On Sum and Restriction of Hypo-EP Operators

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Abstract. An analytic characterization of hypo-EP operator is given. Using this characterization it is proved that sum of hypo-EP operators and restriction of hypo-EP operator are again hypo-EP under some conditions.

1. Introduction

A square matrix $A$ over the complex field is said to be an EP matrix if ranges of $A$ and $A^*$ are equal. The EP matrix was defined by Schwerdtfeger \cite{14}. But it did not get any greater attention until Pearl \cite{13} gave characterization through Moore-Penrose inverse. Let $\mathcal{H}$ be a complex Hilbert space. A bounded operator $A$ with closed range is said to be an EP operator (hypo-EP) if $AA^* = A^*A$ ($A^*A - AA^*$ is a positive operator). Here $A^*$ denotes the Moore-Penrose inverse of $A$. EP matrices and operators have been studied by many authors \cite{3–5, 7, 9, 11}. Hypo-EP operator was defined by Masuo Itoh and it has been studied in \cite{8, 12}. In this paper we have given a characterization of hypo-EP operator. Using this characterization we give necessary and sufficient conditions for sum of hypo-EP operators to be hypo-EP and under some conditions restriction of hypo-EP operator to be hypo-EP.

Throughout this paper, $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ denotes the set of all bounded linear operators from $\mathcal{H}_1$ into $\mathcal{H}_2$ and we write $\mathcal{B}(\mathcal{H}, \mathcal{H}) = \mathcal{B}(\mathcal{H})$. The class $\mathcal{B}_c(\mathcal{H})$ denotes the set of all operators in $\mathcal{B}(\mathcal{H})$ having closed range. For any operator $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2), \mathcal{R}(A)$ and $\mathcal{N}(A)$ denote the range and kernel of $A$ respectively. $A \in \mathcal{B}(\mathcal{H})$ is said to be positive if $\langle Ax, x \rangle \geq 0$ for all $x \in \mathcal{H}$. $A$ is said to be invertible if its inverse exists and bounded.

2. Preliminaries

We start with some known characterizations of hypo-EP operators.

Theorem 2.1. \cite{8} Let $A \in \mathcal{B}_c(\mathcal{H})$. Then the following are equivalent:

\begin{itemize}
  \item[(a)] $A$ is positive.
  \item[(b)] $A$ is invertible.
  \item[(c)] $A - AA^*$ is positive.
\end{itemize}
1. A is hypo-EP.
2. \( \mathcal{R}(A) \subseteq \mathcal{R}(A^*) \).
3. \( \mathcal{N}(A) \subseteq \mathcal{N}(A^*) \).
4. \( A = A^*C \), for some \( C \in \mathcal{B}(\mathcal{H}) \).

Example 2.2. Let \( A : \ell_2 \to \ell_2 \) be defined by \( A(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, \ldots) \) (the right shift operator). Then \( A^*(x_1, x_2, x_3, \ldots) = (x_2, x_3, x_4, \ldots) \). Here \( \mathcal{R}(A) \subseteq \mathcal{R}(A^*) \) and \( \mathcal{R}(A) \) is closed. Hence \( A \) is a hypo-EP operator.

Remark 2.3. The class of all hypo-EP operators contains the class of all EP operators. Hence it contains all normal, self-adjoint and invertible operators having closed range. In the case of finite dimensional, EP and hypo-EP are same.

Theorem 2.4 (Douglas’ Theorem). [6] Let \( \mathcal{H}_1, \mathcal{H}_2, \mathcal{H} \) be Hilbert spaces and let \( A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}), B \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}) \). Then the following are equivalent:

1. \( A = BC \), for some \( C \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2) \).
2. \( \|A^*x\| \leq k\|B^*x\| \), for some \( k > 0 \) and for all \( x \in \mathcal{H} \).
3. \( \mathcal{R}(A) \subseteq \mathcal{R}(B) \).


Several characterizations of hypo-EP operators available in literature are algebraic in nature. The following is a characterization for a bounded closed range operator to be hypo-EP which does not involve \( A^+ \) and \( A^* \).

Theorem 3.1. Let \( A \in \mathcal{B}_c(\mathcal{H}) \). Then \( A \) is hypo-EP if and only if for each \( x \in \mathcal{H} \), there exists \( k > 0 \) such that

\[
\langle Ax, y \rangle \leq k\|Ay\|, \text{ for all } y \in \mathcal{H}. \tag{1}
\]

Proof. Suppose \( A \) is hypo-EP. Let \( x \in \mathcal{H} \) such that \( Ax = 0 \), then the result is trivial. Let \( x \in \mathcal{H} \) such that \( Ax \neq 0 \). Then \( Ax \in \mathcal{R}(A) \subseteq \mathcal{R}(A^*) \). Therefore there exists a non-zero \( z \in \mathcal{H} \) such that \( A^*z = Ax \). Then for all \( y \in \mathcal{H} \),

\[
\langle Ax, y \rangle = \langle A^*z, y \rangle = \langle z, Ay \rangle \leq \|z\|\|Ay\|.
\]

Taking \( k = \|z\| \), we get \( \langle Ax, y \rangle \leq k\|Ay\| \), for all \( y \in \mathcal{H} \).

Conversely, assume that for each \( x \in \mathcal{H} \), there exists \( k > 0 \) such that \( \|Ax, y\| \leq k\|Ay\| \) for all \( y \in \mathcal{H} \). Let \( x \in \mathcal{H} \) be fixed. Then for all \( y \in \mathcal{H} \), \( k\|Ay\| \geq |\langle x, A^*y \rangle| = |f_x(A^*y)| \) setting \( f_x(A^*y) = (A^*y, x) \). Hence \( |(A^*f_x^*)y| \leq k\|A^*y\| \) for some \( k > 0 \), for all \( y \in \mathcal{H} \). By Douglas’ theorem, \( A^*f_x = A^*D \), \( D \in \mathcal{B}(\mathcal{C}, \mathcal{H}) \). Taking adjoint on both sides gives \( f_xA^* = g_xA \) where \( g_x = D^* \in \mathcal{B}(\mathcal{H}, \mathcal{C}) \). By Riesz representation theorem, there exists \( x' \in \mathcal{H} \) such that \( g_x(Az) = \langle Az, x' \rangle \) for all \( z \in \mathcal{H} \). Hence for \( z \in \mathcal{H} \), \( f_xA^*z = g_xAz \) implies that \( \langle A^*z, x' \rangle = \langle Az, x' \rangle \). Therefore for each \( x \in \mathcal{H} \) there exists \( x' \in \mathcal{H} \) such that \( Ax = A^*x' \). Thus \( \mathcal{R}(A) \subseteq \mathcal{R}(A^*) \). \( \square \)

There is an example in [1] for a bounded operator \( A \in \mathcal{B}_c(\mathcal{H}) \) such that \( A^2 \notin \mathcal{B}_c(\mathcal{H}) \). But we prove that if \( A \) is hypo-EP, then \( A^2 \) has closed range always. Moreover any natural power of \( A \) has closed range. The following characterization for closed range operator is used to prove the results.

Theorem 3.2. [2] Let \( A \in \mathcal{B}(\mathcal{H}) \). \( A \) has closed range if and only if there is a positive \( \delta \) such that \( \|Ax\| \geq \delta \|x\| \) for all \( x \in \mathcal{N}(A)^\perp \).

Theorem 3.3. If \( A \) is hypo-EP, then \( A^n \) has closed range for any \( n \in \mathbb{N} \).
Proof. Suppose that $A$ is hypo-EP. Then for any $m, n \in \mathbb{N}$ with $m \leq n$,
\[ A^m [N(A^n)^{-1}] \subseteq A^m [N(A)] \subseteq \mathcal{R}(A) \subseteq \mathcal{R}(A') = N(A)^\perp. \tag{2} \]
As $A$ has closed range, there exists $k > 0$ such that $\|Ax\| \geq k\|x\|$ for all $x \in N(A)^\perp$. Let $x \in N(A^n)^\perp$. Then by (2),
\[ \|A^nx\| = \|A(A^{n-1}x)\| \geq k\|A^{n-1}x\| \geq \cdots \geq k^n\|x\|. \]
Thus $A^n$ has closed range, for any $n \in \mathbb{N}$. \hfill \Box

We proved that if $A$ is hypo-EP, then it is redundant that $\mathcal{R}(A^n)$ is closed for any $n \in \mathbb{N}$. It is observed that the right shift operator $A$ on $\ell_2$ is hypo-EP, but $\mathcal{R}(A) \neq \mathcal{R}(A^n)$ for any $n > 1$.

If we start with any $A \in \mathcal{B}(\mathcal{H})$, the null spaces of $A^n$ are growing in nature along with increasing values of $n$. But interestingly, all null spaces are same when $A$ is hypo-EP.

**Theorem 3.4.** If $A$ is hypo-EP, then $N(A^n) = N(A)$, for each $n \in \mathbb{N}$. Moreover, if $A$ is nilpotent, then $A = 0$.

*Proof.* It is enough to prove that $N(A^n) = N(A^{n+1})$ for each $n \in \mathbb{N}$. Let $z \in \mathcal{H}$ be fixed. If we apply Theorem 3.1 to an element $x = A^{n-1}z$, there exists $k > 0$ such that
\[ |\langle A^{n-1}z, y \rangle| \leq k\|Ay\|, \text{ for all } y \in \mathcal{H}. \]
In particular taking $y = A^n z$, we get $|\langle A^n z, A^n z \rangle| \leq k\|A^n z\|$. If $z \in N(A^{n+1})$, then $z \in N(A^n)$. Hence $N(A^n) = N(A^{n+1})$ for each $n \in \mathbb{N}$. Thus $N(A^n) = N(A)$, for each $n \in \mathbb{N}$. \hfill \Box

**Remark 3.5.** The condition $N(A) = N(A^n)$, for each $n \in \mathbb{N}$ is necessary for $A$ to be hypo-EP. It is not a sufficient condition for $A$ to be hypo-EP. For example, the matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ is not hypo-EP, but $N(A^n) = N(A)$ for each $n \in \mathbb{N}$.

**Theorem 3.6.** If $A$ is hypo-EP, then $A^n$ is hypo-EP, for any $n \in \mathbb{N}$.

*Proof.* Suppose that $A$ is hypo-EP. Then for any $n \in \mathbb{N}$, $N(A^n) = N(A) \subseteq N(A') \subseteq N(AA^{(n-1)^\perp})$, so $N(AA^{(n-1)^\perp}) \subseteq N(A^n)^\perp$. Since $\mathcal{R}(A^n)$ is closed and $\mathcal{R}(A^{(n-1)}A^*) \subseteq \mathcal{R}(A^{(n-1)}A^*), \mathcal{R}(A^{(n-1)}A^*) \subseteq \mathcal{R}(A^n)$. Then by Douglas’ theorem
\[ \|AA^{(n-1)}x\| \leq \ell \|A^n x\|, \text{ for some } \ell > 0, \text{ for all } x \in \mathcal{H} \text{ and } n \in \mathbb{N}. \]
By Theorem 3.1, for each $x \in \mathcal{H}$, there exists $k > 0$ such that
\[ |\langle A^n x, y \rangle| = |\langle Ax, A^{(n-1)^\perp} y \rangle| \leq k\|AA^{(n-1)^\perp}y\| \leq k\ell\|A^n y\|, \text{ for all } y \in \mathcal{H} \text{ and } n \in \mathbb{N}. \]
Thus for any natural number $n$, $A^n$ is hypo-EP. \hfill \Box

**Remark 3.7.** Theorems 3.3, 3.4, 3.6 have been observed by Patel [12], but our characterization given in Theorem 3.1 was used to prove the results.

4. **Sum of Hypo-EP Operators**

In general the sum two hypo-EP operators is not necessarily hypo-EP.

**Example 4.1.** Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. Then $A$ and $B$ are hypo-EP, but $A + B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is not hypo-EP.

**Theorem 4.2.** Let $A, B$ be hypo-EP operators such that $A + B$ has closed range. If

$$
\| Ax \| \leq k \|(A + B)x\|, \text{ for some } k > 0 \text{ and for all } x \in \mathcal{H},
$$

then $A + B$ is hypo-EP.

**Proof.** From (3), for all $x \in \mathcal{H}$,

$$
\| Bx \| \leq \|(A + B)x\| + \| Ax \| \leq \|(A + B)x\| + k\|(A + B)x\| \leq (k + 1)\|(A + B)x\|.
$$

Since $A$ and $B$ are hypo-EP, for each $x \in \mathcal{H}$ there exist $k_1, k_2 > 0$ such that $\|(Ax, y)\| \leq k_1\|Ay\|$ and $\|(Bx, y)\| \leq k_2\|Bx\|$ for all $y \in \mathcal{H}$.

Thus $\|(A + B)x, y)\| \leq [k_1 k_2 (k + 1)] \|(A + B)y\|$. Hence $A + B$ is hypo-EP. □

**Corollary 4.3.** Let $A, B$ be hypo-EP operators such that $A + B$ has closed range. If $A^*B + B^*A = 0$, then $A + B$ is hypo-EP.

**Proof.** The assumption $A^*B + B^*A = 0$ gives $(A + B)(A + B)^* = A^*A + B^*B$. Then

$$
\|(A + B)x\|^2 = \langle (A + B)x, (A + B)x \rangle = \langle (A^*A + B^*B)x, x \rangle \geq \|Ax\|^2
$$

From Theorem 4.2, $A + B$ is hypo-EP. □

**Remark 4.4.** In the above theorem the condition (3) is equivalent to “$\mathcal{N}(A + B) \subseteq \mathcal{N}(A)^\perp$”. But the condition (3) is not necessary for the sum of $A$ and $B$ to be hypo-EP. For example, let $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Then $A, B$ and $A + B$ are hypo-EP. But $\mathcal{N}(A + B) \not\subseteq \mathcal{N}(A)$.

Suppose $A$ and $B$ are hypo-EP. Then by Douglas’ theorem $A^* = D_A^*A$ and $B^* = D_B^*B$ for some operators $D_A, D_B \in \mathcal{B} (\mathcal{H})$. The next theorem shows that the condition (3) is both necessary and sufficient condition for the sum to be hypo-EP under the assumption that $D_A - D_B$ is invertible.

**Theorem 4.5.** Let $A, B \in \mathcal{B}_c(\mathcal{H})$ be hypo-EP operators such that $A + B$ has closed range and $D_A - D_B$ be invertible where $D_A, D_B$ as defined above. Then $A + B$ is hypo-EP if and only if $\|Ax\| \leq k\|(A + B)x\|$ for some $k > 0$ and for all $x \in \mathcal{H}$.

**Proof.** Assume $A + B$ is hypo-EP. Then $A^* + B^* = (A + B)^* = E(A + B)$ for some $E \in \mathcal{B}(\mathcal{H})$. Hence $D_A^*A + D_B^*B = E(A + B)$ which implies that $(D_A - E)A = (E - D_B)B$.

Taking $K = D_A - E, L = E - D_B$, we have $KA = LB$ and $(K + L)A = L(A + B)$. Then $A = (K + L)^{-1}L(A + B)$, since $K + L = D_A - D_B$ is invertible. Hence $\|Ax\| \leq k\|(A + B)x\|$ for all $x \in \mathcal{H}$, where $k = \|(K + L)^{-1}L\|$. The converse follows from Theorem 4.2. □
5. Restriction of Hypo-EP Operators

Let $A \in \mathcal{B}(H)$. A closed subspace $M$ of $H$ is said to be an invariant subspace for $A$ if $A(M) \subseteq M$. $M$ is said to be a reducing subspace for $A$ if both $M$ and $M^\perp$ are invariant subspaces for $A$. In this section we discuss restriction of hypo-EP operator. For $A \in \mathcal{B}(H)$ the restriction to an invariant subspace $M$ for $A$ is denoted by $A|_M$. The adjoint of $A|_M$ is denoted by $(A|_M)^*$ and defined by $(A|_M)^* = PA^*|_M$ where $P$ is the orthogonal projection onto $M$. The restriction operator $A|_M$ coincides with the following properties as in the operator $A \in \mathcal{B}(H)$. The proof of the following proposition are obvious from the definition.

**Proposition 5.1.** Let $A, B \in \mathcal{B}(H)$ and $M$ be an invariant subspace for both $A$ and $B$. Then

1. $(A|_M)^* = A|_M$.
2. $(AB|_M)^* = (B|_M)^*(A|_M)^*$.

From the definition of hypo-EP operator, for any $A \in \mathcal{B}(H)$, we say $A|_M$ is hypo-EP if $\mathcal{R}(A|_M)$ is closed and $\mathcal{R}(A|_M) \subseteq \mathcal{R}(A|_M)^*$.

**Theorem 5.2.** Let $A \in \mathcal{B}(H)$ and $M$ be an invariant subspace for $A$ such that $A|_M$ has closed range. Then $A|_M$ is hypo-EP if and only if for each $x \in M$ there exists $k > 0$ such that

$$\langle A|_M x, y \rangle \leq k\|A|_M y\|, \text{ for all } y \in M.$$

Proof of this theorem is direct using Theorem 3.1 and Proposition 5.1.

**Corollary 5.3.** Let $A$ be a hypo-EP operator and $M$ be an invariant subspace for $A$ such that $A|_M$ has closed range. Then $A|_M$ is hypo-EP.

**Remark 5.4.** There are sufficient conditions available in literature that range of $A|_M$ is closed when $A \in \mathcal{B}(H)$. In [11] Barnes gave a sufficient condition that “$\mathcal{R}(A|_M) = \mathcal{R}(A) \cap M$”. The following example tells that the condition is not necessary.

**Example 5.5.** Let $A$ be the right shift operator on $\ell_2$ and $M = \mathcal{R}(A)$. Then $A|_M$ is hypo-EP operator, but $\mathcal{R}(A|_M) \not\subseteq \mathcal{R}(A) \cap M$.

**Theorem 5.6.** Let $A \in \mathcal{B}(H)$ and $\mathcal{R}(A)$ be a reducing subspace for $A$. If $A|_{\mathcal{R}(A)}$ is hypo-EP, then $A$ is hypo-EP.

Proof. Let $x \in H$. Then $x$ can be expressed as $x = x_1 + x_2$ such that $x_1 \in \mathcal{R}(A)$ and $x_2 \in \mathcal{R}(A)^\perp$. For all $y \in H$, $\langle Ax, y \rangle = \langle Ax_1 + Ax_2, y \rangle$ where $y_1 \in \mathcal{R}(A)$, $y_2 \in \mathcal{R}(A)^\perp$. Since $A|_{\mathcal{R}(A)}$ is hypo-EP, there exists $k > 0$ such that $\langle Ax_1, y_1 \rangle = \langle Ax_1, y_1 \rangle \leq k\|Ay_1\|$. Since $\mathcal{R}(A)$ is a reducing subspace for $A$, $\|Ay_1\|^2 = \|Ay_1\|^2 + \|Ay_2\|^2$, so $\|Ay_1\| \leq \|Ay\|$. Hence $A$ is hypo-EP.

References