) \cap \eta\sigma_{aw}(A)^C\} \cup \{\sigma_a(A) \cap \eta'\sigma_{aw}(A)\}$, equivalently, $\sigma_a(A) \cap \eta'\sigma_{aw}(A) \subseteq \text{iso}\sigma(A)$.

(ii). The implication

$$A \in (w) \Longrightarrow A \in (b), E_0(A) \cap \sigma_{aw}(A) = \emptyset(\Longleftrightarrow E_0(A) \cap \text{iso}\sigma_{aw}(A) = \emptyset)$$

being evident from the argument of the proof of **(i)** above, we prove the reverse implication. As seen above, the hypothesis $A \in (b)$ implies A^* has SVEP on $\sigma_a(A) \cap \sigma_{aw}(A)^C$; hence $\sigma_a(A) \cap \sigma_{aw}(A)^C = \Pi_0(A) \subseteq E_0(A)$. Consider a $\lambda \in E_0(A)$ such that $\lambda \notin \Pi_0(A)$. Since $0 < \alpha(A - \lambda) < \infty$, $(A - \lambda)^d(\mathcal{X})$ is not closed for any integer $d > 0$. Hence $\lambda \in \sigma_{aw}(A)$. Since already λ is isolated in $\sigma(A)$, $\lambda \in \text{iso}\sigma_{aw}(A)$. This being a contradiction of our assumption $E_0(A) \cap \text{iso}\sigma_{aw}(A) = \emptyset$, we must have $\lambda \in \Pi_0(A)$. Hence $E_0(A) \subseteq \Pi_0(A)$. □

An argument similar to the one above proves the following:

Corollary 5.2. *Given an operator $A \in B(\mathcal{X})$:*
(i). *$A \in (gw) \Longrightarrow A \in (gb) \iff A^*$ has SVEP on $\sigma_a(A) \cap \eta'\sigma_{aBw}(A)$ (equivalently, $\sigma_a(A) \cap \eta'\sigma_{aBw}(A) \subseteq \text{iso}\sigma(A)$).*
(ii). *$A \in (gw) \iff A \in (gb)$ and $E(A) \cap \text{iso}\sigma_{aBw}(A) = \emptyset$.*

More is true, as we now prove.

Theorem 5.3. *For operators $A \in B(\mathcal{X})$:*

(i). *$A \in (gb) \Longrightarrow A \in (b)$.*

(ii). *$A \in (gw) \Longrightarrow A \in (w)$.*

(iii). *$A \in (b)$ and A^* has SVEP on $\sigma(A) \cap \eta'\sigma_{aw}(A)$ implies $A \in (gb)$.*

(iv). *$A \in (w)$ is left polaroid (or, polaroid) and $\Pi_0^a(A) \subseteq \Pi_0(A)$ implies $A \in (gw)$.*

Proof. **(i).** We start by observing that the hypothesis $A \in (b)$ is equivalent to $(A \in (a - Bt), \Pi_0^a(A) = \Pi_0(A)$, equivalently) $A \in (a - Bt), A^*$ has SVEP on $\sigma_a(A) \cap \sigma_{aw}(A)^C$, and the hypothesis $A \in (gb)$ is equivalent to $A \in (a - gBt), A^*$ has SVEP on $\sigma_a(A) \cap \sigma_{aBw}(A)^C$. Since $A \in (a - Bt)$ if and only if $A \in (a - gBt)$ [10], and since $\sigma_a(A) \cap \sigma_{aw}(A)^C \subseteq \sigma_a(A) \cap \sigma_{aBw}(A)^C$, $A \in (gb)$ implies $A \in (a - Bt)$ and A^* has SVEP on $\sigma_a(A) \cap \sigma_{aw}(A)^C$, i.e., $A \in (b)$.

(ii). If $A \in (gw)$, then $A \in (gb)$ (implies $A \in (b)$) and $E(A) \cap \text{iso}\sigma_{aBw}(A) = \emptyset$. Assume now that there exists a $\lambda \in E_0(A) \cap \text{iso}\sigma_{aw}(A)$. Then $(A - \lambda)^d(\mathcal{X})$ is not closed for any integer $d > 0$. Consequently, if $A \in (gw)$, then $\lambda \in \sigma_{aBw}(A)$ and hence, since $\lambda \in E_0(A)$ implies $\lambda \in E_0(A) \cap \text{iso}\sigma_{aBw}(A) \subseteq E(A) \cap \text{iso}\sigma_{aBw}(A)$, $E(A) \cap \text{iso}\sigma_{aBw}(A) \neq \emptyset$. This being a contradiction, we must have $E_0(A) \cap \text{iso}\sigma_{aw}(A) = \emptyset$. Conclusion: $A \in (w)$.

(iii). Evidently, A^* has SVEP on $\sigma(A) \cap \sigma_{aw}(A)^C$ if and only if A^* has SVEP on $\sigma(A) \cap \eta'\sigma_{aw}(A) = \sigma(A^*) \cap \eta'\sigma_{sw}(A^*)$, and A^* has SVEP on $\sigma(A) \cap \sigma_{aBw}(A)^C$ if and only if A^* has SVEP on $\sigma(A) \cap \eta'\sigma_{aBw}(A) = \sigma(A^*) \cap \eta'\sigma_{sBw}(A^*)$. Consider a $\lambda \in \eta'\sigma_{sBw}(A^*)$. There exists a large enough integer $n > 0$ such that $\lambda + \frac{1}{n} \in \sigma(A^*) \cap \sigma_{sw}(A^*)^C$ [23], and hence if A^* has SVEP on $\sigma(A) \cap \eta'\sigma_{aw}(A)$, then A^* has SVEP at $\lambda + \frac{1}{n}$. The operators $A^* - \lambda$ and $A^* - \lambda - \frac{1}{n}$ being quasinilpotent equivalent, A^* has SVEP at λ [27, Proposition 3.4.11]. The choice of the point λ having been arbitrary, it follows that A^* has SVEP on $\sigma(A) \cap \sigma_{aBw}(A)^C$. Combining this with the fact that $A \in (b)$ implies

$$A \in (a - Bt) \iff A \in (a - gBt) \iff \sigma_a(A) \cap \sigma_{aBw}(A)^C = \Pi^a(A),$$

we conclude that $A \in (gb)$ (whenever A^* has SVEP on $\sigma_a(A) \cap \eta'\sigma_{aw}(A)$, equivalently, whenever A^* has SVEP on $\Pi^a(A)$).

(iv). We prove that $A \in (b)$ and $\Pi(A) = E(A)$: This would then imply $A \in (gw)$. If $A \in (w)$, then $A \in (b)$ and A^* has SVEP at points in $\sigma_a(A) \cap \sigma_{aw}(A)^C$ where A has SVEP. Since

$$\sigma(A) \cap \sigma_{aw}(A)^C = \{\sigma_a(A) \cap \sigma_{aw}(A)^C\} \cup \{\{\sigma(A) \setminus \sigma_a(A)\} \cap \sigma_{aw}(A)^C\},$$

and since

$$\lambda \in \{\sigma(A) \setminus \sigma_a(A)\} \cap \sigma_{aw}(A)^C \Longrightarrow \lambda \in \Pi_0^a(A) \setminus \sigma_a(A) = \emptyset,$$

the hypothesis $\Pi_0^a(A) \subseteq \Pi_0(A)$ (equivalently, $\Pi_0^a(A) = \Pi_0(A)$) implies A^* has SVEP on $\Pi_0^a(A)$. Hence A^* has SVEP at points in $\sigma_a(A) \cap \sigma_{aw}(A)^C$ where A has SVEP, and this (by **(iii)** above) implies

$$A \in (gb) \Longleftrightarrow \sigma_a(A) \cap \sigma_{aBw}(A)^C = \Pi(A) \subseteq E(A).$$

Consider now a $\lambda \in E(A)$. Since A is left polaroid (resp., polaroid) by hypothesis, $\lambda \in E(A)$ implies $\lambda \in \Pi^a(A)$ and A^* has SVEP at λ (resp., $\lambda \in \Pi(A)$). In either case $\lambda \in \Pi(A)$, which then implies $E(A) \subseteq \Pi(A)$. □

Remark 5.4. *(I)* The hypothesis A^* has SVEP on $\sigma(A) \cap \sigma_{aw}(A)^C$ (equivalently on $\sigma(A) \cap \eta'\sigma_{aw}(A)$) in Theorem 5.3(iii) can not be replaced by A^* has SVEP on $\sigma_a(A) \cap \sigma_{aw}(A)^C$. (Thus $A \in (b)$ does not imply $A \in (gb)$.) To see this, let $A = U \oplus 0$, where $U \in B(\mathcal{H})$ is the forward unilateral shift and $0 \in B(\mathcal{H})$ is the zero operator. Then $\sigma_a(A) = \sigma_{aw}(A) = \partial\mathcal{D} \cup \{0\}, \Pi(A) = \emptyset, A \notin (gb)$. Notice that A^* has SVEP on $\sigma_a(A) \cap \sigma_{aw}(A)^C = \emptyset$, and A^* does not have SVEP on $\sigma(A) \cap \sigma_{aw}(A)^C = \mathcal{D} \setminus \{0\} = \sigma(A) \cap \eta'\sigma_{aw}(A)$. Notice further that $\Pi_0^a(A) = \{0\}$ and A^* does not have SVEP on $\Pi_0^a(A)$.

(II) The reverse implication $A \in (w) \Longrightarrow A \in (gw)$ in Theorem 5.3 fails for the reason that $E_0(A) \cap \text{iso}\sigma_{aw}(A) = \emptyset$ does not, in general, imply $E(A) \cap \text{iso}\sigma_{aBw}(A) = \emptyset$. Consider the operator $A = Q \oplus 0 \in B(\mathcal{X} \oplus \mathcal{X})$, where Q is an injective quasinilpotent operator. Then $A \in (w)$ and $E(A) \cap \text{iso}\sigma_{aBw}(A) = \{0\} \neq \emptyset$. Observe that A is not left polaroid. Indeed, the example $A = U \oplus 0$ of part (I) of the remark shows that the condition A is left polaroid is essential. We have $\sigma_a(A) \cap \sigma_{aw}(A)^C = \emptyset = E_0(A)$, $\sigma_a(A) \cap \sigma_{aBw}(A)^C = \{0\} \neq E_0(A)$, $\Pi_0^a(A) = \Pi_0(A) = \emptyset$ and A is not left polaroid.

The following theorem characterizes operators $A \in B(\mathcal{X})$ such that $A \in (ab)$, or, $A \in (gab)$.

Theorem 5.5. *Let $A \in B(\mathcal{X})$.*

(A). *The following conditions are mutually equivalent:*

(i) $A \in (ab)$.

(ii) A^* has SVEP on $\eta'\sigma_w(A)$ and $\sigma_w(A) \cap \Pi_0^a(A) = \emptyset$.

(iii) A^* has SVEP on $\Pi_0^a(A) \cap \eta'\sigma_w(A)$.

(iv) A^* has SVEP on iso$\sigma_a(A) \cap \eta'\sigma_{aw}(A)$.

(B). *The following conditions are mutually equivalent:*

(i) $A \in (gab)$.

(ii) A^* has SVEP on $\eta'\sigma_{Bw}(A)$ and $\sigma_{Bw}(A) \cap \Pi^a(A) = \emptyset$.

(iii) A^* has SVEP on $\Pi^a(A) \cap \eta'\sigma_{Bw}(A)$.

(iv) A^* has SVEP on iso$\sigma_a(A) \cap \eta'\sigma_{aBw}(A)$.

Proof. The proof in both the cases is the same (simply replace $\Pi_0^a(A)$, $\Pi_0(A)$, and $\sigma_w(A)$ by $\Pi^a(A)$, $\Pi(A)$ and $\sigma_{Bw}(A)$, respectively, in the following). We prove **(A)**. Recall that A has SVEP on $\Pi_0^a(A)$, and that A has SVEP at a point in $\sigma_w(A)^C$ if and only if A and A^* have SVEP at the point. If $A \in (ab)$, then

$$\begin{aligned}
& \sigma(A) \cap \sigma_w(A)^C = \Pi_0^a(A) \\
\Longleftrightarrow\ & \Pi_0^a(A) \cap \sigma_w(A) = \emptyset, A^* \text{ has SVEP on } \sigma_w(A)^C \\
\Longleftrightarrow\ & \Pi_0^a(A) \cap \sigma_w(A) = \emptyset, A^* \text{ has SVEP on } \eta' \sigma_w(A)^C \\
\Longrightarrow\ & \Pi_0^a(A) \subseteq \sigma_w(A)^C, A^* \text{ has SVEP on } \sigma_w(A)^C \\
\Longrightarrow\ & \Pi_0^a(A) \subseteq \sigma_w(A)^C \cap \sigma(A) = \Pi_0(A) \subseteq \Pi_0^a(A) \\
\Longrightarrow\ & \sigma(A) \cap \sigma_w(A)^C = \Pi_0^a(A),
\end{aligned}$$

i.e., $(i) \Longleftrightarrow (ii)$. To prove $(i) \Longleftrightarrow (iii)$, we start by observing that (i) implies A^* has SVEP on $\Pi_0^a(A) \cap \{\sigma(A) \cap \sigma_w(A)^C\} = \Pi_0^a(A) \cap \sigma_w(A)^C$. Hence A^* has SVEP on $\Pi_0^a(A) \cap \eta'\sigma_w(A)^C$, and $(i) \Longrightarrow (iii)$. To prove the reverse implication $(iii) \Longrightarrow (i)$, we note that if A^* has SVEP on $\Pi_0^a(A) \cap \eta'\sigma_w(A)^C$, equivalently if A^* has SVEP on $\Pi_0^a(A) \cap \sigma_w(A)^C$, then $\Pi_0^a(A) \cap \sigma_w(A)^C = \Pi_0^a(A) \cap \sigma_{aw}(A)^C = \Pi_0(A) \cap \sigma_w(A)^C$. Hence (iii) implies

$$\Pi_0^a(A) = \Pi_0^a(A) \cap \sigma_{aw}(A)^C = \Pi_0(A) \cap \sigma_w(A)^C = \sigma(A) \cap \sigma_w(A)^C,$$

i.e., $(iii) \Longrightarrow (i)$. To complete the proof, we prove next that $(iii) \Longleftrightarrow (iv)$. Recalling A^* has SVEP at a $\lambda \in \sigma_{aw}(A)^C$ if and only if $\lambda \in \sigma_w(A)^C$ (and A^* has SVEP at λ), we have

$$\begin{aligned}
(iii) \Longleftrightarrow\ & A^* \text{ has SVEP on } \Pi_0^a(A) \cap \eta'\sigma_w(A) \\
\Longleftrightarrow\ & A^* \text{ has SVEP on } \Pi_0^a(A) \cap \sigma_w(A)^C \\
\Longleftrightarrow\ & A^* \text{ has SVEP on } \{\mathrm{iso}\sigma_a(A) \cap \sigma_{aw}(A)^C\} \cap \sigma_w(A)^C \\
\Longleftrightarrow\ & A^* \text{ has SVEP on } \mathrm{iso}\sigma_a(A) \cap \sigma_w(A)^C \\
\Longrightarrow\ & A^* \text{ has SVEP on } \mathrm{iso}\sigma_a(A) \cap \sigma_{aw}(A)^C \\
\Longleftrightarrow\ & A^* \text{ has SVEP on } \mathrm{iso}\sigma_a(A) \cap \eta'\sigma_{aw}(A),
\end{aligned}$$

i.e., $(iii) \Longleftrightarrow (iv)$. □

Remark 5.6. Operators $A \in B(\mathcal{X})$ such that $A \in (b)$ have SVEP on $\sigma_a(A) \cap \sigma_{aw}(A)^C$ and satsify $\sigma_a(A) \cap \sigma_{aw}(A)^C = \sigma_a(A) \cap \sigma_w(A)^C$. Since A has SVEP on $\sigma_a(A) \cap \sigma_{aw}(A)^C$ implies A has SVEP on $\sigma(A) \cap \sigma_w(A)^C$ (this is simply $(a-Bt) \Longrightarrow (Bt)$), $A \in (b)$ implies $\Pi_0^a(A) = \sigma_a(A) \cap \sigma_{aw}(A)^C = \sigma_a(A) \cap \sigma_w(A)^C \subseteq \sigma(A) \cap \sigma_w(A)^C = \Pi_0(A) \subseteq \Pi_0^a(A)$. Hence $A \in (b) \Longrightarrow A \in (ab)$. The reverse implication does not hold: Consider the operator $A = U \oplus U^*$, where $U \in B(\mathcal{H})$ is the forward unilateral shift, when it is seen that $\sigma(A) = \sigma_w(A)$, $\sigma(A) \cap \sigma_w(A)^C = \Pi_0^a(A) = \emptyset$ (implies $A \in (ab)$) and $A \notin (a-Bt)$ (implies $A \notin (b)$). The reverse implication requires additional hypotheses. For example, if $\sigma_w(A) \setminus \sigma_{aw}(A) = \emptyset$, then

$$\begin{aligned}
A \in (ab) \Longleftrightarrow\ & \sigma(A) \cap \sigma_w(A)^C = \Pi_0^a(A) \\
\Longrightarrow\ & \sigma_a(A) \cap \sigma_w(A)^C = \Pi_0(A) \ \ (\text{since } A^* \text{ has SVEP on } \Pi_0^a(A)) \\
\Longleftrightarrow\ & \sigma_a(A) \cap \sigma_{aw}(A)^C = \Pi_0(A) \ \ (\text{since } \sigma_w(A) = \sigma_{aw}(A)) \\
\Longleftrightarrow\ & A \in (b).
\end{aligned}$$

6. Variations: Perturbation by commuting Riesz operators

Unlike the Browder theorems, properties (b), (w) and their generalized versions do not survive perturbation by commuting Riesz operators (even finite rank and quasinilpotent variety). Property (ab) is, however, inherited by $A + R$ from A for commuting Riesz operators R.

Theorem 6.1. *If $A, R \in B(\mathcal{X})$, where R is Riesz and $[A, R] = 0$, then:*

(A) *$A \in (ab) \Longleftrightarrow A + R \in (ab)$.*

(B) *$A \in (b) \Longrightarrow A + R \in (b)$ if and only if one of the following mutually equivalent conditions holds:*

(i) A^* has SVEP on iso$\sigma_a(A+R)\cap\sigma_{aw}(A)^C$.

(ii) A^* has SVEP on $\Pi_0^a(A+R)$.

(iii) $\Pi_0^a(A+R)\subseteq\Pi_0(A+R)$.

(iv) A^* has SVEP on $\{\sigma_a(A+R)\setminus\sigma_a(A)\}\cap\eta'\sigma_{aw}(A)$.

(C) *$A\in(w)\Longrightarrow A+R\in(w)$ if and only if $E_0(A+R)\cap iso\sigma_{aw}(A+R)=\emptyset$, and one of the equivalent conditions (i) to (iv) of* **(B)** *above holds.*

Proof. **(A).** Recall from Theorem 5.5**(A)** that $A\in(ab)$ if and only if A^* has SVEP on $\eta'\sigma_w(A)$ and $\sigma_w(A)\cap\Pi_0^a(A)=\emptyset$. Since $\eta'\sigma_w(A)=\eta'\sigma_w(A+R)$, and since A^* has SVEP at a point if and only if $(A+R)^*$ has SVEP at the point, A^* has SVEP on $\eta'\sigma_w(A)$ if and only if A^* has SVEP on $\eta'\sigma_w(A+R)$. Again, since

$$\begin{aligned} & \sigma_w(A+R)\cap\Pi_0^a(A+R)\\ = \;& \sigma_w(A+R)\cap\{\lambda\in\sigma_{aw}(A+R)^C : A+R \text{ has SVEP at } \lambda\}\\ = \;& \sigma_w(A)\cap\{\lambda\in\sigma_{aw}(A)^C : A \text{ has SVEP at } \lambda\}=\sigma_w(A)\cap\Pi_0^a(A),\\ & \sigma_w(A+R)\cap\Pi_0^a(A+R)=\emptyset\Longleftrightarrow\sigma_w(A)\cap\Pi_0^a(A)=\emptyset.\end{aligned}$$

Hence $A\in(ab)\Longleftrightarrow A+R\in(ab)$.

(B). In the following we start by proving the necessity of condition (*ii*), prove next the equivalence of conditions (*i*) to (*iv*), and then prove that condition (*iv*) is sufficient.

If $A\in(b)$, then $(\sigma_a(A)\cap\sigma_{aw}(A)^C=\Pi_0^a(A)=\Pi_0(A)$ implies) $A\in(a-Bt)$, which then implies that $A+R\in(a-Bt)$, i.e.,

$$\sigma_a(A+R)\cap\sigma_{aw}(A+R)^C=\Pi_0^a(A+R)\ \ (\supseteq\Pi_0(A+R)).$$

Hence, if $A+R\in(b)$, then necessarily $(A+R)^*$, equivalently A^*, has SVEP on $\Pi_0^a(A+R)$.

The equivalences $(i)\Longleftrightarrow(ii)\Longleftrightarrow(iii)$ follows from the following:

$$\begin{aligned} & \text{iso}\sigma_a(A+R)\cap\sigma_{aw}(A)^C=\text{iso}\sigma_a(A+R)\cap\sigma_{aw}(A+R)^C\\ = \;& \{\lambda\in\sigma_{aw}(A+R)^C : A+R \text{ has SVEP at } \lambda\}=\Pi_0^a(A+R), \text{ and}\end{aligned}$$

$$\begin{aligned} & A^* \text{ has SVEP on } \text{iso}\sigma(A+R)\cap\sigma_{aw}(A)^C\Longleftrightarrow A^* \text{ has SVEP on } \Pi_0^a(A+R)\\ \Longleftrightarrow\;& \Pi_0^a(A+R)=\Pi_0(A+R)\ \ (\Longleftrightarrow\Pi_0^a(A+R)\subseteq\Pi_0(A+R)).\end{aligned}$$

We prove next that $(ii)\Longleftrightarrow(iv)$. The hypothesis $A\in(b)$ implies A^* has SVEP on

$$\begin{aligned}\sigma_a(A)\cap\sigma_{aw}(A)^C &= \{\sigma_a(A)\cap\eta\sigma_w(A)^C\}\cup\{\sigma_a(A)\cap\eta'\sigma_{aw}(A)\}\\ &= \{\sigma_a(A)\cap\eta\sigma_w(A+R)^C\}\cup\{\sigma_a(A)\cap\eta'\sigma_{aw}(A+R)\}\end{aligned}$$

Since $\sigma_a(A+R)\cap\eta'\sigma_{aw}(A)=\{(\sigma_a(A+R)\cap\sigma_a(A))\cap\eta'\sigma_{aw}(A)\}\cup\{(\sigma_a(A+R)\setminus\sigma_a(A))\cap\eta'\sigma_{aw}(A)\}$, A^* has SVEP on $\sigma_a(A)\cap\sigma_{aw}(A)^C$, equivalently A^* has SVEP on $\pi_0^a(A+R)$, if and only if condition (*iv*) is satisfied. To complete the proof, we prove next that $A\in(b)$ and condition (*iv*) imply $A+R\in(b)$. If $A\in(b)$, then (as seen above, A^* and hence) $(A+R)^*$ has SVEP on $\sigma_a(A)\cap\eta'\sigma_{aw}(A+R)$. We have:

$$\begin{aligned} A+R\in(b) &\Longleftrightarrow (A+R)^* \text{ has SVEP on } \sigma_a(A+R)\cap\eta'\sigma_{aw}(A+R)\\ &\Longleftrightarrow (A+R)^* \text{ has SVEP on } \{\{\sigma_a(A+R)\cap\sigma_a(A)\}\cap\eta'\sigma_{aw}(A+R)\}\\ &\quad\cup\ \{\{\sigma_a(A+R)\setminus\sigma_a(A)\}\cap\eta'\sigma_{aw}(A+R)\}\\ &\Longleftrightarrow (A+R)^* \text{ has SVEP on } \{\sigma_a(A+R)\setminus\sigma_a(A)\}\cap\eta'\sigma_{aw}(A+R).\end{aligned}$$

(C). *Sufficiency.* Recall from Theorem 5.1(ii) that

$$A\in(w)\Longleftrightarrow A\in(b),\ \ E_0(A)\cap\text{iso}\sigma_{aw}(A)=E_0(A)\cap\sigma_{aw}(A)=\emptyset,$$

and from **(B)** above that $A\in(b)$ if and only if one of the hypotheses (*i*) to (*iv*) is satified. Since

$$\Pi_0^a(A+R)\cap\sigma_{aw}(A)^C\subseteq\Pi_0(A+R)\Longrightarrow\sigma_a(A+R)\cap\sigma_{aw}(A+R)^C=\Pi_0(A+R)\subseteq E_0(A+R),$$

the sufficiency would follow once we have proved

$$E_0(A+R) \cap \text{iso}\sigma_{aw}(A+R) = \emptyset \Longrightarrow E_0(A+R) \subseteq \Pi_0(A+R).$$

For every $\lambda \in E_0(A+R)$ such that $\lambda \in E_0(A)$,

$$\begin{array}{rl} & \lambda \notin E_0(A) \cap \text{iso}\sigma_{aw}(A) = E_0(A) \cap \text{iso}\sigma_{aw}(A+R) \\ \Longrightarrow & \lambda \notin E_0(A+R) \cap \text{iso}\sigma_{aw}(A+R) \\ (\Longleftrightarrow & \lambda \notin E_0(A+R) \cap \sigma_{aw}(A+R) \Longleftrightarrow \lambda \in \Pi_0(A+R)). \end{array}$$

Consider next $\lambda \in E_0(A+R)$ such that $\lambda \notin E_0(A)$ $(= \sigma_a(A) \cap \sigma_{aw}(A)^C)$. Since

$$\lambda \notin \sigma_{aw}(A)^C \Longrightarrow \lambda \notin \sigma_{aw}(A+R)^C \Longrightarrow \lambda \notin E_0(A+R) \cap \sigma_{aw}(A+R)^C,$$

we must have ($\lambda \in \sigma_{aw}(A)^C$ and) $\lambda \notin \sigma_a(A)$. But then ($A$, consequently) $A+R$ has SVEP at λ, which then forces $\lambda \in \Pi_0^a(A+R)$. Since $\lambda \in E_0(A+R)$ implies $(A+R)^*$ has SVEP at λ, $\lambda \in \Pi_0^a(A+R)$ implies $\lambda \in \Pi_0(A+R)$.

Necessity. The necessity of the condition $E_0(A+R) \cap \text{iso}\sigma_{aw}(A+R) = \emptyset$ is immediate from the equivalence

$$A+R \in (w) \Longleftrightarrow A+R \in (b), E_0(A+R) \cap \text{iso}\sigma_{aw}(A+R) = \emptyset.$$

Furthermore, since

$$\begin{array}{rcl} A+R \in (b) & \Longleftrightarrow & \sigma_a(A+R) \cap \sigma_{aw}(A+R)^C = \Pi_0(A+R) \\ & \Longrightarrow & \sigma_a(A+R) \cap \sigma_{aw}(A+R)^C = \Pi_0^a(A+R) = \Pi_0(A+R) \\ & \Longrightarrow & \Pi_0^a(A+R) \subseteq \Pi_0(A+R), \end{array}$$

the condition $\Pi_0^a(A+R) \cap \sigma_{aw}(A+R)^C \subseteq \Pi_0(A+R)$ too is necessary. □

Theorem 6.1 extends to operators $A \in (gab)$, $A \in (gb)$ and $A \in (gw)$

Theorem 6.2. *Let $A, R \in B(\mathcal{X})$, where R is a Riesz operator such that $[A, R] = 0$. Then:*

(A) *$A \in (gb) \Longrightarrow A+R \in (gb)$ if and only if $\Pi^a(A+R) \cap \sigma_{aBw}(A+R)^C \subseteq \Pi(A+R)$.*

(B) *$A \in (gab) \Longrightarrow A+R \in (gab)$ if and only if $\text{iso}\sigma_a(A+R) \cap \sigma_{aBw}(A+R)^C \subseteq \text{iso}\sigma_a(A+R) \cap \sigma_{Bw}(A+R)^C$ (equivalently, $\Pi^a(A+R) \subseteq \Pi(A+R)$).*

(C) *For left polaroid operators A, $A \in (gw) \Longrightarrow A+R \in (gw)$ if and only if $(A+R)^*$ has SVEP on $\Pi^a(A+R)$ and $E(A+R) \cap \text{iso}\sigma_{aw}(A+R) = \emptyset$ (equivalently, if and only if $\Pi^a(A+R) \subseteq \Pi(A+R)$ and $E(A+R) \cap \text{iso}\sigma_{aw}(A+R) = \emptyset$).*

Proof. **(A).** Since $\Pi_0^a(A+R) \subseteq \Pi^a(A+R)$ and $\sigma_{aBw}(A+R) \subseteq \sigma_{aw}(A+R)$,

$$\Pi_0^a(A+R) \cap \sigma_{aw}(A+R)^C \subseteq \Pi^a(A+R) \cap \sigma_{aBw}(A+R)^C.$$

Hence the hypothesis $\Pi^a(A+R) \cap \sigma_{aBw}(A+R)^C \subseteq \Pi(A+R)$ implies $(A+R)^*$ has SVEP on $\Pi_0^a(A+R) \cap \sigma_{aw}(A+R)^C = \Pi_0^a(A+R) \cap \sigma_{aw}(A)^C$ $(= \Pi_0^a(A+R))$. Consequently,

$$A \in (gb) \Longrightarrow A \in (b) \Longrightarrow A+R \in (b).$$

Recall from Theorem 5.3(iii) that if $(A+R)^*$ has SVEP on $\Pi^a(A+R) = (\text{iso}\sigma_a(A+R) \cap \sigma_{aBw}(A+R)^C =)$ $\Pi^a(A+R) \cap \sigma_{aBw}(A+R)^C$, then

$$A+R \in (b) \Longrightarrow A+R \in (gb).$$

Hence the condition $\Pi^a(A+R) \cap \sigma_{aBw}(A+R)^C \subseteq \Pi(A+R)$ is sufficient for $A \in (gb)$ to imply $A+R \in (gb)$. To see the necessity of the condition, assume $A \in (gb)$ implies $A+R \in (gb)$. Then

$$\begin{array}{rl} & A \in (gb) \Longrightarrow A+R \in (gb) \Longrightarrow A+R \in (a-gBt) \\ \Longleftrightarrow & \sigma_a(A+R) \cap \sigma_{aBw}(A+R)^C = \Pi^a(A+R), \end{array}$$

and

$$A + R \in (gb) \Longrightarrow \sigma_a(A + R) \cap \sigma_{aBw}(A + R)^C = \Pi(A + R).$$

Hence

$$\Pi^a(A + R) = \sigma_a(A + R) \cap \sigma_{aBw}(A + R)^C = \Pi(A + R).$$

(B). The hypothesis $A \in (gab)$ implies $A \in (ab)$, and hence $A + R \in (ab)$ (see Theorem 6.1**(A)**). Since

$$\begin{aligned} & A + R \in (ab) \Longrightarrow A + R \in (Bt) \Longleftrightarrow A + R \in (gBt) \\ \Longleftrightarrow \quad & \sigma(A + R) \cap \sigma_{Bw}(A + R)^C = \Pi(A + R) \subseteq \Pi^a(A + R), \end{aligned}$$

and since

$$\Pi^a(A + R) = \text{iso}\sigma_a(A + R) \cap \sigma_{aBw}(A + R)^C,$$

the hypothesis $\text{iso}\sigma_a(A + R) \cap \sigma_{aBw}(A + R)^C \subseteq \text{iso}\sigma_a(A + R) \cap \sigma_{Bw}(A + R)^C$ implies

$$\Pi^a(A + R) = \text{iso}\sigma_a(A + R) \cap \sigma_{Bw}(A + R)^C = \Pi(A + R).$$

(Recall that $\lambda \in \text{iso}\sigma_a(A + R) \cap \sigma_{Bw}(A + R)^C$ if and only if $\lambda \in \text{iso}\sigma(A + R) \cap \sigma_{Bw}(A + R)^C$.) This proves the sufficiency of the condition.

Conversely, if $A \in (gab)$ implies $A + R \in (gab)$, then

$$\begin{aligned} \sigma(A) \cap \sigma_{Bw}(A)^C &= \Pi^a(A) \Longrightarrow \sigma(A + R) \cap \sigma_{Bw}(A + R)^C \\ &= \text{iso}\sigma_a(A + R) \cap \sigma_{aBw}(A + R)^C = \Pi^a(A + R). \end{aligned}$$

Hence, since

$$\begin{aligned} A \in (gab) \quad &\Longrightarrow \quad A \in (gBt) \Longleftrightarrow A + R \in (gBT) \\ &\Longleftrightarrow \quad \sigma(A + R) \cap \sigma_{Bw}(A + R)^c = \Pi(A + R) \\ &= \quad \{\lambda : \lambda \in \text{iso}\sigma_a(A + R) \cap \sigma_{Bw}(A + R)^C\}, \end{aligned}$$

we must have

$$\begin{aligned} & \{\lambda : \lambda \in \text{iso}\sigma_a(A + R) \cap \sigma_{aBw}(A + R)^C\} \\ = \quad & \{\lambda : \lambda \in \text{iso}\sigma_a(A + R) \cap \sigma_{Bw}(A + R)^C\} \\ \Longleftrightarrow \quad & \{\lambda : \lambda \in \text{iso}\sigma_a(A + R) \cap \sigma_{aBw}(A + R)^C\} \\ \subseteq \quad & \{\lambda : \lambda \in \text{iso}\sigma_a(A + R) \cap \sigma_{Bw}(A + R)^C\}. \end{aligned}$$

(C). The hypothesis $A \in (gw)$ implies $A \in (w)$. Since $\Pi_0^a(A + R) \subseteq \Pi^a(A + R)$, the hypothesis $(A + R)^*$ has SVEP on $\Pi^a(A + R)$ implies $(A + R)^*$ has SVEP on $\Pi_0^a(A + R)$ (and hence $\Pi_0^a(A + R) \cap \sigma_w(A + R)^C = \Pi_0(A + R)$). Again, since $E_0(A + R) \subseteq E(A + R)$, $E(A + R) \cap \text{iso}\sigma_{aw}(A + R) = \emptyset$ implies $E_0(A + R) \cap \text{iso}\sigma_{aw}(A + R) = \emptyset$. Consequently,

$$A \in (gw) \Longrightarrow A \in (w) \Longrightarrow A + R \in (w).$$

Assume now that $A + R$ is left polaroid. Then, since (already $(A + R)^*$ has SVEP on $\Pi^a(A + R)$, and therefore) $\Pi^a(A + R) = \Pi(A + R)$, Theorem 5.3(iv) applies and we conclude that $A + R \in (gw)$. This proves the sufficiency.

The necessity of the conditions $(A + R)^*$ has SVEP on $\Pi^a(A + R)$ and $E(A + R) \cap \sigma_{aw}(A + R) = \emptyset$ is immediate from $A + R \in (gw)$ if and only if $\sigma_a(A + R) \cap \sigma_{aBw}(A + R)^C = E(A + R)$ (once one observes the fact that $E(A + R) \cap \sigma_{aBw}(A + R) = \emptyset \Longrightarrow E(A + R) \cap \sigma_{aw}(A + R) = \emptyset$.) $\square$

7. Perturbation by commuting finite rank, nilpotent and quasinilpotent operators

Theorems 6.1 and 6.2 subsume a number of extant results on the perturbation of operators satisfying either of the properties (b), (ab), (w) and their generalizations. In the following we consider but a few of these results on perturbation by commuting finite rank, nilpotent and quasinilpotent operators , starting with the result that $A \in (b) \Longleftrightarrow A+R \in (b)$ for R either quasinilpotent or finite rank such that $\text{iso}\sigma_a(A) = \text{iso}\sigma_a(A+R)$. But before that we recall that if $[A,R] = 0$ and: (i) R is quasinilpotent, then $\sigma_x(A+R) = \sigma_x(A)$, $\sigma_x = \sigma$ or σ_a; (ii) R^n is finite rank, then $\text{acc}\sigma(\text{A} + \text{R}) = \text{acc}\sigma(\text{A})$.

Proposition 7.1. *Given $A, R \in B(\mathcal{X})$ such that $[A,R] = 0$, if:*

(A) *R is quasinilpotent, then $A \in (b) \Longleftrightarrow A+R \in (b)$.*

(B) *R^n is finite rank for some integer $n > 0$ and $\text{iso}\sigma_a(A) = \text{iso}\sigma_a(A+R)$, then $A \in (b) \Longleftrightarrow A+R \in (b)$.*

Proof. The proof in both the cases is almost a direct consequence of the fact that if $[A,R] = 0$ and R is quasinilpotent, or, R^n is finite rank with $\text{iso}\sigma_a(A) = \text{iso}\sigma_a(A+R)$, then $\sigma_a(A) = \sigma_a(A+R)$. Since A^* has SVEP at a point if and only if $(A+R)^*$ has SVEP at the point, and since $\sigma_a(A) \cap \eta'\sigma_{aw}(A) = \sigma_a(A+R) \cap \eta'\sigma_{aw}(A+R)$,

$$\begin{aligned} A \in (b) &\Longleftrightarrow A^* \text{ has SVEP on } \sigma_a(A) \cap \eta'\sigma_{aw}(A) \\ &\Longleftrightarrow (A+R)^* \text{ has SVEP on } \sigma_a(A+R) \cap \eta'\sigma_{aw}(A+R), \end{aligned}$$

i.e., $A \in (b) \Longleftrightarrow A+R \in (b)$. □

Remark 7.2. Since $\sigma_a(A) \cap \sigma_{aw}(A+R)^C = \{(\sigma_a(A) \cap \sigma_a(A+R)) \cap \sigma_{aw}(A+R)^C\} \cup \{(\sigma_a(A) \setminus \sigma_a(A+R)) \cap \sigma_{aw}(A+R)^C\}$, and if A is polaroid $\sigma_a(A) \setminus \sigma_a(A+R) \cap \sigma_{aw}(A+R)^C = \text{iso}\sigma(A) \cap \sigma_a(A+R)^C$ whenever $(A+R)^*$ has SVEP on $\sigma_a(A) \cap \sigma_{aw}(A)^C$, the hypothesis $\text{iso}\sigma_a(A) = \text{iso}\sigma_a(A+R)$, in the case in which R^n is finite rank in Proposition 7.1, may be replaced by a hypothesis guaranteeing $\text{iso}\sigma(A) \cap \sigma_a(A+R)^C = \emptyset$.

The equivalence of Proposition 7.1 extends to property (gb) for commuting nilpotent, and finite rank operators satisfying $\sigma_a(A) = \sigma_a(A+R)$.

Proposition 7.3. *Given operators $A, R \in B(\mathcal{X})$ such that $[A,R] = 0$ and either (i) R is nilpotent, or, (ii) R^n is finite rank with $\sigma_a(A) = \sigma_a(A+R)$, then $A \in (gb)$ if and only if $A+R \in (gb)$.*

Proof. If (ii) is satisfied, then $\sigma_a(A+R) = \sigma_a(A)$ and (since semi B-Fredholm spectrum is stable under perturbations by finite rank operators [11, Proposition 2.7]) $\sigma_{aBw}(A+R) = \sigma_{aBw}(A)$. Hence A^* has SVEP on $\sigma_a(A) \cap \eta'\sigma_{aBw}(A)$ if and only if $(A+R)^*$ has SVEP on $\sigma_a(A+R) \cap \eta'\sigma_{aBw}(A+R)$. This ensures $A \in (gb) \Longleftrightarrow A+R \in (gb)$. If A is nilpotent, then A (A^*) has finite ascent if and only if $A+R$ (resp., $(A+R)^*$) has finite ascent [19, Lemma 2.5]. Hence, for $\lambda \in \sigma_{aBw}(A)^C$, λ is a pole of the resolvent of A if and only if λ is a pole of the resolvent of $A+R$. This implies $A \in (gb) \Longleftrightarrow A+R \in (gb)$. □

The equivalence $A \in (gb) \Longleftrightarrow A+R \in (gb)$ fails for commuting quasinilpotents R. Let $T = U \oplus 0 \in B(\ell^2 \oplus \ell^2)$, where U is the forward unilateral shift and 0 is the zero operator. Let $Q \in B(\ell^2)$ be a non-nilpotent quasnilpotent operator, and $R = 0 \oplus Q$. Define the operator $A \in B(\ell^2 \oplus \ell^2)$ by $A = U \oplus Q$. Then $\sigma_a(A) = \partial\mathcal{D} \cup \{0\} = \sigma_{aBw}(A), \Pi(A) = \emptyset$ and $A \in (gb)$ and the operator $T = A - R$ satisfies $\sigma_a(T) = \partial\mathcal{D} \cup \{0\}, \sigma_{aBw}(T) = \partial\mathcal{D}, \Pi^a(T) = \{0\}$ and $\Pi(T) = \emptyset$. Hence $\sigma_a(A-R) \cap \sigma_{aBw}(A-R)^C = \Pi^a(A-R)$ and $A - R \notin (gb)$.

Remark 7.4. The above example is in a way typical of operators for which $A \in (gb)$ does not imply $A+R \in (gb)$ for comuting quasinilpotent (indeed, Riesz) operators R (see Theorem 6.2**(A)**, which says that we must have $\Pi^a(A+R) \subseteq \Pi(A+R)$).

We consider next implications $A \in (w) \Longrightarrow A+R \in (w)$ and $A \in (gw) \Longrightarrow A+R \in (gw)$ for commuting finite rank and quasinilpotent operators R. If R is either nilpotent, or if R^n is finite rank with $\text{iso}\sigma_a(A) = \text{iso}\sigma_a(A+R)$, and $[A,R] = 0$, then $\sigma_a(A) = \sigma_a(A+R)$ and $\sigma_{aw}(A) = \sigma_{aw}(A+R)$. Hence, if $A \in (w)$, then $E_0(A) = \sigma_a(A) \cap \sigma_{aw}(A)^C = \sigma_a(A+R) \cap \sigma_{aw}(A+R)^C = \Pi_0(A+R) \subseteq E_0(A+R)$. The reverse inclusion $E_0(A+R) \subseteq E_0(A)$ fails. Consider, for example, the operator $A+R$, where $A \in B(\ell^2)$ is the operator $A(x_1, x_2, x_3, ...) = (0, \frac{1}{2}x_1, \frac{1}{3}x_2, ...)$ and $R \in B(\ell^2)$ is the nilpotent, finite rank operator $R(x_1, x_2, x_3, ...) = (0, -\frac{1}{2}x_1, 0, ...)$, when it is seen that $(E_0(A+R) \cap \text{iso}\sigma_{aw}(A+R) \neq \emptyset$ and) $E_0(A+R) \not\subseteq E_0(A)$.

Proposition 7.5. *If A, R are the operators of Proposition 7.3, then $A \in (w) \iff A + R \in (w)$ if and only if $E_0(A + R) = E_0(A)$.*

Proof. The proof is immediate from the argument above which shows that $A \in (w)$ implies $E_0(A) = \sigma_a(A + R) \cap \sigma_{aw}(A + R)^C \subseteq E_0(A + R)$, and $A + R \in (w)$ implies $E_0(A + R) = \sigma_a((A + R) - R) \cap \sigma_{aw}((A + R) - R)^C \subseteq E_0(A)$. □

The equality $E_0(A) = E_0(A + R)$ in the above proposition may be achieved in a number of ways. Thus, if A is left polaroid (or, polaroid) and $A \in (w)$, then ($\sigma_x(A + R) = \sigma_x(A)$, $\sigma_x = \sigma$ or σ_a, and)

$$\begin{aligned} \lambda \in E_0(A + R) &\implies \lambda \in \text{iso}\sigma_a(A + R) = \text{iso}\sigma_a(A), (A + R)^* \text{ has SVEP at } \lambda \\ &\quad\ \text{(resp., } \lambda \in \text{iso}\sigma(A + R) = \text{iso}\sigma(A) \text{ if } A \text{ is polaroid)} \\ &\implies \lambda \in \Pi(A) \subseteq E(A); \end{aligned}$$

since $0 < \alpha(A + R - \lambda) < \infty \implies 0 < \alpha(A - \lambda) < \infty$, $\lambda \in E_0(A)$. Again, if $A \in (w)$ and $A + R$ is left polaroid (or, polaroid), then

$$\lambda \in E_0(A + R) \implies \lambda \in \Pi_0(A + R) \implies \lambda \in \Pi_0(A) \subseteq E_0(A).$$

The hypothesis $A \in (w)$ is isoloid (Recall: A is isoloid if $\lambda \in \text{iso}\sigma(A) \implies \lambda \in E(A)$) implies $E_0(A + R) = E_0(A)$ (and, by symmetry, the hypothesis $A + R \in (w)$ is isoloid implies $E_0(A) = E_0(A + R)$).

Perturbation by commuting quasinilpotents fails to satisfy the condition $E_0(A + R) \cap \text{iso}\sigma_{aw}(A + R) = \emptyset$. Consider the operators $A = 0$ and $R(x_1, x_2, x_3, ...) = (\frac{1}{2}x_2, \frac{1}{3}x_3, ...)$. Then $A \in (w)$, $R \in B(\ell^2)$ is (a non-injective) quasinilpotent, $\sigma_a(A + R) = \sigma_{aw}(A + R) = \{0\}$, $\Pi_0^a(A + R) = \Pi_0(A + R) = \emptyset$, $E_0(A + R) = \{0\}$ ($\neq E_0(A) = \emptyset$) and $A + R \notin (w)$. The following proposition shows that $A \in (w)$ implies $A + R \in (w)$ for commuting injective quasinilpotent operators R, and for commuting quasinilpotent operators R for which the operator A is finitely left polaroid.

Proposition 7.6. *Given operators $A, R \in B(\mathcal{X})$ such that R is a quasinilpotent which commutes with A, if:*
(i) Either R is injective, or A is finitely left polaroid, then $A \in (w)$ if and only if $A + R \in (w)$. (ii) A is left polaroid, then $A \in (w)$ implies $A + R \in (w)$ if and only if $\sigma_{aBw}(A)^C \cap \text{iso}\sigma_{aw}(A) = \emptyset$.

Proof. (i). We prove the forward implication for the case in which R is an injective quasinilpotent; the proof of the backward implication follows by symmetry. For this we prove that

$$\Pi_0^a(A + R) = \Pi_0(A + R), E_0(A + R) \cap \text{iso}\sigma_{aw}(A + R) = \emptyset$$

(see Theorem 6.1**(C)**). The operator R being quasinilpotent, $[A, R] = 0$ implies $\sigma_a(A) = \sigma_a(A + R)$, $\sigma_{aw}(A) = \sigma_{aw}(A + R)$, and A (A^*) has SVEP at a point if and only if $A + R$ (resp., $(A + R)^*$) has SVEP at the point. If $A \in (w)$, then

$$\begin{aligned} E_0(A) &= \sigma_a(A) \cap \sigma_{aw}(A)^C = \sigma_a(A + R) \cap \sigma_{aw}(A + R)^C \\ &= \Pi_0^a(A + R) \ \text{ (since } A + R \text{ has SVEP on } E_0(A)) \\ &= \Pi_0(A + R) \ \text{ (since } (A + R)^* \text{ has SVEP on } E_0(A)). \end{aligned}$$

Recall now from [18, Theorem 8.4.8, Page 133] (else, see the proof of [2, Theorem 2.13]) that if R is injective, then $\Pi_0(A) = \Pi_0(A + R) = E_0(A + R) = \emptyset$; hence $E_0(A + R) \cap \sigma_{aw}(A + R) = \emptyset$. (Indeed, $A \in (w) \implies E_0(A) = E_0(A + R) = \sigma_a(A) \cap \sigma_{aw}(A)^C = \sigma_w(A + R) \cap \sigma_{aw}(A + R)^C = \emptyset$.)

We prove next the case in which A is finitely left polaroid. Here, since A is finitely left polaroid if and only if $A + R$ is finitely left polaroid (see Proposition 3.2), it would suffice to prove the forward implication $A \in (w) \implies A + R \in (w)$. Furthermore, since $A \in (w)$ implies $\Pi_0^a(A + R) = \Pi_0(A + R)$ (a straightforward consequence, as seen above, of the fact that the definition of $A \in (w)$ implies both A and A^* have SVEP on $\sigma_a(A) \cap \sigma_{aw}(A)^C$), we have only to prove that $E_0(A + R) \cap \text{iso}\sigma_{aw}(A + R) = \emptyset$. Assume that there exists a $\lambda \in E_0(A + R) \cap \text{iso}\sigma_{aw}(A + R)$. Then , since

$$\begin{aligned} &\lambda \in \text{iso}\sigma_a(A + R) \implies \lambda \in \text{iso}\sigma_a(A) \\ \implies\ &\lambda \in \Pi_0^a(A) = \sigma_a(A) \cap \sigma_{aw}(A)^C = \sigma_a(A + R) \cap \sigma_{aw}(A + R)^C, \end{aligned}$$

$\lambda \notin \sigma_{aw}(A+R)$. This being a contradiction, we must have $E_0(A+R) \cap \text{iso}\sigma_{aw}(A+R) = \emptyset$.

(ii). It is clear from the argument of the proof of (i) above that if $A \in (w)$ is left polaroid, then $\Pi_0^a(A+R) = \Pi_0(A+R)$, $E_0(A) \subseteq E_0(A+R)$ and $\lambda \in E_0(A+R)$ implies $\lambda \in \Pi^a(A) = \text{iso}\sigma_a(A) \cap \sigma_{aBw}(A)^C$, A^* has SVEP at λ. Since $\text{iso}\sigma_a(A) \cap \sigma_{aBw}(A)^C = \{(\text{iso}\sigma_a(A) \cap \sigma_{aBw}(A)^C) \cap (\sigma_{aw}(A) \cup \sigma_{aw}(A)^C)\} = \{(\text{iso}\sigma_a(A) \cap \sigma_{aBw}(A)^C) \cap \sigma_{aw}(A)^C\} = \text{iso}\sigma_a(A) \cap \sigma_{aw}(A)^C$ if and only if

$$(\text{iso}\sigma_a(A) \cap \sigma_{aBw}(A)^C) \cap \sigma_{aw}(A) = \emptyset \iff \sigma_{aBw}(A)^C \cap \text{iso}\sigma_{aw}(A) = \emptyset,$$

we have $\lambda \in E_0(A+R)$ implies $\lambda \in \Pi_0^a(A)$ $(= \Pi_0(A) = E_0(A)$, since $A \in (w))$ if and only if $\sigma_{aBw}(A)^C \cap \text{iso}\sigma_{aw}(A) = \emptyset$. Thus $E_0(A+R) \subseteq E_0(A)$ (implies $E_0(A+R) = E_0(A)$ and $E_0(A+R) \cap \text{iso}\sigma_{aw}(A+R) = \emptyset$, implies $A+R \in (w)$) if and only if $\sigma_{aBw}(A)^C \cap \text{iso}\sigma_{aw}(A) = \emptyset$. □

Property (gw) does not, as one would expect, survive perturbation by commuting finite rank and nilpotent operators . If we let $A \in B(\ell^2)$ be the quasinilpotent operator $A(x_1, x_2, x_3, ...) = (0, \frac{x_1}{2}, \frac{x_2}{3}, \cdots)$, then $\sigma_a(A) = \sigma_{aBw}(A) = \{0\}$, $E(A) = \emptyset$ and $A \in (gw)$. Let $R \in B(\ell^2)$ be the finite rank (nilpotent) operator $R(x_1, x_2, x_3, \cdots) = (0, -\frac{x_1}{2}, 0, \cdots)$, then $\sigma_a(A+R) = \sigma_{aBw}(A+R) = E(A+R) = \{0\}$ and $A+R \notin (gw)$. However, just as for property (w), a satisfactory situation prevails for the case in which either A is left polaroid and R is the finite rank operator of Proposition 7.3, or, A is polaroid and R is the operator of Proposition 7.3.

Proposition 7.7. *Let $A, R \in B(\mathcal{X})$, where $[A, R] = 0$.*
(i) If A is left polaroid and R is the finite rank operator of Proposition 7.3, then $A \in (gw)$ if and only if $A+R \in (gw)$.
(ii) If A is polaroid and R is the operator of Proposition 7.3, then $A \in (gw)$ if and only if $A+R \in (gw)$.
(iii) If $\mathcal{X} = \mathcal{H}$ is a Hilbert space and the operator R is nilpotent, then $A \in (gw)$ if and only if $A+R \in (gw)$.

Proof. (i). We prove the forward implication; the reverse implication follows by symmetry (since A is left polaroid if and only if $A+R$ is left polaroid by Proposition 3.3). Since the left polaroid hypothesis on A implies

$$\begin{aligned} \lambda \in E(A+R) &\implies \lambda \in \Pi^a(A+R), (A+R)^* \text{ has SVEP at } \lambda, 0 < \alpha(A+R-\lambda) \\ &\implies \lambda \in \Pi(A+R), 0 < \alpha(A+R-\lambda) \implies \lambda \in E(A+R), \end{aligned}$$

we have

$$\begin{aligned} & \lambda \in E(A) \implies \lambda \in \text{iso}\sigma(A) = \text{iso}\sigma(A+R), 0 < \alpha(A-\lambda) \\ \implies & \lambda \in \Pi(A+R), 0 < \alpha(A+R-\lambda) \implies \lambda \in E(A+R). \end{aligned}$$

(We remark here that if $\alpha(A+R-\lambda) = 0$, then $\lambda \notin \sigma_a(A+R) = \sigma_a(A)$.) Thus

$$E(A) \subseteq E(A+R) \subseteq E((A+R)-R) = E(A) \implies E(A) = E(A+R).$$

Since $A \in (gw)$ implies $E(A) \cap \text{iso}\sigma_{aw}(A) = \emptyset$, we must have $E(A+R) \cap \text{iso}\sigma_{aw}(A+R) = \emptyset$. Already we have $\Pi^a(A+R) = \Pi(A+R)$. Hence $A+R \in (gw)$.

(ii). Again, we prove the forward implication. (Recall from Proposition 3.3 that A is polaroid if and only if $A+R$ is polaroid.) Suppose $A \in (gw)$. If R is the finite rank operator of the statement, then $\sigma_{aBw}(A) = \sigma_{aBw}(A+R)$ (see the proof of Proposition 7.3), and if R is the nilpotent operator of the statement, then $\sigma_{Bb}(A) = \sigma_{Bb}(A+R)$ [19, Theorem 2.6]. Since $\sigma_x(A) = \sigma_x(A+R)$, $\sigma_x = \sigma$ or σ_a, if R is the finite rank operator of the statement, then

$$\begin{aligned} \Pi^a(A+R) &= \{\lambda : \lambda \in \sigma_a(A+R) \cap \sigma_{aBw}(A+R)^C, A+R \text{ has SVEP at } \lambda\} \\ &= \{\lambda : \lambda \in \sigma_a(A) \cap \sigma_{aBw}(A)^C, A \text{ and } A^* \text{ have SVEP at } \lambda \\ &\qquad (\text{since } A \in (gw))\} \\ &= \{\lambda : \lambda \in \sigma_a(A+R) \cap \sigma_{aBw}(A+R)^C, A+R \text{ and } (A+R)^* \\ &\qquad \text{have SVEP at } \lambda\} = \Pi(A+R) \end{aligned}$$

and if R is the nilpotent operator of the statement, then

$$\begin{aligned}
\Pi^a(A+R) &= \{\lambda : \lambda \in \sigma_a(A+R) \cap \sigma_{aBw}(A+R)^C, A+R \text{ has SVEP at } \lambda\}\\
&= \{\lambda : \lambda \in \sigma_a(A) \cap \sigma_{aBw}(A)^C, A \text{ and } A^* \text{ have SVEP at } \lambda\}\\
&= \{\lambda : \lambda \in \sigma_a(A) \cap \sigma_{Bb}(A)^C\} = \{\lambda : \lambda \in \sigma_a(A+R) \cap \sigma_{Bb}(A+R)^C\}\\
&= \{\lambda : \lambda \in \Pi^a(A+R), (A+R)^* \text{ has SVEP at } \lambda\} = \Pi(A+R).
\end{aligned}$$

The polaroid property of A implies

$$\begin{aligned}
\lambda \in E(A+R) &\Longrightarrow \lambda \in \text{iso}\sigma(A+R), 0 < \alpha(A+R-\lambda)\\
&\Longleftrightarrow \lambda \in \text{iso}\sigma(A), 0 < \alpha(A+R-\lambda) \Longrightarrow \lambda \in \text{iso}\sigma(A), 0 < \alpha(A-\lambda)\\
&\Longrightarrow \lambda \in E(A) = E((A+R)-R) \Longrightarrow \lambda \in E(A+R).
\end{aligned}$$

Hence $E(A+R) = E(A)$, $E(A+R) \cap \text{iso}\sigma_{aw}(A+R) = E(A) \cap \text{iso}\sigma_{aw}(A) = \emptyset$, and $A+R \in (gw)$.

(iii). If $\mathcal{X}$ is a Hilbert space, then $\sigma_{aBw} = \sigma_{aBw}(A+R)$ (see [6, Theorem 4.4] and [18, Corollary 2.9]). □

Property (gw) does not travel from A to $A+R$ for commuting quasinilpotents (even, commuting injective quasinilpotents). Let, for example, $A = U \oplus Q \in B(\ell^2 \oplus \ell^2)$, where U is the forward unilateral shift and Q is an injective quasinilpotent. Let $R = -(Q \oplus Q)$. Then $\sigma_a(A) = \partial\mathcal{D} \cup \{0\} = \sigma_{aBw}(A)$, $E(A) = \emptyset$, $A \in (gw)$ and $\sigma_a(A+R) = \partial\mathcal{D} \cup \{0\}, \sigma_{aBw}(A+R) = \partial\mathcal{D}, E(A+R) = \emptyset, A+R \notin (gw)$. Requiring A is finitely polaroid, however, does the trick.

Proposition 7.8. *If the operators $A, R \in B(\mathcal{X})$ are such that $[A,R] = 0$, A is finitely polaroid and R is a quasinilpotent, then $A \in (gw)$ if and only if $A+R \in (gw)$.*

Proof. In view of the fact that A is finitely polaroid if and only if $A+R$ is finitely polaroid, Proposition 3.2, it would suffice to prove the forward implication. If A is finitely polaroid, then

$$A \in (gw) \Longleftrightarrow \sigma_a(A) \cap \sigma_{aBw}(A)^C = E(A) = E_0(A).$$

Since $\lambda \in \sigma_{aBw}(A)^C$ and $0 < \alpha(A-\lambda) < \infty$ imply $\lambda \in \sigma_{aw}(A)^C$,

$$\begin{aligned}
A \in (gw) &\Longrightarrow \sigma_a(A) \cap \sigma_{aw}(A)^C = E_0(A)\\
&\Longleftrightarrow \sigma_a(A+R) \cap \sigma_{aw}(A+R)^C = E_0(A+R) \quad \text{(Proposition 7.6)}\\
&\Longrightarrow \sigma_a(A+R) \cap \sigma_{aBw}(A+R)^C = E(A+R)
\end{aligned}$$

(since $A+R$ is finitely polaroid). □

We end this section with the result that the hypotheses of Proposition 7.6 (ii) are sufficient for the transfer of property (gw) from A to $A+R$ for commuting quasinilpotents R.

Proposition 7.9. *Let $A, R \in B(\mathcal{X})$, where R is a quasinilpotent operator such that $[A,R] = 0$. If $\text{iso}\sigma_{aw}(A) \cap \sigma_{aBw}(A) = \emptyset$, then $A \in (gw)$ implies $A+R \in (gw)$.*

.

Proof. If $\text{iso}\sigma_{aw}(A) \cap \sigma_{aBw}(A) = \emptyset$, then (the argument of the proof of Proposition 7.6 (ii) implies that) $\text{iso}\sigma_a(A) \cap \sigma_{aBw}(A)^C = \text{iso}\sigma_a(A) \cap \sigma_{aw}(A)^C$. The hypothesis $A \in (gw)$ implies

$$\begin{aligned}
E(A) &= \sigma_a(A) \cap \sigma_{aBw}(A)^C = \{\lambda : \lambda \in \text{iso}\sigma_a(A) \cap \sigma_{aBw}(A)^C, A^* \text{ has SVEP at } \lambda\}\\
&= \{\lambda : \lambda \in \text{iso}\sigma_a(A) \cap \sigma_{aw}(A)^C, A^* \text{ has SVEP at } \lambda\}\\
&= \{\lambda : \lambda \in \text{iso}\sigma_a(A+R) \cap \sigma_{aw}(A+R)^C, (A+R)^* \text{ has SVEP at } \lambda\}\\
&= \{\lambda : \lambda \in \Pi_0^a(A+R), (A+R)^* \text{ has SVEP at } \lambda\}\\
&= \{\lambda : \lambda \in \Pi_0(A+R), 0 < \alpha(A+R-\lambda)\\
&\quad (\text{since } \alpha(A+R-\lambda) = 0 \text{ forces } \lambda \notin \sigma(A) = \sigma(A+R))\}\\
&\subseteq \{\lambda : \lambda \in \Pi(A+R), 0 < \alpha(A+R-\lambda)\} \subseteq E(A+R).
\end{aligned}$$

Again, since

$$\begin{aligned} E(A+R) &= \{\lambda : \lambda \in \mathrm{iso}\sigma(A+R), 0 < \alpha(A+R-\lambda)\} \\ &= \{\lambda : \lambda \in \mathrm{iso}\sigma(A), 0 < \alpha(A+R-\lambda)\} \\ &= \{\lambda : \lambda \in \Pi^a(A), A^* \text{ has SVEP at } \lambda, 0 < \alpha(A-\lambda) \\ &\quad (\text{since } \alpha(A-\lambda) = 0 \Longrightarrow \lambda \notin \sigma(A) = \sigma(A+R))\} \\ &= \{\lambda : \lambda \in \Pi(A), 0 < \alpha(A-\lambda)\} \subseteq E(A), \end{aligned}$$

we have

$$E(A+R) = \Pi(A+R) = E(A) \Longrightarrow E(A+R) \cap \sigma_{aBw}(A+R) = \emptyset.$$

Finally, since $\lambda \in \Pi^a(A+R)$ implies ($\lambda \in \mathrm{iso}\sigma_a(A+R)$ implies) $\lambda \in \mathrm{iso}\sigma_a(A)$, $\lambda \in \Pi^a(A) = \mathrm{iso}\sigma_a(A) \cap \sigma_{aBw}(A)^C = E(a) = E(A+R) = \Pi(A+R)$. $\square$

8. Perturbation by non-commuting compact operators

Compact operators $K \in B(\mathcal{X})$ being Riesz, the results above cover the transfer of propertise (Bt), (Wt), (gw) etc. from $A \in B(\mathcal{X})$ to $A+K$ for commuting K. The commutativity here is essential, (in a number of cases) for the reason that SVEP does not survive perturbation by (not necessarily commuting) compact operators. A pertinent example here is that of the unitary operator $A = \begin{pmatrix} U & 1-UU^* \\ 0 & U^* \end{pmatrix} \in B(\mathcal{H} \oplus \mathcal{H})$, U the forward unilateral shift, and its perturbation by the compact operator $K = \begin{pmatrix} 0 & -1+UU^* \\ 0 & 0 \end{pmatrix} \in B(\mathcal{H} \oplus \mathcal{H})$. Since the operator A has SVEP and is polaroid (with $\sigma(A) = \sigma_a(A) = \sigma_w(A) = \sigma_{aw}(A) = \partial\mathcal{D}$ and $\Pi_0^a(A) = \Pi^a(A) = \Pi_0(A) = \Pi(A) = E_0^a(A) = E_0(A) = E^a(A) = E(A) = \emptyset$, A satisfies (all) the properties (thus far) considered. However, the operator $A+K$, which satisfies $\sigma(A+K) = \sigma_a(A+K) = \overline{\mathcal{D}}$ and $\sigma_w(A+K) = \sigma_{aw}(A+K) = \partial\mathcal{D}$, does not satisfy any of the properties (Bt), (Wt), (w), (gw) etc. Observe that $\eta'\sigma_w(A) = \eta'\sigma_{aw}(A) = \mathcal{D}$ (the open unit disc), i.e., the Weyl and the a-Weyl spectra of A have holes. The absence of holes leads to (some) positive results (see [21, 22, 28, 33]). Thus, if an operator $A \in B(\mathcal{X})$ is such that $\sigma_w(A)$ (resp., $\sigma_{aw}(A)$) has no holes, then $A+K \in (Bt)$ (resp., $A+K \in (a-Bt)$) for every compact operator $K \in B(\mathcal{X})$ [21, Theorem 4.1]; if also $\mathrm{iso}\sigma_w(A) = \emptyset$ (resp., $\mathrm{iso}\sigma_{aw}(A) = \emptyset$), then $A+K \in (Wt)$ (resp., $A+K \in (a-Wt)$) for every compact operator $K \in B(\mathcal{X})$ [21, Theorem 6.4]. For operators $A \in B(\mathcal{X})$ such that $\sigma(A) = \sigma_w(A)$, a sufficient condition for $A+K \in (Bt)$ for every compact operator $K \in B(\mathcal{X})$ is that $\eta'\sigma(A)$ is a finite union of the holes of $\sigma(A)$ [22, Theorem 3.2]. Again, if the component $\Omega_a(A) = \{\lambda \in \Phi_+(A) : \mathrm{ind}(A-\lambda) \leq 0\}$ is connected for an $A \in B(\mathcal{X})$, then $A+K \in (gb)$ for every compact operator $K \in B(\mathcal{X})$ if and only if A^* has SVEP on $\sigma_a(A) \cap \sigma_{aw}(A)^C$ [21, Theorem 5.2]. Observe from Theorem 5.1 above that $A+K \in (b)$ (resp., $A+K \in (w)$), for operators $A, K \in B(\mathcal{X})$ with K compact, if and only if $(A+K)^*$ has SVEP on $\sigma_a(A+K) \cap \eta'\sigma_{aw}(A)$ (resp., if and only if $(A+K) \in (b)$ and $E_0(A+K) \cap \mathrm{iso}\sigma_{aw}(A) = \emptyset$). The example of the operator $B = U \oplus U^*$, $U \in B(\mathcal{H})$ the forward unilateral shift, the compact operator $K = \begin{pmatrix} 0 & -1+UU^* \\ 0 & 0 \end{pmatrix}$ and $A = B-K$ shows that $A \in (b)$ does not imply $A+K \in (b)$. Observe that $\eta'\sigma_{aw}(A+K) = \eta'\sigma_{aw}(A) = \mathcal{D}$ and $\sigma_a(A+K) \cap \sigma_{aw}(A+K)^C \nsubseteq \mathrm{iso}\sigma_a(A+K)$ $(= \emptyset)$. Again, if we let $A \in U \oplus 0 \in B(\ell^2 \oplus \ell^2)$, then $\sigma_a(A) = \sigma_{aw}(A) = \partial\mathcal{D} \cup \{0\}$, $\Pi_0(A) = \emptyset$ and $A \in (b)$. Let $E \in B(\ell^2)$ be the compact operator $E(x_1, x_2, x_3, \ldots) = (0, \frac{x_2}{2}, \frac{x_3}{3}, \ldots)$. Then the operator $K = 0 \oplus E$ is compact and the operator $A+K$ satisfies $\sigma_a(A+K) = \partial\mathcal{D} \cup \{0, \frac{1}{2}, \frac{1}{3}, \ldots\}$, $\sigma_{aw}(A+K) = \partial\mathcal{D} \cup \{0\}$, $\Pi_0(A) = \emptyset$ and $A+K \notin (b)$. Observe that $\sigma_a(A+K) \cap \sigma_{aw}(A+K)^C \nsubseteq \mathrm{iso}\sigma_a(A+K)$ $(= \emptyset)$. The following theorem shows that the hypotheses $\sigma_a(A+K) \cap \sigma_{aw}(A+K)^C \subseteq \mathrm{iso}\sigma_a(A+K)$ and $iso\sigma_a(A+K) \cap \sigma_a(A)^C \subseteq \mathrm{iso}\sigma(A+K)$ are both necessary and sufficient for $A \in (b)$ to imply $A+K \in (b)$.

Theorem 8.1. *Given operators $A, K \in B(\mathcal{X})$ with B compact, $A \in (b)$ implies $A+K \in (b)$ if and only if $\sigma_a(A+K) \cap \sigma_{aw}(A+K)^C \subseteq iso\sigma_a(A+K)$ and $iso\sigma_a(A+K) \cap \sigma_a(A)^C \subseteq iso\sigma(A+K)$.*

Proof. Sufficiency. Assume that $\sigma_a(A+K)\cap\sigma_{aw}(A+K)^C \subseteq \mathrm{iso}\sigma_a(A+K)$ and $\mathrm{iso}\sigma_a(A+K)\cap\sigma_a(A)^C \subseteq \mathrm{iso}\sigma(A+K)$. The hypothesis $A \in (b)$ implies both A and A^* have SVEP on $\sigma_a(A) \cap \eta'\sigma_{aw}(A)$. Since A^* has SVEP at a point $\lambda \in \eta'\sigma_{aw}(A)$ implies $\lambda \in \eta'\sigma_w(A)$, if $A \in (b)$, then

$$\begin{aligned} & \{\lambda : \lambda \in \sigma_a(A) \cap \eta'\sigma_{aw}(A), A^* \text{ has SVEP at } \lambda\} \\ = & \{\lambda : \lambda \in \sigma_a(A) \cap \eta'\sigma_w(A), A \text{ has SVEP at } \lambda\} \\ = & \{\lambda : \lambda \in \sigma_a(A) \cap \eta'\sigma_w(A+K), A \text{ has SVEP at } \lambda\}. \end{aligned}$$

Assume now that $A \in (b)$, and consider the set

$$\begin{aligned} & \{\lambda : \lambda \in \sigma_a(A+K) \cap \eta'\sigma_{aw}(A+K)\} \\ = & \{\lambda : \lambda \in (\sigma_a(A+K) \cap \sigma_a(A)) \cap \eta'\sigma_w(A+K)\} \\ & \cup\{\lambda : \lambda \in (\sigma_a(A+K) \cap \sigma_a(A)^C) \cap \eta'\sigma_w(A+K)\} \\ = & S_1 \cup S_2 \quad \text{(say)}. \end{aligned}$$

Since $\sigma_a(A) \cap \eta'\sigma_{aw}(A+K) = \sigma_a(A) \cap \eta'\sigma_{aw}(A) \subseteq \Pi_0(A) \subseteq \sigma_w(A+K)^C$,

$$\begin{aligned} S_1 & \subseteq \{\lambda : \lambda \in \sigma_a(A+K) \cap \sigma_w(A+K)^C\} \\ & \subseteq \{\lambda : \lambda \in \mathrm{iso}\sigma_a(A+K) \cap \sigma_w(A+K)^C\} \subseteq \Pi_0(A+K). \end{aligned}$$

Again, if $\mathrm{iso}\sigma_a(A+K) \setminus \sigma_a(A) \subseteq \mathrm{iso}\sigma(A+K)$, then

$$S_2 \subseteq \{\lambda : \lambda \in \mathrm{iso}\sigma(A+K) \cap \eta'\sigma_{aw}(A+K)\} \subseteq \Pi_0(A+K).$$

Hence

$$\begin{aligned} \sigma_a(A+K) \cap \sigma_{aw}(A+K)^C & \subseteq \Pi_0(A+K) = \mathrm{iso}\sigma_a(A+K) \cap \sigma_w(A+K)^C \\ & \subseteq \sigma_a(A+K) \cap \sigma_{aw}(A+K)^C, \end{aligned}$$

i.e., $A + K \in (b)$.

Necessity. If $A + K \in (b)$, then $\mathrm{iso}\sigma_a(A+K) \cap \sigma_a(A)^C \subseteq \mathrm{iso}\sigma_a(A+K) \cap \sigma_{aw}(A)^C \subseteq \Pi_0(A+K) \subseteq \mathrm{iso}\sigma(A+K)$ and

$$\begin{aligned} & \{\lambda : \lambda \in \sigma_a(A+K) \cap \sigma_{aw}(A+K)^C, (A+K)^* \text{ has SVEP at } \lambda\} \\ = & \{\lambda : \lambda \in \sigma_a(A+K) \cap \sigma_w(A+K)^C, (A+K) \text{ has SVEP at } \lambda\} \\ = & \Pi_0(A+K) \subseteq \Pi_0^a(A+K) \subseteq \mathrm{iso}\sigma_a(A+K). \end{aligned}$$

This completes the proof. □

Operators $A \in B(\mathcal{X})$ such that $\sigma_a(A) = \sigma_{aw}(A)$ satisfy property (b). If also such an operator A satisfies $\sigma_a(A+K) \cap \sigma_{aw}(A)^C \subseteq \mathrm{iso}\sigma(A+K)$ for a compact operator $K \in B(\mathcal{X})$, then $A + K \in (b)$: This follows from Theorem 8.1, since $\sigma_a(A) = \sigma_{aw}(A)$ implies $\sigma_a(A+K)\cap\sigma_a(A)\cap\eta'\sigma_{aw}(A) = \emptyset$ and $\sigma_a(A+K)\cap\sigma_a(A)^C\cap\sigma_{aw}(A)^C = \sigma_a(A+K) \cap \sigma_{aw}(A)^C$.

Corollary 8.2. *Operators $A \in B(\mathcal{X})$ such that $\sigma_a(A) = \sigma_{aw}(A)$ satisfy property (b). Furthermore, if $K \in B(\mathcal{X})$ is a compact operator such that $\sigma_a(A+K) \cap \eta'\sigma_{aw}(A) \subseteq \mathrm{iso}\sigma(A+K)$, then $A + K \in (b)$.*

An important example of the class of operators $A \in B(\mathcal{X})$ satisfying $\sigma_a(A) = \sigma_{aw}(A)$ is that of the operators satisfying *the abstract shift condition $A^\infty(\mathcal{X}) = \bigcup_{n=1}^\infty A^n(\mathcal{X}) = \{0\}$* [27, Page 78]. (*Weighted right shift operators $A \in B(\ell^p)$*, $1 \le p < \infty$, are an interesting subclass of the class of operators satisfying the abstract shift condition [27].) Let $\mathcal{A}$ denote the class of non-quasinilpotent operators $A \in B(\mathcal{X})$ satisfying the abstract shift condition. Then $\sigma_w(A)$ is connected for operators $A \in \mathcal{A}$, and it follows from [21, Theorem 6.4] that $A + K$ is polaroid for compact operators $K \in B(\mathcal{X})$. Hence:

Corollary 8.3. *If $A \in \mathcal{A}$ and $K \in B(\mathcal{X})$ is a compact operator such that $\sigma_a(A+K) \cap \eta'\sigma_{aw}(A) \subseteq \mathrm{iso}\sigma(A+K)$, then $A + K \in (w)$.*

Proof. Corollary 8.2 implies that $A+K \in (b)$. Since $\text{iso}\sigma_w(A) = \emptyset$, $A+K$ is polaroid. Hence $E_0(A+K) \subseteq \Pi_0(A+K)$ ($\Longleftrightarrow E_0(A+K) = \Pi_0(A+K)$). This implies $A+K \in (w)$. $\square$

The argument above leads us to:

Theorem 8.4. *Given operators $A, K \in B(\mathcal{X})$, with K compact, a sufficient condition for $A \in (w)$ implies $A+K \in (w)$ is that $A \in (b)$ implies $A+K \in (b)$ and $\text{iso}\sigma_w(A) = \emptyset$.*

Proof. Evident, since $A \in (w)$ implies ($A \in (b)$, hence) $A+K \in (b)$, and $\text{iso}\sigma_w(A) = \emptyset$ implies $\Pi_0(A+K) = E_0(A+K)$. $\square$

The example of the unitary operator $A \in B(\mathcal{H} \oplus \mathcal{H})$ and the compact operator $K \in B(\mathcal{H} \oplus \mathcal{H})$ considered above (at the begining of this section) shows that $A \in (ab)$ does not imply $A+K \in (ab)$ (i.e., property (ab) is not stable under perturbation by compact operators). The following theorem gives a sufficient condition.

Theorem 8.5. *Given $A \in B(\mathcal{X})$, a sufficient condition for $A \in (ab)$ to impliy $A+K \in (ab)$ for every compact operator $K \in B(\mathcal{X})$ is that $\sigma_{aw}(A)^C$ is connected.*

Proof. We start by proving that $A+K$ has SVEP on $\sigma_{aw}(A)^C = \sigma_{aw}(A+K)^C$ for all compact operators K. Suppose to the contrary that there exists a compact operator K such that $A+K$ does not have SVEP at a point $\lambda \in \sigma_{aw}(A+K)^C$. Then $\text{asc}(A+K-\lambda) = \infty$. The component $\sigma_{aw}(A+K)^C$ being connected $\rho_a(A+K) = \mathbb{C}\setminus\sigma_a(A+K)$ intersects $\sigma_{aw}(A+K)^C$. Hence the continuity of the index at points $\lambda \in \sigma_{aw}(A+K)^C$ implies that $\alpha(A+K-\lambda) = 0$ at every $\lambda \in \sigma_{aw}(A+K)^C$, except perhaps for a countable subset (where $\alpha(A+K-\lambda) > 0$). In any case, $A+K$ has SVEP at λ, consequently $\text{asc}(A+K-\lambda) < \infty$. This being a contradiction, we must have $A+K$ has SVEP on $\sigma_{aw}(A+K)^C$, and hence ($A+K \in (a-Bt)$, i.e.,) $\sigma_a(A+K) \cap \sigma_{aw}(A+K)^C = \Pi_0^a(A+K)$. Assume now that $A \in (ab)$. Then $\sigma(A) \cap \sigma_w(A)^C = \Pi_0^a(A)$, and hence A^* has SVEP on $\Pi_0^a(A)$. Since

$$\begin{aligned} & A^* \text{ has SVEP on } \Pi_0^a(A) \Longleftrightarrow A^* \text{ has SVEP on } \sigma_{aw}(A)^C \\ \Longrightarrow\ & \sigma_{aw}(A+K)^C = \sigma_{aw}(A)^C = \sigma_w(A)^C = \sigma_w(A+K)^C, \end{aligned}$$

and since $A+K$ has SVEP on ($\sigma_{aw}(A+K)^C =$) $\sigma_w(A+K)^C$,

$$\begin{aligned} \Pi_0^a(A+K) &= \{\lambda : \lambda \in \text{iso}\sigma_a(A+K) \cap \sigma_{aw}(A+K)^C\} \\ &= \{\lambda : \lambda \in \text{iso}\sigma_a(A+K) \cap \sigma_w(A+K)^C\} \\ &= \{\lambda : \lambda \in \text{iso}\sigma_a(A+K) \cap (\text{iso}\sigma(A+K) \cap \sigma_w(A+K)^C)\} \\ &= \{\lambda : \lambda \in \text{iso}\sigma(A+K) \cap \sigma_w(A+K)^C\} \\ &= \{\lambda \in \Pi_0^a(A+K) : (A+K)^* \text{ has SVEP at } \lambda\}. \end{aligned}$$

Hence $A+K \in (ab)$. $\square$

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