Functional Analysis,
Approximation and
Computation
9 (1) (2017), 25–35

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/faac

Numerical Range of a Simple Compression

Philip G Spain[a]

[a] *School of Mathematics and Statistics, University of Glasgow, Glasgow G12 8QW, United Kingdom*

Abstract. The numerical range of the contraction $K : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$ acting on $L(\mathbb{C}^2)$ is identified, so allowing one to exhibit a hermitian projection that is not ultrahermitian.

An explicit formula for the norm of the operator $\kappa_m := \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} ma & b \\ c & d \end{bmatrix}$ $(m \in \mathbb{C})$. translates into a family of inequalities in four complex variables.

Introduction

Although the product of hermitian operators on a Hilbert space is also hermitian if (and only if) they commute, this does not extend to hermitian operators on a Banach space. Indeed, the square of a hermitian need not be hermitian: and even the product of two commuting hermitian *projections* need not be hermitian.

Here I identify the numerical range of the simplest nontrivial compression operator $K : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$, and so can exhibit hermitian projections that are not ultrahermitian.

The norms of the related operators $\kappa_m := \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} ma & b \\ c & d \end{bmatrix}$ are calculated explicitly (as m varies in the complex plane).

Perhaps surprisingly, the quantity $a^2 + b^2 + c^2 + d^2 + \sqrt{(a^2 + b^2 + c^2 + d^2)^2 - 4(ad \pm bc)^2}$ does not necessarily decrease when one replaces a by 0 (a, b, c and d being arbitrary real numbers), but may increase by up to the factor $\|\kappa_0\|$.

1. Numerical range

I follow the standard notation and rehearse only a few salient details, referring the reader to [BD], for example, for a full exposition and other references.

Given a Banach space X we say that

$$f \in X' \textit{ supports } x \in X \text{ if } \langle x, f \rangle = \|x\| = \|f\| = 1.$$

2010 *Mathematics Subject Classification*. 47A12, 47B15.
Keywords. Numerical range; Compression; Ultrahermitian projection.
Received: 12 Dececmber 2016; Accepted: 27 January 2017
Communicated by Dragan S. Djordjević
Email address: Philip.Spain@glasgow.ac.uk (Philip G Spain)

The *supporting set* for X is

$$\Pi_X := \{(x, f) \in X \times X' : \langle x,\ f\rangle = \|x\| = \left\|f\right\| = 1\}.$$

The *(spatial) numerical range* of the operator $T(\in L(X))$ is

$$V(T) := \left\{\langle Tx,\ f\rangle : (x, f) \in \Pi_X\right\}.$$

Definition 1.1. *H in $L(X)$ is* hermitian *if its numerical range is real: equivalently, if* $\|e^{irH}\| = 1 \quad (\forall r \in \mathbb{R})$*: equivalently, if* $\|I_X + irH\| \le 1 + o(r) \quad (\mathbb{R} \ni r \to 0)$.

2. The Banach space $L(\mathbb{C}^2)$ and some linear algebra

My example lives on $L(\mathbb{C}^2)$ with the operator norm. Facts to notice about this Banach space:

- Given $f \in L(\mathbb{C}^2)$ we can define a functional $\omega_f : y \mapsto \mathrm{tr}(yf)$ in $L(\mathbb{C}^2)'$: here tr is the *unnormalised* trace: and

  $$\left\|\omega_f\right\| = \mathrm{tr}\left|f\right| = \mathrm{tr}(f^*f)^{\frac{1}{2}}.$$

  Since any functional must be of this form we see that the [pre]dual of $L(\mathbb{C}^2)$ is, as a set, the same space as $L(\mathbb{C}^2)$: but with the trace norm.

- $\Pi_{L(\mathbb{C}^2)}$ is *biunitarily invariant* in the sense that

  $$(uxv, v^*fu^*) \in \Pi_{L(\mathbb{C}^2)} \iff (x, f) \in \Pi_{L(\mathbb{C}^2)}$$

  for any unitaries u and v.

- $\Pi_{L(\mathbb{C}^2)}$ is invariant under complex conjugation too — so $V(T)$ is symmetric in the real axis when T has real entries.

Given an element $x = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ of $L(\mathbb{C}^2)$ define

$$\sigma^2 = |a|^2 + |b|^2 + |c|^2 + |d|^2, \quad \nu^2 = |ad - bc|, \quad \text{and} \quad \rho^4 = \sigma^4 - 4\nu^4.$$

Then (routine computation!) the eigenvalues of x^*x are $(\sigma^2 \pm \rho^2)/2$ from which we have

$$\|x\|^2_{L(\mathbb{C}^2)} = \frac{\sigma^2 + \rho^2}{2} \qquad \text{and} \qquad \mathrm{tr}\,|x| = \left[\sigma^2 + 2\nu^2\right]^{\frac{1}{2}}.$$

Singular value decomposition

Given $x \in L(\mathbb{C}^2)$ there are unitaries u and v such that

$$uxv = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

where λ_1 and λ_2 ($\lambda_1 \ge \lambda_2$) are the eigenvalues of $|x|$. In particular, if $\|x\| = 1$, there are u, v such that

$$uxv = \begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix} =: \quad x_\lambda$$

with $0 \le \lambda \le 1$: and $\lambda = 1$ precisely when x itself is unitary.

The supporting set $\Pi_{L(\mathbb{C}^2)}$

Define

$$f_{(\alpha)} = \begin{bmatrix} \alpha & 0 \\ 0 & 1-\alpha \end{bmatrix}.$$

Lemma 2.1. *The functionals* $f_{(\alpha)}$ $(0 \le \alpha \le 1)$ *support* x_1*: and only these.*
The functional $f_{(1)}$ *is the only support of* x_λ *when* $0 \le \lambda < 1$. □

Hence

Lemma 2.2.

$$\Pi_{L(\mathbb{C}^2)} = \{(u^* x_\lambda v^*, v f_{(\alpha)} u)\}$$

where u, v *are unitary,* $0 \le \lambda \le 1$, *&* $\alpha \left\{ \begin{array}{ll} \in [0,1] & \lambda = 1 \\ = 1 & 0 \le \lambda < 1 \end{array} \right\}$.

3. The compression K

Consider the selfadjoint projection $P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ in $L(\mathbb{C}^2)$. Then the left and right multiplication operators

$$L = L_P \qquad \& \qquad R = R_P$$

are hermitian projections in $L(L(\mathbb{C}^2))$, for $\| e^{irL_P} \| = \| e^{irR_P} \| = \| e^{irP} \| = 1$ $(r \in \mathbb{R})$.

They commute, and their product

$$K = LR = RL$$

is a norm 1 projection on $L(\mathbb{C}^2)$, the *compression* $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$.

Theorem 3.1. *K is not hermitian.*

Proof. Note that $\|I - 2Q\| = \| e^{i\pi Q} \| = 1$ for any hermitian projection Q. However, $\|I - 2K\| \ge \sqrt{2}$ — for $(I - 2K) \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $\left\| \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \right\| = \sqrt{2}$ while $\left\| \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\| = 2$. (In fact, $\|I - 2K\| = \|\kappa_{-1}\| = \sqrt{2}$: see §5 below.) □

[AF] showed, also explicitly, that $\left\| \exp(3\pi i K/2) \right\| > 1$.

Ultrahermitian projections

Consider the following two properties that may hold for a projection E on a Banach space X. Note that they are symmetrical in E and its complement $\overline{E}$ $(= I - E)$. First,

$$(U1) \qquad \|Ex\| \left\|E'\phi\right\| + \left\|\overline{E}x\right\| \left\|\overline{E}'\phi\right\| \le \|x\| \left\|\phi\right\|$$

for $x \in X$, $\phi \in X'$: and, second,

$$(U2) \qquad \left\|EAE + \overline{E}B\overline{E}\right\| \le 1$$

for any contractions $A, B \in L(X)$.

Hermitian projections on Hilbert spaces have both these properties, as is easy to check.

The present author showed, see [S], that the properties (U1) and (U2) are equivalent, and introduced the term *ultrahermitian* for a projection that has either [and so both] of them.

Ultrahermitian projections are automatically hermitian [S, Theorem 4.3] and the product of two hermitian projections of which one is ultrahermitian must be hermitian [S, Corollary 4.8]. Hence

Theorem 3.2. *The left and right multiplication operators* L_P *and* R_P*, though hermitian, are not ultrahermitian.*

4. The numerical range $V(K)$

By Lemma 2.2 this is the convex set of all

$$
\begin{aligned}
\varpi_{\lambda,\alpha} &:= \langle K\, u^* x_\lambda v^*,\ v f_{(\alpha)} u\rangle \\
&= \operatorname{tr}\Big([Pu^* x_\lambda v^* P]\,[v f_{(\alpha)} u]\Big) \\
&= \operatorname{tr}\Big([Pu^* x_\lambda v^* P]\,[Pv f_{(\alpha)} uP]\Big) \\
&= \Big(u^* x_\lambda v^*\Big)_{(1,1)} \Big(v f_{(\alpha)} u\Big)_{(1,1)}
\end{aligned}
$$

where u, v are arbitrary unitaries, $0 \le \lambda \le 1$, and $\alpha \left\{ \begin{array}{ll} \in [0,1] & \lambda = 1 \\ = 1 & 0 \le \lambda < 1 \end{array} \right\}$.

As a full set of unitaries we may take

$$
u := \quad \omega_0 \begin{bmatrix} c & \omega_2\, s \\ \omega_1\, s & -\omega_1\omega_2\, c \end{bmatrix} \qquad \text{and} \qquad v := \quad w_0 \begin{bmatrix} C & w_2\, S \\ w_1\, S & -w_1 w_2\, C \end{bmatrix}
$$

with $|\omega_k| = 1$, $c = \cos\theta$, $s = \sin\theta$, $(0 \le \theta \le \pi/2)$, and $|w_k| = 1$, $C = \cos\varphi$, $S = \sin\varphi$, $(0 \le \varphi \le \pi/2)$. Compute:

$$
\begin{aligned}
Pu^* x_\lambda v^* P &= \overline{\omega_0\, w_0} \begin{bmatrix} cC + \lambda\overline{\omega_1 w_2}\, sS & 0 \\ 0 & 0 \end{bmatrix} \\
Pv f_{(\alpha)} uP &= \omega_0\, w_0 \begin{bmatrix} \alpha\, cC + (1-\alpha)\omega_1 w_2\, sS & 0 \\ 0 & 0 \end{bmatrix}
\end{aligned}
$$

So

$$
\begin{aligned}
\varpi_{\lambda,\alpha} &= \alpha c^2 C^2 + \lambda(1-\alpha) s^2 S^2 + [\alpha\lambda\overline{\omega_1 w_2} + (1-\alpha)\omega_1 w_2] cCsS \\
&= \left\{ \begin{array}{ll} c^2C^2 + \lambda\overline{\omega_1 w_2} cCsS & 0 \le \lambda < 1^* \\ \alpha[c^2C^2 + \overline{\omega_1 w_2} cCsS] + (1-\alpha)[s^2S^2 + \omega_1 w_2 cCsS] & \lambda = 1 \end{array} \right\}
\end{aligned}
$$

(* — also for $\lambda = 1$ — put $\alpha = 1$ in the following line.)

Replace $\overline{\omega_1 w_2}$ by ω. The points $\varpi_{\lambda,1}$, *ie*

$$
c^2C^2 + \lambda\omega\, cCsS \quad (0 \le \lambda \le 1)
$$

form the closed discs

$$
D(\theta,\varphi) := \Big\{\cos^2\theta\cos^2\varphi + \zeta\,\cos\theta\cos\varphi\sin\theta\sin\varphi \ :\ |\zeta| \le 1\Big\}
$$

with boundaries as in Figure 1. This demonstrates

Theorem 4.1.

$$
V(K) = \bigcup_{\substack{0 \le \theta \le \pi/2 \\ 0 \le \varphi \le \pi/2}} D(\theta,\varphi).
$$

Remark 4.2. *Since* $-\frac{1}{8} \in V(K)$ *we see that* $\|I - 2K\| \ge |V(I-2K)| = \frac{5}{4}$, *so, again, K cannot be hermitian.*

Lemma 4.3 (Cosine-geometric mean). *Given θ, φ in the first quadrant define their* cosine-geometric mean

$$
\psi := \cos^{-1}\sqrt{\cos\theta\cos\varphi}.
$$

Then the disc $D(\theta,\varphi)$ lies concentrically inside the disc

$$
D(\psi,\psi) = \Big\{\cos^4\psi + \zeta\,\cos^2\psi\sin^2\psi \ :\ |\zeta| \le 1\Big\}.
$$

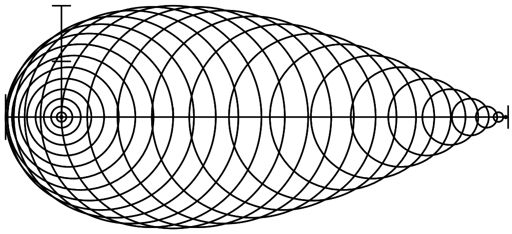

Figure 1: $\left\{\cos^2\theta\cos^2\varphi + \omega\,\cos\theta\cos\varphi\sin\theta\sin\varphi \;:\; |\omega| = 1\right\}$

Proof. Just check that $\sin\theta\sin\varphi = \cos(\theta-\varphi) - \cos^2\psi \le 1 - \cos^2\psi = \sin^2\psi$. □

Next, for $0 < \alpha < 1$, the points $\varpi_{1,\alpha}$ of the numerical range *ie*

$$\alpha[c^2C^2 + \overline{\omega}\,cCsS] + (1-\alpha)[s^2S^2 + \omega\,cCsS]$$

lie in the convex hull of $D(\psi,\psi)$ and $D(\tilde\psi,\tilde\psi)$, where $\tilde\psi$ is the cosine-geometric mean of $\frac{\pi}{2}-\theta$ and $\frac{\pi}{2}-\varphi$. Thus

Theorem 4.4.

$$V(K) = \bigcup_{\substack{0\le\theta\le\pi/2\\ 0\le\varphi\le\pi/2}} D(\theta,\varphi) = \bigcup_{0\le\psi\le\pi/2} D(\psi,\psi).\square$$

The circles $\partial D(\theta,\varphi)$ and $\partial D(\psi,\psi)$ lie as shown in Figure 2; and $V(K)$, the union of the discs $D(\psi,\psi)$, is as in Figure 3.

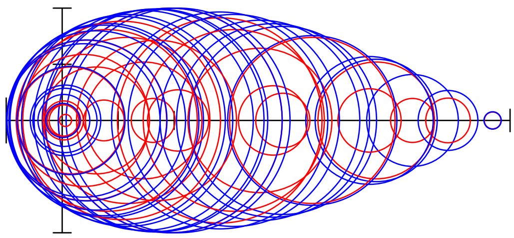

Figure 2: $\partial D(\theta,\varphi)$ (red) & $\partial D(\psi,\psi)$ (blue)

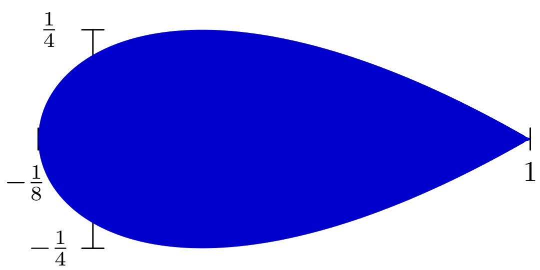

Figure 3: $V(K) = \bigcup_{0\le\theta\le\pi/2} D(\theta,\theta)$

The envelope and cusp

The circumference of the disc $D(\psi, \psi)$ is [setting $\gamma = \cos^2 \psi$]

$$(x - \gamma^2)^2 + y^2 = \gamma^2(1 - \gamma)^2 = \gamma^2 - 2\gamma^3 + \gamma^4.$$

To find the envelope of the $D(\psi, \psi)$ solve this equation simultaneously with its γ-derivative

$$2(x - \gamma^2)[-2\gamma] = 2\gamma - 6\gamma^2 + 4\gamma^3$$

to get

$$\begin{aligned} 2x &= 3\gamma - 1 \\ 2y &= \pm(1 - \gamma)\{4\gamma - 1\}^{\frac{1}{2}} \end{aligned}$$

for $\frac{1}{4} \leq \gamma \leq 1$.

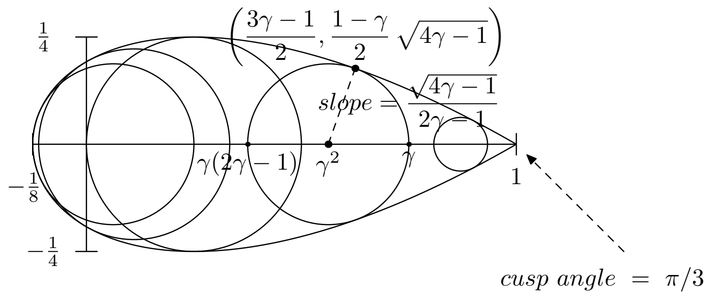

Figure 4: The cusp angle

5. The map κ_m and its norm ($m \in \mathbb{C}$)

The map κ_m is defined as

$$\kappa_m := I + (m - 1)K : L(\mathbb{C}^2) \to L(\mathbb{C}^2) : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} ma & b \\ c & d \end{bmatrix}.$$

As a first estimate $\|\kappa_m\| \geq 1$ and $\|\kappa_m\| \geq |m|$.

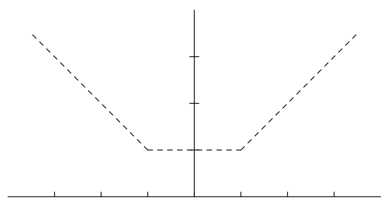

Figure 5: $\|\kappa_m\| \geq \max\{1, |m|\}$

Since κ_m attains its norm on the unit ball of $L(\mathbb{C}^2)$, the convex hull of the unitaries (the Russo-Dye theorem [BD, §38]), we next examine the values $\|\kappa_m u\|$ for unitary u. It will be more convenient to work with the expression $2\|\kappa_m u\|^2$.

With $c = \cos\theta$, $s = \sin\theta$, and $0 \leq \theta \leq \pi/2$, consider a typical unitary

$$u := u(c) = \omega_0 \begin{bmatrix} c & \omega_2\, s \\ \omega_1\, s & -\omega_1\, \omega_2\, c \end{bmatrix}$$

where ω_1 and ω_2 are arbitrary unimodular complex numbers. Calculate:

$$\begin{aligned}\sigma(\kappa_m u)^2 &= 2 + (|m|^2 - 1)c^2 \\ \rho(\kappa_m u)^4 &= c^2 \left\{4\,|m-1|^2 + [(|m|^2-1)^2 - 4\,|m-1|^2]c^2\right\}\end{aligned}$$

$$\begin{aligned}F_m(c) &:= 2\,\|\kappa_m u\|^2 \\ &= \sigma(\kappa_m u)^2 + \rho(\kappa_m u)^2 \\ &= 2 + (|m|^2-1)c^2 + c\left\{4\,|m-1|^2 + [(|m|^2-1)^2 - 4\,|m-1|^2]c^2\right\}^{\frac{1}{2}}\end{aligned}$$

The ω_1 and ω_2 are now seen to be irrelevant, so, without loss of generality, take $\omega_1 = \omega_2 = 1$.

Put

$$\Gamma := 4\,|m-1|^2 - (|m|^2-1)^2.$$

Then

$$F_m(c) = 2 + (|m|^2-1)c^2 + c\left\{4\,|m-1|^2 - \Gamma\, c^2\right\}^{\frac{1}{2}}.$$

Note that

$$\begin{aligned}F_m(0) &= 2, \\ F_m(1) &= 2 + |m|^2 - 1 + \left\{(|m|^2-1)^2\right\}^{\frac{1}{2}}, \\ &= 2\max\{1, |m|^2\} \quad [\geq F_m(0)].\end{aligned}$$

Thus

$$\|\kappa_m\| = \max\{1, |m|\}$$

when F_m has no turning point in $[0, 1]$.

The cardioid $\Gamma = 0$

The locus $\Gamma = 0$, that is, $|m|^2 - 1 = 2\,|m-1|$, is the *cardioid* shown in Figure 6.

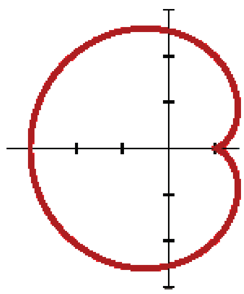

Figure 6: $|m|^2 - 1 = 2\,|m-1|$

In plane polar coordinates (r, ϕ) the equation is $8r\cos\phi = 3 + 6r^2 - r^4$.

Outside the cardioid $\Gamma = 0$

The function $F_m(c)$ certainly increases on $[0, 1]$ if $\Gamma \leq 0$ (which forces $|m| \geq 1$) so $\|\kappa_m\| = \max\{1, |m|\} = |m|$ outside the cardioid.

Inside the cardioid $\Gamma = 0$

To find turning points differentiate with respect to c:

$$\begin{aligned} F'_m(c) &= 2(|m|^2-1)c \;+\; \left\{4\,|m-1|^2-\Gamma\,c^2\right\}^{\frac{1}{2}} \\ &\qquad\qquad -\Gamma\,c^2\left\{4\,|m-1|^2-\Gamma\,c^2\right\}^{-\frac{1}{2}} \\ &= 2(|m|^2-1)c+2\left\{2\,|m-1|^2-\Gamma\,c^2\right\}\left\{4\,|m-1|^2-\Gamma\,c^2\right\}^{-\frac{1}{2}} \end{aligned}$$

Setting $F'_m(c) = 0$ and squaring [so possibly introducing spurious solutions] leads to the equation

$$\Gamma\,c^4 - 4\,|m-1|^2\,c^2 + |m-1|^2 = 0$$

for c^2.

Note that if $|m| = 1$ [leaving $m = 1$ aside] the equation reduces to $(1-2c^2)^2 = 0$, and therefore κ_m attains its norm at $\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$, independently of $\arg m$.

Otherwise the discriminant is

$$\begin{aligned} \Delta &= (2\,|m-1|^2)^2 - [4\,|m-1|^2 - (|m|^2-1)^2]\,|m-1|^2 \\ &= |m-1|^2\,(|m|^2-1)^2 > 0 \end{aligned}$$

and the candidate solutions are

$$\begin{aligned} c_\pm^2 &= \frac{2\,|m-1|^2 \pm |m-1|\,(|m|^2-1)}{\left[2\,|m-1| - (|m|^2-1)\right]\left[2\,|m-1| + (|m|^2-1)\right]} \\ &= \frac{|m-1|}{2\,|m-1| \mp (|m|^2-1)} > 0 \end{aligned}$$

It is straightforward to check that

$$\begin{aligned} 4\,|m-1|^2 - \Gamma\,c_\pm^2 &= \frac{|m-1|^2}{c_\pm^2}, \\ 2\,|m-1|^2 - \Gamma\,c_\pm^2 &= \mp\,|m-1|\,(|m|^2-1). \end{aligned}$$

Thus

$$\begin{aligned} F'_m(c_\pm) &= 2(|m|^2-1)c+2\left\{2\,|m-1|^2-\Gamma\,c^2\right\}\left\{4\,|m-1|^2-\Gamma\,c^2\right\}^{-\frac{1}{2}} \\ &= 2c_\pm\left\{\left[|m|^2-1\right]\mp\left[|m|^2-1\right]\right\}, \end{aligned}$$

which shows that c_+ alone is a possible turning point for F_m: but does c_+ lie in $[0,1]$?

The condition for this is that $|m-1| \le 2\,|m-1| - (|m|^2-1)$ *ie* that

$$|m|^2 - 1 \le |m-1|\,.$$

The cardioidoid $\left||m|^2-1\right| = |m-1|$

The 'edge locus' $\left||m|^2-1\right| = |m-1|$, which, for lack of another name I shall call a *cardioidoid*, bounds the blue region in Figure 7.

In plane polar coordinates it has equation $2r\cos\phi = 3r^2 - r^4$.

However, the set $|m|^2 - 1 \le |m-1|$ includes the unit disc too: I refer to this set as the *filled cardioidoid*.

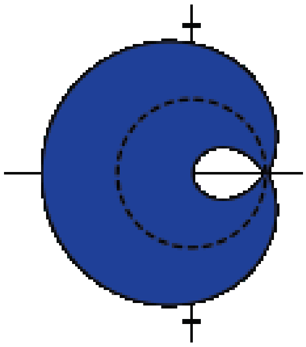

Figure 7: The cardioidoid

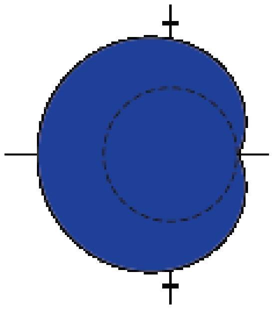

Figure 8: Filled cardioidoid

Inside the filled cardioidoid

Suppose that m lies inside the filled cardioidoid, so that $c_+ \in [0, 1]$.

Then

$$F_m(c_+) = \cdots = 2\,\frac{(|m-1|+1)^2 - |m|^2}{2\,|m-1| + 1 - |m|^2}.$$

Next

$$F_m(c_+) - 2 = \frac{2\,|m-1|^2}{2\,|m-1| + 1 - |m|^2} \geq 0$$

and

$$F_m(c_+) - 2|m|^2 = \frac{2\,(|m|^2 - 1 - |m-1|)^2}{2\,|m-1| + 1 - |m|^2} \geq 0$$

so

$$F_m(c_+) \geq F_m(1) \geq F_m(0).$$

Therefore

$$\|\kappa_m\|^2 = \frac{(|m-1|+1)^2 - |m|^2}{2\,|m-1| + 1 - |m|^2}$$

for m inside the filled cardioidoid. When m is real, within these limits, this expression reduces to $\dfrac{4}{3+m}$.

To sum up:

Theorem 5.1.

$$\|\kappa_m\| = \left\{ \begin{array}{ll} |m| & \textit{outside} \\ \sqrt{\frac{(|m-1|+1)^2-|m|^2}{2|m-1|+1-|m|^2}} & \textit{inside} \\ \sqrt{\frac{4}{3+m}} & \textit{on real axis inside} \end{array} \right\} \textit{the filled cardioidoid.}$$

Graph of $\|\kappa_m\|$ *for m real*

For *real m* inside the filled cardioidoid, *ie* $-2 \le m \le 1$, we have

$$\|\kappa_m\| = \sqrt{\frac{4}{3+m}}.$$

The graph of norm κ_m is shown in Figure 9.

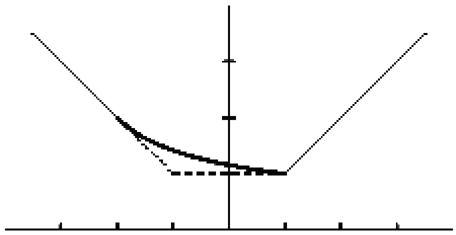

Figure 9: $\|\kappa_m\|$ *is continuous for all m but is not differentiable at* 1, *even as a function of a real variable*

6. An inequality

The inequalities

$$\|\kappa_m A\| \le \|\kappa_m\|\,\|A\|$$

(for complex 2×2 matrices A) are hardly transparent when written out explicitly. However, for $m = 0$, the simplest case, we have $\|I - K\| = \|\kappa_0\| = 2/\sqrt{3}$ so, for any real numbers a, b, c, d, we have

$$3\left(b^2 + c^2 + d^2 + \sqrt{(b^2+c^2+d^2)^2 - 4b^2c^2}\right)$$
$$\le 4\left(a^2 + b^2 + c^2 + d^2 + \sqrt{(a^2+b^2+c^2+d^2)^2 - 4(ad \pm bc)^2}\right)$$

or, on rewriting,

$$3\left(b^2 + c^2 + d^2 + \sqrt{[(b-c)^2 + d^2][(b+c)^2 + d^2]}\right)$$
$$\le 4\left(a^2 + b^2 + c^2 + d^2 + \sqrt{[(a-d)^2 + (b \mp c)^2][(a+d)^2 + (b \pm c)^2]}\right).$$

Acknowledgment

It is a pleasure to thank R.E. Harte for piquing my interest in $V(K)$ and for bringing the paper [AF] to my attention.

References

[AF] J. Anderson & C. Foiaş, *Properties which normal operators share with normal derivations and related operators*, Pacific J Math **61** (1975) 313–325.
[BD] F. F. Bonsall & J. Duncan, Complete Normed Algebras, Springer (1973).
[S] P. G. Spain, *Ultrahermitian Projections on Banach Spaces*, — ResearchGate.