Functional Analysis,
Approximation and
Computation
9 (1) (2017), 53–60

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/faac

A General Fixed Point Theorem of Meir - Keeler Type for Mappings Satisfying an Implicit Relation in Partial Metric Spaces

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Abstract. In this paper a general theorem of Meir - Keeler type for mappings satisfying an implicit relation in partial metric spaces, which generalize Theorem 2.3 and Corollary 2.4 [3] is proved.

1. Introduction

Let f, g be self mappings that are defined on a nonempty set X. We say that $x \in X$ is a coincidence point of f and g if $fx = gx$. We denote by $\mathcal{C}(f, g)$ the set of all coincidence points of f and g.

Let (X, d) be a metric space. Jungck [10] defined f and g to be compatible if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.

In 1994, Pant introduced the notion of pointwise R - weakly commuting mappings. It is proved in [18] that pointwise R - weakly commuting is equivalent to commutativity in coincidence points.

Definition 1.1 ([11]). *Two self mappings f and g that are defined on a nonempty set X are said to be weakly compatible if $fgu = gfu$ for each $u \in \mathcal{C}(f, g)$.*

In 1969, Meir and Keeler [15], established a fixed point theorem for mappings that are defined on a metric space (X, d) satisfying the following condition:
for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\varepsilon < d(x, y) < \varepsilon + \delta \text{ implies } d(fx, fy) < \varepsilon.$$

There exists a vast literature which generalizes the result of Meir-Keeler.

In 1986, Jungck [10] and Pant [16] extend these results for four mappings. It is known by Jungck [10], Pant [16], [17], [19] and other papers that, in the case of theorems for four mappings A, B, S and $T : (X, d) \to (X, d)$, a condition of Meir - Keeler type does not assure the existence of a fixed point.

2010 *Mathematics Subject Classification.* Primary 54H25; Secondary 47H10
Keywords. Fixed point; Meir - Keeler type; altering distance; integral type; occasionally weakly compatible.
Received: 24 November 2016; Accepted: 26 February 2017
Communicated by Dragan S. Djordjević
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In 1994, Matthews [14] introduced the concept of partial metric space as a part of the study of denotional semantics of dataflows networks and proved the Banach contraction principle in such spaces. Many authors studied some contractive conditions in complete partial metric spaces.

Recently, in [1], [2], [6], [7], [12], [13] and in other papers, some fixed point theorems under various contractive conditions are proved. Quite recently, some generalizations of Meir - Keeler type in partial metric spaces are proved in [3], [4], [24].

Several classical fixed point theorems and common fixed point theorems have been unified considering a general condition by an implicit relation in [21], [22] and in other papers.

Recently, the method is used in the study of fixed points in metric spaces, symmetric spaces, quasi - metric spaces, b - metric spaces, Hilbert spaces, ultra - metric spaces, convex metric spaces, compact metric spaces, paracompact metric spaces, in two and three metric spaces, for single - valued mappings, hybrid pairs of mappings and set - valued mappings.

Quite recently, the method is used in the study of fixed points for mappings satisfying a contractive condition of integral type, in fuzzy metric spaces, probabilistic metric spaces, intuitionistic metric spaces, ordered metric spaces and G - metric spaces.

With this method, the proofs of some fixed point theorems are simpler. Also, the method allows the study of local and global properties of fixed point structures.

The study of coincidence and fixed points for mappings satisfying implicit relations in partial metric spaces is initiated in [8], [9], [25], [5].

Fixed point theorems for mappings of Meir - Keeler type satisfying an implicit relation in metric spaces are proved in [23], [20].

In this paper, a general fixed point theorem of Meir - Keeler type for mappings satisfying an implicit relation in partial metric spaces, which generalize Theorem 2.1 and Corollary 2.4 [3] is proved.

2. Preliminaries

Definition 2.1 ([14]). *Let X be a nonempty set. A function $p : X \times X \to \mathbb{R}_+$ is said to be a partial metric on X if for any $x, y, z \in X$, the following conditions hold:*

$(P_1) : p(x,x) = p(y,y) = p(x,y)$ *if and only if* $x = y$,

$(P_2) : p(x,x) \leq p(x,y)$,

$(P_3) : p(x,y) = p(y,x)$,

$(P_4) : p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$.

The pair (X,p) is called a partial metric space.

If $p(x,y) = 0$, then $x = y$, but the converse does not always hold.

If $x \neq y$, then $p(x,y) > 0$.

Each partial metric space on X generates a T_0 - topology τ_p which has as base the family of open p - balls $\{B_p(x,\varepsilon) : x \in X \text{ and } \varepsilon > 0\}$, where $B_p(x,\varepsilon) = \{y \in X : p(x,y) \leq p(x,x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

A sequence $\{x_n\}$ in a partial metric space (X,p) converges to a limit $x \in X$ with respect to τ_p if and only if $p(x,x) = \lim_{n\to\infty} p(x,x_n)$.

If p is a partial metric on X, then the function

$$d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y)$$

is a metric on X.

Definition 2.2 ([14]). *Let (X,p) be a partial metric space.*

a) A sequence $\{x_n\}$ in X is said to be a Cauchy sequence if $\lim_{n,m\to\infty} p(x_n,x_m)$ exists and is finite.

b) (X,p) is said to be complete if every Cauchy sequence in X converges with respect to τ_p to a point $x \in X$, that is $\lim_{m,n\to\infty} p(x_n,x_m) = p(x,x)$.

Lemma 2.3 ([14], [2]). *Let (X,p) be a partial metric space. Then:*

a) A sequence $\{x_n\}$ is Cauchy if and only if $\{x_n\}$ is Cauchy in the metric space (X,d_p).

b) (X,p) is complete if and only if the metric space (X,d_p) is complete. Moreover, $\lim_{n\to\infty} d_p(x_n,x) = 0$ if and only if

$$\lim_{n\to\infty} p(x,x_n) = \lim_{n,m\to\infty} p(x_n,x_m) = p(x,x). \tag{1}$$

Lemma 2.4 ([2], [13]). *Let (X,p) be a partial metric space. If $\lim_{n\to\infty} x_n = x$ and $p(x,x) = 0$, then $\lim_{n\to\infty} p(x_n,y) = p(x,y)$, $\forall y \in X$.*

The following Meir - Keeler common fixed point theorem on partial metric spaces is proved in [3].

Theorem 2.5 (Theorem 2.3 [3]). *Let A,B,S and T be self mappings defined on a partial metric space (X,p) satisfying the following conditions:*

(C_1) $AX \subset TX$ and $BX \subset SX$,

(C_2) for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x,y \in X$,

$$\varepsilon < M(x,y) < \varepsilon + \delta \text{ implies } p(Ax,By) \le \varepsilon,$$

where

$$M(x,y) = \max\left\{p(Sx,Ty), p(Sx,Ax), p(Ty,By), \frac{1}{2}\left[p(Sx,By) + p(Ty,Ax)\right]\right\},$$

(C_3) for all $x,y \in X$ with $M(x,y) > 0$ implies $p(Ax,By) < M(x,y)$,

(C_4) $p(Ax,By) \le \max\{a[p(Sx,Ty) + p(Sx,Ax) + p(Ty,By)], b[p(Sx,By) + p(Ty,Ax)]\}$, for all $x,y \in X$, $0 \le a < \frac{1}{2}, 0 \le b < \frac{1}{2}$.

If one of AX, BX, TX and SX is a closed subset of X, then

(i) A and S have a coincidence point,

(ii) B and T have a coincidence point.

Moreover, if A and S, as well as, B and T are weakly compatible, then A,B,S and T have a unique common fixed point.

Theorem 2.6 (Corollary 2.4 [3]). *Let A,B,S and T be self mappings defined on a partial metric space (X,p) satisfying the following conditions:*

(C_1) $AX \subset TX$ and $BX \subset SX$,

(C_2) for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x,y \in X$,

$$\varepsilon < M(x,y) < \varepsilon + \delta \text{ implies } p(Ax,By) \le \varepsilon,$$

where

$$M(x,y) = \max\left\{p(Sx,Ty), p(Sx,Ax), p(Ty,By), \frac{1}{2}\left[p(Sx,By) + p(Ty,Ax)\right]\right\},$$

(C_3) for all $x,y \in X$ with $M(x,y) > 0$ implies $p(Ax,By) < M(x,y)$,

(C_4) $p(Ax,By) \le k[p(Sx,Ty) + p(Sx,Ax) + p(Ty,By) + p(Sx,By) + p(Ty,Ax)]$, for all $x,y \in X$ and $0 < k < \frac{1}{3}$.

If one of AX, BX, TX and SX is a complete subspace of X, then

(i) A and S have a coincidence point,

(ii) B and T have a coincidence point.

Moreover, if A and S, as well as, B and T are weakly compatible, then A,B,S and T have a unique common fixed point.

Remark 2.7. *It is shown in [24] that in Theorem 2.6, $0 \le k < \frac{1}{4}$.*

3. Implicit relation

Definition 3.1. *Let $\mathcal{F}_{MK}$ be the set of all real continuous mappings $F(t_1, ..., t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ satisfying the following conditions:*
$(F_1) :$ *F is nonincreasing in variables $t_2, t_3, ..., t_6$,*
$(F_2) :$ *$F(t, t, 0, t, t, t) \leq 0$ implies $t = 0$,*
$(F_3) :$ *$F(t, t, t, 0, t, t) \leq 0$ implies $t = 0$.*

In the following examples the property (F_1) is obviously.

Example 3.2. $F(t_1, ..., t_6) = t_1 - \max\{a(t_2 + t_3 + t_4), b(t_5 + t_6)\}$, *where* $0 \leq a < \dfrac{1}{2}$ *and* $0 \leq b < \dfrac{1}{2}$.
$(F_2) :$ $F(t, t, 0, t, t, t) = t\,(1 - \max\{2a, 2b\}) \leq 0$ *implies* $t = 0$.
$(F_3) :$ $F(t, t, t, 0, t, t) = t\,(1 - \max\{2a, 2b\}) \leq 0$ *implies* $t = 0$.

Example 3.3. $F(t_1, ..., t_6) = t_1 - k\,(t_2 + t_3 + t_4 + t_5 + t_6)$, *where* $k \in \left[0, \dfrac{1}{4}\right)$.
$(F_2) :$ $F(t, t, 0, t, t, t) = t\,(1 - 4k) \leq 0$ *implies* $t = 0$.
$(F_3) :$ $F(t, t, t, 0, t, t) = t\,(1 - 4k) \leq 0$ *implies* $t = 0$.

Example 3.4. $F(t_1, ..., t_6) = t_1 - k \max\{t_2, t_3, t_4, t_5, t_6\}$, *where* $k \in [0, 1)$.
$(F_2) :$ $F(t, t, 0, t, t, t) = t\,(1 - k) \leq 0$ *implies* $t = 0$.
$(F_3) :$ $F(t, t, t, 0, t, t) = t\,(1 - k) \leq 0$ *implies* $t = 0$.

Example 3.5. $F(t_1, ..., t_6) = t_1 - k \max\left\{t_2, t_3, t_4, \dfrac{t_5 + t_6}{2}\right\}$, *where* $k \in [0, 1)$.
$(F_2) :$ $F(t, t, 0, t, t, t) = t\,(1 - k) \leq 0$ *implies* $t = 0$.
$(F_3) :$ $F(t, t, t, 0, t, t) = t\,(1 - k) \leq 0$ *implies* $t = 0$.

Example 3.6. $F(t_1, ..., t_6) = t_1 - at_2 - bt_3 - ct_4 - dt_5 - et_6$, *where* $a, b, c, d, e \geq 0$, $a + c + d + e < 1$ *and* $a + b + d + e < 1$.
$(F_2) :$ $F(t, t, 0, t, t, t) = t[1 - (a + c + d + e)] \leq 0$ *implies* $t = 0$.
$(F_3) :$ $F(t, t, t, 0, t, t) = t[1 - (a + b + d + e)] \leq 0$ *implies* $t = 0$.

Example 3.7. $F(t_1, ..., t_6) = t_1 - \alpha \max\{t_2, t_3, t_4\} - (1 - \alpha)\,(at_5 + bt_6)$, *where* $\alpha \in (0, 1)\,, a, b \geq 0$ *and* $a + b < 1$.
$(F_2) :$ $F(t, t, 0, t, t, t) = t\,(1 - \alpha)\,(a + b) \leq 0$ *implies* $t = 0$.
$(F_3) :$ $F(t, t, t, 0, t, t) = t\,(1 - \alpha)\,(a + b) \leq 0$ *implies* $t = 0$.

Example 3.8. $F(t_1, ..., t_6) = t_1 - \max\{ct_2, ct_3, ct_4, at_5 + bt_6\}$, *where* $c \in (0, 1)\,, a, b \geq 0$ *and* $a + b < 1$.
$(F_2) :$ $F(t, t, 0, t, t, t) = t\,(1 - \max\{c, a + b\}) \leq 0$ *implies* $t = 0$.
$(F_3) :$ $F(t, t, t, 0, t, t) = t\,(1 - \max\{c, a + b\}) \leq 0$ *implies* $t = 0$.

Example 3.9. $F(t_1, ..., t_6) = t_1 - at_2 - b \max\{t_3, t_4\} - c \max\{t_3 + t_4, t_5 + t_6\}$, *where* $a, b, c \geq 0$ *and* $a + b + 2c < 1$.
$(F_2) :$ $F(t, t, 0, t, t, t) = t[1 - (a + b + 2c)] \leq 0$ *implies* $t = 0$.
$(F_3) :$ $F(t, t, t, 0, t, t) = t[1 - (a + b + 2c)] \leq 0$ *implies* $t = 0$.

4. Main results

Theorem 4.1. *Let A, B, S and T be self mappings on a complete partial metric space (X, p) satisfying the following conditions:*

$$AX \subset TX \text{ and } BX \subset SX, \tag{2}$$

$$\begin{gathered} \text{for all } \varepsilon > 0, \text{ there exists } \delta > 0 \text{ such that for all } x, y \in X, \\ \varepsilon < M(x,y) < \varepsilon + \delta \text{ implies } p(Ax, By) \leq \varepsilon, \\ \text{where } M(x,y) = \max\left\{p(Sx,Ty), p(Sx,Ax), p(Ty,By), \tfrac{1}{2}\left[p(Sx,By) + p(Ty,Ax)\right]\right\}, \end{gathered} \tag{3}$$

$$\text{for all } x, y \in X \text{ with } M(x,y) > 0 \text{ implies } p(Ax, By) < M(x,y), \tag{4}$$

$$\begin{gathered} F(p(Ax,By), p(Sx,Ty), p(Sx,Ax), p(Ty,By), p(Sx,By), p(Ty,Ax)) \leq 0, \\ \text{for all } x, y \in X \text{ and } F \in \mathfrak{F}_{MK}. \end{gathered} \tag{5}$$

If one of AX, BX, SX, TX is a closed subset of (X,p), then
a) $\mathcal{C}(A,S) \neq \varnothing$,
b) $\mathcal{C}(B,T) \neq \varnothing$.

Moreover, if A and S, as well B and T are weakly compatible, then A, B, S and T have a unique common fixed point.

Proof. Let x_0 be an arbitrary point in X. Since $AX \subset TX$, there exists $x_1 \in X$ such that $Tx_1 = Ax_0$. Since $BX \subset SX$, there exists $x_2 \in X$ such that $Sx_2 = Bx_1$.

Continuing this process we construct two sequences $\{x_n\}$ and $\{y_n\}$ in X defined by

$$y_{2n} = Tx_{2n+1} = Ax_{2n},\ y_{2n+1} = Sx_{2n+2} = Bx_{2n+1}, \forall n \in \mathbb{N}.$$

Suppose that $p(y_{2n}, y_{2n+1}) = 0$ for some $n \in \mathbb{N}$. Then $y_{2n} = y_{2n+1}$ which implies $Ax_{2n} = Tx_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$. Then T and B have a coincidence point. Further, $p(y_{2n+1}, y_{2n+2}) = 0$ for some $n \in \mathbb{N}$, then $Ax_{2n+2} = Tx_{2n+3} = Bx_{2n+1} = Sx_{2n+2}$, so A and S have a coincidence point. For the rest, assume that $p(y_n, y_{n+1}) > 0$, $\forall n \in \mathbb{N}$. If for some $x, y \in X$, $M(x,y) = 0$ then we have $Ax = Sx$ and $By = Ty$, hence $\mathcal{C}(S,A) \neq \varnothing$ and $\mathcal{C}(B,T) \neq \varnothing$.

If $M(x,y) > 0$ for all $x, y \in X$, as in [3] we obtain that for sequence $\{y_n\}$, $\lim_{n\to\infty} p(y_n, y_{n+1}) = 0$, so $\{y_n\}$ is a Cauchy sequence in (X, d_p) and by Lemma 2.3, $\{y_n\}$ is a Cauchy sequence in (X,p). Since (X,p) is complete, there exists $y \in X$ such that from Lemmas 2.3, 2.4 and $\lim_{n\to\infty} p(y_n, y_{n+1}) = 0$ we have

$$p(y,y) = \lim_{n\to\infty} p(y_n, y) = \lim_{n\to\infty} p(y_n, y_n) = 0.$$

This implies that

$$\lim_{n\to\infty} p(y_{2n}, y) = \lim_{n\to\infty} p(y_{2n-1}, y) = 0.$$

Hence,

$$\lim_{n\to\infty} p(Ax_{2n}, y) = \lim_{n\to\infty} p(Tx_{2n-1}, y) = 0$$

and

$$\lim_{n\to\infty} p(Bx_{2n-1}, y) = \lim_{n\to\infty} p(Sx_{2n}, y) = 0.$$

Now we suppose without loss of generality that SX is a closed subset of the partial metric space (X,p). Then there exists $u \in X$ such that $y = Su$.

By (5) we have successively

$$F\left(\begin{array}{c} p(Au, Bx_{2n+1}), p(Su, Tx_{2n+1}), p(Su, Au), \\ p(Tx_{2n+1}, Bx_{2n+1}), p(Su, Bx_{2n+1}), p(Tx_{2n+1}, Au) \end{array}\right) \leq 0,$$

$$F\left(\begin{array}{l} F(p(Au,y_{2n+1}),p\left(Su,y_{2n}\right),p\left(Su,Au\right), \\ p\left(y_{2n},y_{2n+1}\right),p\left(Su,y_{2n+1}\right),p\left(y_{2n},Au\right) \end{array}\right) \leq 0.$$

Letting n tends to infinity we obtain

$$F(p(y,Au),p\left(y,Su\right),p\left(Su,Au\right),0,p\left(y,Su\right),p\left(y,Au\right)) \leq 0,$$

$$F(p(Au,Su),p\left(Su,Su\right),p\left(Su,Au\right),0,p\left(Su,Su\right),p\left(Su,Au\right)) \leq 0.$$

By (P_2) and (F_1) we have

$$F(p(Au,Su),p\left(Su,Au\right),p\left(Su,Au\right),0,p\left(Su,Au\right),p\left(Su,Au\right)) \leq 0.$$

By (F_3) we have $p(Au,Su)=0$, i.e. $Au=Su$. Hence, $y=Au=Su$ and $\mathcal{C}(A,S)\neq\varnothing$.
Since $AX\subset TX$, $y\in TX$. Hence, there exists $v\in X$ such that $y=Tv$.
Again, by (5) we obtain

$$F\left(\begin{array}{l} p(Au,Bv),p\left(Su,Tv\right),p\left(Su,Au\right), \\ p\left(Tv,Bv\right),p\left(Su,Bv\right),p\left(Tv,Au\right) \end{array}\right) \leq 0,$$

$$F(p(y,Bv),0,0,p\left(y,Bv\right),p\left(y,Bv\right),0) \leq 0,$$

which implies by (F_1) and (P_2) that

$$F(p(Tv,Bv),p(Tv,Bv),0,p(Tv,Bv),p(Tv,Bv),0) \leq 0.$$

By (F_2), $p(Tv,Bv)=0$, i.e. $Tv=Bv=y$ and $\mathcal{C}(B,T)\neq\varnothing$.
Hence, $y=Su=Au=Tv=Bv$.
Suppose that A and S are weakly compatible and B and T are, also, weakly compatible.
By $y=Su=Au$ we obtain $Sy=SAu=ASu=Ay$ and by $y=Tv=Bv$ we obtain $Ty=TBv=BTv=By$.
By (5) we obtain

$$F\left(\begin{array}{l} p(Au,By),p\left(Su,Ty\right),p\left(Su,Au\right), \\ p\left(Ty,By\right),p\left(Su,By\right),p\left(Ty,Au\right) \end{array}\right) \leq 0,$$

$$F(p(y,By),p\left(y,By\right),0,p\left(By,By\right),p\left(y,By\right),p\left(y,By\right)) \leq 0.$$

Since by (P_2) $p(By,By)\leq p(y,By)$, then by (F_1) we obtain

$$F(p(y,By),p\left(y,By\right),0,p\left(y,By\right),p\left(y,By\right),p\left(y,By\right)) \leq 0,$$

which implies by (F_2) that $p\left(y,By\right)=0$, i.e. $y=By=Ty$ and y is a common fixed point of B and T.
Again, by (5) we obtain

$$F\left(\begin{array}{l} p(Ay,Bv),p\left(Sy,Tv\right),p\left(Sy,Ay\right), \\ p\left(Tv,Bv\right),p\left(Sy,Bv\right),p\left(Tv,Ay\right) \end{array}\right) \leq 0,$$

$$F(p(Ay,y),p\left(Ay,y\right),p\left(Ay,Ay\right),0,p\left(Ay,y\right),p\left(y,Ay\right)) \leq 0.$$

By (F_1) and (P_2) we obtain

$$F(p(Ay,y),p\left(Ay,y\right),p\left(Ay,y\right),0,p\left(Ay,y\right),p\left(y,Ay\right)) \leq 0,$$

which implies by (F_3) that $p(Ay, y) = 0$, i.e. $y = Ay = Sy$. Hence, y is a fixed point of A and S. Therefore, y is a common fixed point of A, B, S and T.

Suppose that w is another common fixed point of A, B, S and T. Then by (5) we have

$$F\left(\begin{array}{c} p(Ay,Bw), p(Sy,Tw), p(Sy,Ay), \\ p(Tw,Bw), p(Sy,Bw), p(Tw,Ay) \end{array}\right) \leq 0,$$

$$F(p(y,w), p(y,w), 0, p(w,w), p(y,w), p(y,w))$$

By (F_1) and (P_2) we obtain

$$F(p(y,w), p(y,w), 0, p(y,w), p(y,w), p(y,w)) \leq 0,$$

which implies by (F_2) that $p(y, w) = 0$, i.e. $y = w$. Hence, y is a unique common fixed point of A, B, S and T. $\square$

Theorem 4.2. *Let A, B, S and T be self mappings on a partial metric space (X, p) satisfying the following conditions:*

$$AX \subset TX \text{ and } BX \subset SX, \tag{6}$$

$$\begin{array}{c} \text{for all } \varepsilon > 0, \text{ there exists } \delta > 0 \text{ such that for all } x, y \in X, \\ \varepsilon < M(x,y) < \varepsilon + \delta \text{ implies } p(Ax, By) \leq \varepsilon, \\ \text{where } M(x,y) = \max\left\{p(Sx,Ty), p(Ax,Sx), p(By,Ty), \frac{1}{2}[p(Sx,By) + p(Ty,Ax)]\right\}, \end{array} \tag{7}$$

$$\text{for all } x, y \in X \text{ with } M(x,y) > 0 \text{ implies } p(Ax,By) < M(x,y), \tag{8}$$

$$\begin{array}{c} F(p(Ax,By), p(Sx,Ty), p(Sx,Ax), p(Ty,By), p(Sx,By), p(Ty,Ax)) \leq 0, \\ \text{for all } x, y \in X \text{ and } F \in \mathfrak{F}_{MK}. \end{array} \tag{9}$$

If one of AX, BX, SX, TX is a complete subspace of (X, p), then

a) $\mathcal{C}(A, S) \neq \varnothing$,

b) $\mathcal{C}(B, T) \neq \varnothing$.

Moreover, if A and S, as well B and T are weakly compatible, then A, B, S and T have a unique common fixed point.

Proof. The proof is similar to the proof of Theorem 4.1. $\square$

Remark 4.3. *1) By Theorem 4.1 and Example 3.3 we obtain Theorem 2.5.*

2) By Theorems 4.1 and Example 3.3 we obtain Theorem 2.6.

3) By Theorems 4.1, 4.2 and Examples 3.4 - 3.9 we obtain new particular results.

References

[1] T. Abdeljawad, E. Karapinar and K. Taş, *Existence and uniqueness of common fixed points on partial metric spaces*, Appl. Math. Lett. 24 (1) (2011) 1900 - 1904.

[2] I. Altun, F. Sola and H. Simsek, *Generalized contractions in partial metric spaces*, Topology Appl. 197 (18) (2010) 2778 - 2785.

[3] H. Aydi and E. Karapinar, *A Meir - Keeler common fixed point theorem on partial metric spaces*, Fixed Point Theory Appl. 2012 2012:26.

[4] H. Aydi, E. Karapinar and S. Rezapur, *A generalized Meir - Keeler type contractions on partial metric spaces*, Abstr. and Appl. Analysis 2012, Article ID 287127.

[5] H. Aydi, M. Jellali and E. Karapinar, *Common fixed points for alpha - implicit contractions in partial metric spaces. Consequences and applications*, RACSAM, Ser. A Math. (accepted in 2014).

[6] R. P. Chi, E. Karapinar and T. D. Thanh, *A generalized contraction principle in partial metric spaces*, Math. Comput. Modelling 55 (2012) 1673 - 1681.

[7] L. Ćirić, P. Samet, H. Aydi and C. Vetro, *Common fixed point of generalized contraction on partial metric spaces and applications*, Appl. Math. Comput. 218 (2011) 2398 - 2406.

[8] S. Gülyaz, E. Karapinar, *Coupled fixed point results in partially ordered metric spaces through implicit function*, Hacet. J. Math. Stat. 43 (4) (2013) 347 - 357.

[9] S. Gülyaz, E. Karapinar and I. S. Yüce, *A coupled coincidence point theorem in partially ordered metric spaces with an implicit relation*, Fixed Point Theory Appl. 2013 2013:38.

[10] G. Jungck, *Compatible mappings and common fixed points*, Intern. J. Math. Math. Sci. 9 (1986) 771 - 779.

[11] G. Jungck, *Common fixed points for noncontinuous nonself maps on a nonnumeric space*, Far East J. Math. Sci. 4 (2) (1996) 199 - 215.

[12] Z. Kadelburg, H. K. Nashine and S. Radenović, *Fixed point results under various contractive conditions in partial metric spaces*, RACSAM 107 (2013) 241 - 256.

[13] E. Karapinar and U. Yüksel, *Some common fixed point theorems in partial metric spaces*, J. Appl. Math. Volume 2011, Article ID 263621.

[14] S. Matthews, *Partial metric topology*, Proc. 8th Summer Conf. General Topology Appl. New York Acad. Sci. 728 (1994) 183 - 192.

[15] A. Meir and E. Keeler, *A theorem of contraction mappings*, J. Math. Anal. Appl. 28 (1969) 326 - 329.

[16] R. P. Pant, *Common fixed point of two pairs of commuting mappings*, Indian J. Pure Appl. Math. 17 (2) (1986) 187 - 192.

[17] R. P. Pant, *Common fixed points of weakly commuting mappings*, Math. Student 62, 1 - 4 (1993) 97 - 102.

[18] R. P. Pant, *Common fixed points for four mappings*, Calcutta Math. Soc. 9 (1998) 281 - 287.

[19] R. P. Pant, *Common fixed point theorems for contractive maps*, J. Math. Anal. Appl. 226 (1998) 281 - 286.

[20] A.-M. Patriciu and V. Popa, *Altering distances, some generalizations of Meir - Keeler theorems and applications*, Anal. Şt. Univ. Ovidius Constanţa 21 (2) (2013) 207 - 222.

[21] V. Popa, *Fixed point theorems for implicit contractive mappings*, Stud. Cercet. Ştiinţ., Ser. Mat., Univ. Bacău 7 (1997) 129 - 133.

[22] V. Popa, *Some fixed point theorem for four compatible mappings satisfying an implicit relation*, Demontr. Math. 32 (1999) 157 - 163.

[23] V. Popa, *A general common fixed point theorem of Meir – Keeler type for noncontinuous weak compatible mappings*, Filomat 18 (2004) 33 - 40.

[24] K. P. R. Rao, G. N. V. Kishare and P. R. S. Babu, *A unique common fixed point theorem of Meir - Keeler type in partial metric spaces*, Math. Stat. 1 (2) (2013) 19 - 24.

[25] C. Vetro and F. Vetro, *Common fixed points for mappings satisfying implicit relations in partial metric spaces*, J. Nonlinear Sci. Appl. 6 (2013) 152 - 161.