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On Sum and Restriction of Hypo-*EP* Operators

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Abstract. An analytic characterization of hypo-*EP* operator is given. Using this characterization it is proved that sum of hypo-*EP* operators and restriction of hypo-*EP* operator are again hypo-*EP* under some conditions.

1. Introduction

A square matrix A over the complex field is said to be an *EP* matrix if ranges of A and A^* are equal. The *EP* matrix was defined by Schwerdtfeger [14]. But it did not get any greater attention until Pearl [13] gave characterization through Moore-Penrose inverse. Let $\mathcal{H}$ be a complex Hilbert space. A bounded operator A with closed range is said to be an *EP* operator (hypo-*EP*) if $AA^\dagger = A^\dagger A$ ($A^\dagger A - AA^\dagger$ is a positive operator). Here $A^\dagger$ denotes the Moore-Penrose inverse of A. *EP* matrices and operators have been studied by many authors [3–5, 7, 9, 11]. Hypo-*EP* operator was defined by Masuo Itoh and it has been studied in [8, 12]. In this paper we have given a characterization of hypo-*EP* operator. Using this characterization we give necessary and sufficient conditions for sum of hypo-*EP* operators to be hypo-*EP* and under some conditions restriction of hypo-*EP* operator to be hypo-*EP*.

Throughout this paper, $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ denotes the set of all bounded linear operators from $\mathcal{H}_1$ into $\mathcal{H}_2$ and we write $\mathcal{B}(\mathcal{H}, \mathcal{H}) = \mathcal{B}(\mathcal{H})$. The class $\mathcal{B}_c(\mathcal{H})$ denotes the set of all operators in $\mathcal{B}(\mathcal{H})$ having closed range. For any operator $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$, $\mathcal{R}(A)$ and $\mathcal{N}(A)$ denote the range and kernel of A respectively. $A \in \mathcal{B}(\mathcal{H})$ is said to be positive if $\langle Ax, x \rangle \geq 0$ for all $x \in \mathcal{H}$. A is said to be invertible if its inverse exists and bounded.

2. Preliminaries

We start with some known characterizations of hypo-*EP* operators.

Theorem 2.1. *[8] Let $A \in \mathcal{B}_c(\mathcal{H})$. Then the following are equivalent:*

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1. *A is hypo-EP.*
2. $\mathcal{R}(A) \subseteq \mathcal{R}(A^*)$.
3. $\mathcal{N}(A) \subseteq \mathcal{N}(A^*)$.
4. $A = A^*C$, *for some* $C \in \mathcal{B}(\mathcal{H})$.

Example 2.2. *Let* $A : \ell_2 \to \ell_2$ *be defined by* $A(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, \ldots)$ *(the right shift operator). Then* $A^*(x_1, x_2, x_3, \ldots) = (x_2, x_3, x_4, \ldots)$. *Here* $\mathcal{R}(A) \subseteq \mathcal{R}(A^*)$ *and* $\mathcal{R}(A)$ *is closed. Hence A is a hypo-EP operator.*

Remark 2.3. *The class of all hypo-EP operators contains the class of all EP operators. Hence it contains all normal, self-adjoint and invertible operators having closed range. In the case of finite dimensional, EP and hypo-EP are same.*

Theorem 2.4 (Douglas' Theorem). *[6] Let* $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}$ *be Hilbert spaces and let* $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}), B \in \mathcal{B}(\mathcal{H}_2, \mathcal{H})$. *Then the following are equivalent:*

1. $A = BC$, *for some* $C \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$.
2. $\|A^*x\| \leq k\|B^*x\|$, *for some* $k > 0$ *and for all* $x \in \mathcal{H}$.
3. $\mathcal{R}(A) \subseteq \mathcal{R}(B)$.

3. Characterizations of Hypo-*EP* Operators

Several characterizations of hypo-*EP* operators available in literature are algebraic in nature. The following is a characterization for a bounded closed range operator to be hypo-*EP* which does not involve $A^\dagger$ and A^*.

Theorem 3.1. *Let* $A \in \mathcal{B}_c(\mathcal{H})$. *Then A is hypo-EP if and only if for each* $x \in \mathcal{H}$, *there exists* $k > 0$ *such that*

$$|\langle Ax, y\rangle| \leq k\|Ay\|, \text{ for all } y \in \mathcal{H}. \tag{1}$$

Proof. Suppose A is hypo-*EP*. Let $x \in \mathcal{H}$ such that $Ax = 0$, then the result is trivial. Let $x \in \mathcal{H}$ such that $Ax \neq 0$. Then $Ax \in \mathcal{R}(A) \subseteq \mathcal{R}(A^*)$. Therefore there exists a non-zero $z \in \mathcal{H}$ such that $A^*z = Ax$. Then for all $y \in \mathcal{H}$,

$$|\langle Ax, y\rangle| = |\langle A^*z, y\rangle| = |\langle z, Ay\rangle| \leq \|z\|\|Ay\|.$$

Taking $k = \|z\|$, we get $|\langle Ax, y\rangle| \leq k\|Ay\|$, for all $y \in \mathcal{H}$.

Conversely, assume that for each $x \in \mathcal{H}$, there exists $k > 0$ such that $|\langle Ax, y\rangle| \leq k\|Ay\|$ for all $y \in \mathcal{H}$. Let $x \in \mathcal{H}$ be fixed. Then for all $y \in \mathcal{H}$, $k\|Ay\| \geq |\langle x, A^*y\rangle| = |f_x(A^*y)|$ setting $f_x(A^*y) = \langle A^*y, x\rangle$. Hence $|(Af_x^*)^*y| \leq k\|(A^*)^*y\|$ for some $k > 0$, for all $y \in \mathcal{H}$. By Douglas' theorem, $Af_x^* = A^*D$, $D \in \mathcal{B}(\mathbb{C}, \mathcal{H})$. Taking adjoint on both sides gives $f_xA^* = g_xA$ where $g_x = D^* \in \mathcal{B}(\mathcal{H}, \mathbb{C})$. By Riesz representation theorem, there exists $x' \in \mathcal{H}$ such that $g_x(Az) = \langle Az, x'\rangle$ for all $z \in \mathcal{H}$. Hence for $z \in \mathcal{H}$, $f_xA^*z = g_xAz$ implies that $\langle A^*z, x\rangle = \langle Az, x'\rangle$. Therefore for each $x \in \mathcal{H}$ there exists $x' \in \mathcal{H}$ such that $Ax = A^*x'$. Thus $\mathcal{R}(A) \subseteq \mathcal{R}(A^*)$. □

There is an example in [1] for a bounded operator A in $\mathcal{B}_c(\mathcal{H})$ such that $A^2 \notin \mathcal{B}_c(\mathcal{H})$. But we prove that if A is hypo-*EP*, then A^2 has closed range always. Moreover any natural power of A has closed range. The following characterization for closed range operator is used to prove the results.

Theorem 3.2. *[2] Let* $A \in \mathcal{B}(\mathcal{H})$. *A has closed range if and only if there is a positive* δ *such that* $\|Ax\| \geq \delta\|x\|$ *for all* $x \in \mathcal{N}(A)^\perp$.

Theorem 3.3. *If A is hypo-EP, then* A^n *has closed range for any* $n \in \mathbb{N}$.

Proof. Suppose that A is hypo-*EP*. Then for any $m, n \in \mathbb{N}$ with $m \leq n$,

$$A^m[\mathcal{N}(A^n)^\perp] \subseteq A^m[\mathcal{N}(A)^\perp] \subseteq \mathcal{R}(A) \subseteq \mathcal{R}(A^*) = \mathcal{N}(A)^\perp. \tag{2}$$

As A has closed range, there exists $k > 0$ such that $\|Ax\| \geq k\|x\|$, for all $x \in \mathcal{N}(A)^\perp$. Let $x \in \mathcal{N}(A^n)^\perp$. Then by (2),

$$\|A^n x\| = \|A(A^{n-1}x)\| \geq k\|A^{n-1}x\| \geq \cdots \geq k^n\|x\|.$$

Thus A^n has closed range, for any $n \in \mathbb{N}$. □

We proved that if A is hypo-*EP*, then it is redundant that $\mathcal{R}(A^n)$ is closed for any $n \in \mathbb{N}$. It is observed that the right shift operator A on ℓ_2 is hypo-*EP*, but $\mathcal{R}(A) \neq \mathcal{R}(A^n)$ for any $n > 1$.

If we start with any $A \in \mathcal{B}(\mathcal{H})$, the null spaces of A^n are growing in nature along with increasing values of n. But interestingly, all null spaces are same when A is hypo-*EP*.

Theorem 3.4. *If A is hypo-EP, then $\mathcal{N}(A^n) = \mathcal{N}(A)$, for each $n \in \mathbb{N}$. Moreover, if A is nilpotent, then $A = 0$.*

Proof. It is enough to prove that $\mathcal{N}(A^n) = \mathcal{N}(A^{n+1})$ for each $n \in \mathbb{N}$. Let $z \in \mathcal{H}$ be fixed. If we apply Theorem 3.1 to an element $x = A^{n-1}z$, there exists $k > 0$ such that

$$|\langle A(A^{n-1}z), y\rangle| \leq k\|Ay\|, \text{ for all } y \in \mathcal{H}.$$

In particular taking $y = A^n z$, we get $|\langle A^n z, A^n z\rangle| \leq k\|A^{n+1}z\|$. If $z \in \mathcal{N}(A^{n+1})$, then $z \in \mathcal{N}(A^n)$. Hence $\mathcal{N}(A^n) = \mathcal{N}(A^{n+1})$ for each $n \in \mathbb{N}$. Thus $\mathcal{N}(A^n) = \mathcal{N}(A)$, for each $n \in \mathbb{N}$. □

Remark 3.5. *The condition $\mathcal{N}(A) = \mathcal{N}(A^n)$, for each $n \in \mathbb{N}$ is necessary for A to be hypo-EP. It is not a sufficient condition for A to be hypo-EP. For example, the matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ is not hypo-EP, but $\mathcal{N}(A^n) = \mathcal{N}(A)$ for each $n \in \mathbb{N}$.*

Theorem 3.6. *If A is hypo-EP, then A^n is hypo-EP, for any $n \in \mathbb{N}$.*

Proof. Suppose that A is hypo-*EP*. Then for any $n \in \mathbb{N}$, $\mathcal{N}(A^n) = \mathcal{N}(A) \subseteq \mathcal{N}(A^*) \subseteq \mathcal{N}(AA^{(n-1)*})$, so $\mathcal{N}(AA^{(n-1)*})^\perp \subseteq \mathcal{N}(A^n)^\perp$. Since $\mathcal{R}(A^{n*})$ is closed and $\mathcal{R}(A^{(n-1)}A^*) \subseteq \overline{\mathcal{R}(A^{(n-1)}A^*)}$, $\mathcal{R}(A^{(n-1)}A^*) \subseteq \mathcal{R}(A^{n*})$. Then by Douglas' theorem

$$\|AA^{(n-1)*}x\| \leq \ell\|A^n x\|, \text{ for some } \ell > 0, \text{ for all } x \in \mathcal{H} \text{ and } n \in \mathbb{N}.$$

By Theorem 3.1, for each $x \in \mathcal{H}$, there exists $k > 0$ such that

$$|\langle A^n x, y\rangle| = |\langle Ax, A^{(n-1)*}y\rangle| \leq k\|AA^{(n-1)*}y\| \leq k\ell\|A^n y\|, \text{ for all } y \in \mathcal{H} \text{ and } n \in \mathbb{N}.$$

Thus for any natural number n, A^n is hypo-*EP*. □

Remark 3.7. *Theorems 3.3, 3.4, 3.6 have been observed by Patel [12], but our characterization given in Theorem 3.1 was used to prove the results.*

4. Sum of Hypo-*EP* Operators

In general the sum two hypo-*EP* operators is not necessarily hypo-*EP*.

Example 4.1. *Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. Then A and B are hypo-EP, but $A + B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is not hypo-EP.*

Meenakshi [10] discussed results on sum of *EP* matrices. The next theorem gives a sufficient condition for the sum of hypo-*EP* operators to be a hypo-*EP* operator.

Theorem 4.2. *Let A, B be hypo-EP operators such that $A + B$ has closed range. If*

$$\|Ax\| \leq k\|(A+B)x\|, \text{ for some } k > 0 \text{ and for all } x \in \mathcal{H}, \tag{3}$$

then $A + B$ is hypo-EP.

Proof. From (3), for all $x \in \mathcal{H}$,

$$\|Bx\| \leq \|(A+B)x\| + \|Ax\| \leq \|(A+B)x\| + k\|(A+B)x\| \leq (k+1)\|(A+B)x\|.$$

Since A and B are hypo-*EP*, for each $x \in \mathcal{H}$ there exist $k_1, k_2 > 0$ such that $|\langle Ax, y\rangle| \leq k_1\|Ay\|$ and $|\langle Bx, y\rangle| \leq k_2\|By\|$ for all $y \in \mathcal{H}$.

$$\begin{aligned} |\langle (A+B)x, y\rangle| &\leq |\langle Ax, y\rangle| + |\langle Bx, y\rangle| \\ &\leq k_1\|Ay\| + k_2\|By\| \\ &\leq k_1k\|(A+B)y\| + k_2(k+1)\|(A+B)y\|. \end{aligned}$$

Thus $|\langle (A+B)x, y\rangle| \leq [k_1k + k_2(k+1)]\, \|(A+B)y\|$. Hence $A + B$ is hypo-*EP*. □

Corollary 4.3. *Let A, B be hypo-EP operators such that $A + B$ has closed range. If $A^*B + B^*A = 0$, then $A + B$ is hypo-EP.*

Proof. The assumption $A^*B + B^*A = 0$ gives $(A+B)^*(A+B) = A^*A + B^*B$. Then

$$\|(A+B)x\|^2 = \langle (A+B)x, (A+B)x\rangle = \langle (A^*A + B^*B)x, x\rangle \geq \|Ax\|^2$$

From Theorem 4.2, $A + B$ is hypo-*EP*. □

Remark 4.4. *In the above theorem the condition (3) is equivalent to "$\mathcal{N}(A+B) \subseteq \mathcal{N}(A)$". But the condition (3) is not necessary for the sum of A and B to be hypo-EP. For example, let $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Then A, B and $A + B$ are hypo-EP. But $\mathcal{N}(A+B) \nsubseteq \mathcal{N}(A)$.*

Suppose A and B are hypo-*EP*. Then by Douglas' theorem $A^* = D_AA$ and $B^* = D_BB$ for some operators $D_A, D_B \in \mathcal{B}(\mathcal{H})$. The next theorem shows that the condition (3) is both necessary and sufficient condition for the sum to be hypo-*EP* under the assumption that $D_A - D_B$ is invertible.

Theorem 4.5. *Let $A, B \in \mathcal{B}_c(\mathcal{H})$ be hypo-EP operators such that $A + B$ has closed range and $D_A - D_B$ be invertible where D_A, D_B as defined above. Then $A + B$ is hypo-EP if and only if $\|Ax\| \leq k\|(A+B)x\|$ for some $k > 0$ and for all $x \in \mathcal{H}$.*

Proof. Assume $A + B$ is hypo-*EP*. Then $A^* + B^* = (A+B)^* = E(A+B)$ for some $E \in \mathcal{B}(\mathcal{H})$. Hence $D_AA + D_BB = E(A+B)$ which implies that $(D_A - E)A = (E - D_B)B$.

Taking $K = D_A - E, L = E - D_B$, we have $KA = LB$ and $(K+L)A = L(A+B)$. Then $A = (K+L)^{-1}L(A+B)$, since $K + L = D_A - D_B$ is invertible. Hence $\|Ax\| \leq k\|(A+B)x\|$ for all $x \in \mathcal{H}$, where $k = \|(K+L)^{-1}L\|$. The converse follows from Theorem 4.2. □

5. Restriction of Hypo-*EP* Operators

Let $A \in \mathcal{B}(\mathcal{H})$. A closed subspace $\mathcal{M}$ of $\mathcal{H}$ is said to be an invariant subspace for A if $A(\mathcal{M}) \subseteq \mathcal{M}$. $\mathcal{M}$ is said to be a reducing subspace for A if both $\mathcal{M}$ and $\mathcal{M}^\perp$ are invariant subspaces for A. In this section we discuss restriction of hypo-*EP* operator. For $A \in \mathcal{B}(\mathcal{H})$ the restriction to an invariant subspace $\mathcal{M}$ for A is denoted by $A|_{\mathcal{M}}$. The adjoint of $A|_{\mathcal{M}}$ is denoted by $(A|_{\mathcal{M}})^*$ and defined by $(A|_{\mathcal{M}})^* = PA^*|_{\mathcal{M}}$ where P is the orthogonal projection onto $\mathcal{M}$. The restriction operator $A|_{\mathcal{M}}$ coincides with the following properties as in the operator $A \in \mathcal{B}(\mathcal{H})$. The proof of the following proposition are obvious from the definition.

Proposition 5.1. *Let $A, B \in \mathcal{B}(\mathcal{H})$ and $\mathcal{M}$ be an invariant subspace for both A and B. Then*

1. $(A|_{\mathcal{M}})^{**} = A|_{\mathcal{M}}$.
2. $(AB|_{\mathcal{M}})^* = (B|_{\mathcal{M}})^*(A|_{\mathcal{M}})^*$.

From the definition of hypo-*EP* operator, for any $A \in \mathcal{B}(\mathcal{H})$, we say $A|_{\mathcal{M}}$ is hypo-*EP* if $\mathcal{R}(A|_{\mathcal{M}})$ is closed and $\mathcal{R}(A|_{\mathcal{M}}) \subseteq \mathcal{R}((A|_{\mathcal{M}})^*)$.

Theorem 5.2. *Let $A \in \mathcal{B}(\mathcal{H})$ and $\mathcal{M}$ be an invariant subspace for A such that $A|_{\mathcal{M}}$ has closed range. Then $A|_{\mathcal{M}}$ is hypo-EP if and only if for each $x \in \mathcal{M}$ there exists $k > 0$ such that*

$$|\langle A|_{\mathcal{M}}x, y\rangle| \leq k\|A|_{\mathcal{M}}y\|, \text{ for all } y \in \mathcal{M}.$$

Proof of this theorem is direct using Theorem 3.1 and Proposition 5.1.

Corollary 5.3. *Let A be a hypo-EP operator and $\mathcal{M}$ be an invariant subspace for A such that $A|_{\mathcal{M}}$ has closed range. Then $A|_{\mathcal{M}}$ is hypo-EP.*

Remark 5.4. *There are sufficient conditions available in literature that range of $A|_{\mathcal{M}}$ is closed when $A \in \mathcal{B}_c(\mathcal{H})$. In [1] Barnes gave a sufficient condition that "$\mathcal{R}(A|_{\mathcal{M}}) = \mathcal{R}(A) \cap \mathcal{M}$". The following example tells that the condition is not necessary.*

Example 5.5. *Let A be the right shift operator on ℓ_2 and $\mathcal{M} = \mathcal{R}(A)$. Then $A|_{\mathcal{M}}$ is hypo-EP operator, but $\mathcal{R}(A|_{\mathcal{M}}) \subsetneq \mathcal{R}(A) \cap \mathcal{M}$.*

Theorem 5.6. *Let $A \in \mathcal{B}_c(\mathcal{H})$ and $\mathcal{R}(A)$ be a reducing subspace for A. If $A|_{\mathcal{R}(A)}$ is hypo-EP, then A is hypo-EP.*

Proof. Let $x \in \mathcal{H}$. Then x can be expressed as $x = x_1 + x_2$ such that $x_1 \in \mathcal{R}(A)$ and $x_2 \in \mathcal{R}(A)^\perp$. For all $y \in \mathcal{H}$, $|\langle Ax, y\rangle| = |\langle Ax, y_1\rangle + \langle Ax, y_2\rangle|$ where $y_1 \in \mathcal{R}(A), y_2 \in \mathcal{R}(A)^\perp$. Since $A|_{\mathcal{R}(A)}$ is hypo-*EP*, there exists $k > 0$ such that $|\langle Ax, y\rangle| = |\langle Ax, y_1\rangle| \leq k\|Ay_1\|$. Since $\mathcal{R}(A)$ is a reducing subspace for A, $\|Ay\|^2 = \|Ay_1\|^2 + \|Ay_2\|^2$, so $\|Ay_1\| \leq \|Ay\|$. Hence A is hypo-*EP*. □

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