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Quasinormalty and subscalarity of class p-$wA(s,t)$ operators

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Abstract. Let T be a bounded linear operator on a complex Hilbert space $\mathcal{H}$ and let $T = U|T|$ be the polar decomposition of T. An operator T is called a class p-$wA(s,t)$ operator if $(|T^*|^t|T|^{2s}|T^*|^t)^{\frac{tp}{s+t}} \geq |T^*|^{2tp}$ and $(|T|^s|T^*|^{2t}|T|^s)^{\frac{sp}{s+t}} \leq |T|^{2sp}$ where $0 < s,t$ and $0 < p \leq 1$. We investigate quasinormality and subscalarity of class p-$wA(s,t)$ operators.

Dedicated to the memory of Professor Takayuki Furuta with deep gratitude.

1. Introduction

Let $B(\mathcal{H})$ denote the algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$. Aluthge [1] introduced p-hyponormal operator T which is defined as $(T^*T)^p \geq (TT^*)^p$ where $0 < p \leq 1$, and proved interesting properties of p-hyponormal operators by using Furuta's inequality [8]. If $p = 1$, T is called hyponormal. Hence p-hyponormality is a generalization of hyponormality. It is known that p-hyponormal operators have many interesting properties as hyponormal operators, for example, Putnam's inequality, Fuglede-Putnam type theorem, Bishop's property (β), Weyl's theorem and polaroid property. After this discovery, many authors are investigating new generalizations of hyponormal operator.

Let $T \in B(\mathcal{H})$ and $|T| = (T^*T)^{\frac{1}{2}}$. By taking $U|T|x = Tx$ for $x \in \mathcal{H}$ and $Ux = 0$ for $x \in \ker|T|$, T has a unique polar decomposition $T = U|T|$ with condition $\ker U = \ker|T|$. We say that $T = U|T|$ is the polar decomposition of T in this paper.

The authors [14] introduced class p-$wA(s,t)$ operator as follows:

Definition 1.1. *An operator T is called a class p-$wA(s,t)$ operator if*

$$(|T^*|^t|T|^{2s}|T^*|^t)^{\frac{tp}{s+t}} \geq |T^*|^{2tp}$$

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and

$$|T|^{2sp} \geq (|T|^s|T^*|^{2t}|T|^s)^{\frac{sp}{s+t}}$$

where $0 < s, t$ *and* $0 < p \leq 1$.

It is known that p-hyponormal operators and log-hyponormal operators are class 1-$wA(s,t)$ operators for any $0 < s, t$. Class 1-$wA(1,1)$ is called class A and class 1-$wA(\frac{1}{2}, \frac{1}{2})$ is called w-hyponormal [7, 9, 10, 15]. Hence the class of p-$wA(s,t)$ operators is a generalization of the class of A and w-hyponormal operators.

C. Yang and J. Yuan [16–18] studied a class of $wF(p,r,q)$ operators T, i.e.,

$$\left(|T^*||T|^{2p}|T^*|^r\right)^{\frac{1}{q}} \geq |T^*|^{\frac{2(p+r)}{q}}$$

and

$$|T|^{2(p+r)(1-\frac{1}{q})} \geq \left(|T|^p|T^*|^{2r}|T|^p\right)^{1-\frac{1}{q}}$$

where $0 < p, 0 < r, 1 \leq q$. If we take small p_1 such that $0 < p_1 \leq \frac{p+r}{qr}$ and $p_1 \leq \frac{(p+r)(q-1)}{pq}$, then T is a class p_1-$wA(p,r)$ operator. Hence the class of p_1-$wA(p,r)$ opertors is a generalization of the class of $wF(p,r,q)$ operators.

Aluthge transformation [1] is a good tool in operator theory. I. B. Jung, E. Ko and C. Pearcy [11] studied spectral properties of Aluthge transformation.

Definition 1.2. *Let* $T = U|T|$ *be the polar decomposition of* $T \in B(\mathcal{H})$. *Then generalized Aluthge transformation is defined by*

$$T(s,t) = |T|^s U |T|^t$$

where $0 < s, t$.

2. Main Results

It is known that an operator T is a class p-$wA(s,t)$ operator if and only if $|T(s,t)|^{\frac{2tp}{s+t}} \geq |T|^{2tp}$ and $|T|^{2sp} \geq |T(s,t)^*|^{\frac{2sp}{s+t}}$ by [14]. As a continuation of [14], we investigate quasinormality and subscalarity of class p-$wA(s,t)$ operators. To prove main results, we need the following Lemma. The proof is essentially due to C. Yang and J. Yuan (Proposition 3.4 of [18]). For completeness, we prove the following Lemma.

Lemma 2.1. [18] *If* T *is a class* p-$wA(s,t)$ *operator and* $0 < s \leq s_1, 0 < t \leq t_1, 0 < p_1 \leq p \leq 1$, *then* T *is a class* p_1-$wA(s_1,t_1)$ *operator.*

Proof. Let T be class p-$wA(s,t)$. Then

$$(|T^*|^t|T|^{2s}|T^*|^t)^{\frac{tp}{s+t}} \geq |T^*|^{2tp} \tag{1}$$

and

$$|T|^{2sp} \geq (|T|^s|T^*|^{2t}|T|^s)^{\frac{sp}{s+t}}. \tag{2}$$

We prove that T is a p-$wA(s_1,t_1)$ operator. Then T is a p_1-$wA(s_1,t_1)$ operator by Lowner-Heinz's inequality.

Let $A_1 = (|T^*|^t|T|^{2s}|T^*|^t)^{\frac{tp}{s+t}}$ and $B_1 = |T^*|^{2tp}$. Since (1) implies $A_1 \geq B_1$, we have

$$\left(B_1^{\frac{r_2}{2}} A_1^{p_2} B_1^{\frac{r_2}{2}}\right)^{\frac{1+r_2}{p_2+r_2}} \geq B_1^{1+r_2}$$

for any $r_2 > 0$ and $p_2 \geq 1$ by Furuta's inequality [8]. Let

$$\beta \geq t, p_2 = \frac{s+t}{tp} \geq 1, r_2 = \frac{\beta - t}{tp} \geq 0.$$

Then

$$\left(|T^*|^\beta |T|^{2s} |T^*|^\beta\right)^{\frac{tp+\beta-t}{s+t_1}} \geq |T^*|^{2tp+2\beta-2t}.$$

Hence we have

$$\left(|T^*|^\beta |T|^{2s} |T^*|^\beta\right)^{\frac{w}{s+\beta}} \geq |T^*|^{2w}$$

for any $0 < w \leq tp + \beta - t$.

Let

$$f_s(\beta) = \left(|T|^s |T^*|^{2\beta} |T|^s\right)^{\frac{s}{s+\beta}}$$

for $\beta \geq t$. Then

$$\begin{aligned} f_s(\beta) &= \left\{\left(|T|^s |T^*|^{2\beta} |T|^s\right)^{\frac{s+\beta+w}{s+\beta}}\right\}^{\frac{s}{s+\beta+w}} \\ &= \left\{|T|^s |T^*|^\beta \left(|T^*|^\beta |T|^{2s} |T^*|^\beta\right)^{\frac{w}{s+\beta}} |T^*|^\beta |T|^s\right\}^{\frac{s}{s+\beta+w}} \\ &\geq \left\{|T|^s |T^*|^\beta |T^*|^{2w} |T^*|^\beta |T|^s\right\}^{\frac{s}{s+\beta+w}} \\ &= \left\{|T|^s |T^*|^{2(\beta+w)} |T|^s\right\}^{\frac{s}{s+\beta+w}} \\ &= f_s(\beta + w). \end{aligned}$$

Hence $f_s(\beta)$ is decreasing for $\beta \geq t$.

Then, by (2),

$$\begin{aligned} |T|^{2sp} &\geq (|T|^s |T^*|^{2t} |T|^s)^{\frac{sp}{s+t}} \\ &= \{f_s(t)\}^p \\ &\geq \{f_s(t_1)\}^p = \left(|T|^s |T^*|^{2t_1} |T|^s\right)^{\frac{sp}{s+t_1}}. \end{aligned}$$

Let $A_2 = |T|^{2sp}$ and $B_2 = \left(|T|^s |T^*|^{2t_1} |T|^s\right)^{\frac{sp}{s+t_1}}$. Then

$$A_2^{1+r_3} \geq \left(A_2^{\frac{r_3}{2}} B_2^{p_3} A_2^{\frac{r_3}{2}}\right)^{\frac{1+r_3}{p_3+r_3}}$$

for any $r_3 \geq 0$ and $p_3 \geq 1$ by Furuta's inequality [8]. Let

$$p_3 = \frac{s+t_1}{sp} \geq 1, r_3 = \frac{s_1 - s}{sp} \geq 0.$$

Then

$$|T|^{2sp+2s_1-2s} \geq \left(|T|^{s_1} |T^*|^{2t_1} |T|^{s_1}\right)^{\frac{sp+s_1-s}{s_1+t_1}}.$$

Since

$$sp + s_1 - s - s_1 p = (s_1 - s)(1 - p) \geq 0,$$

we have

$$|T|^{2s_1 p} \geq \left(|T|^{s_1} |T^*|^{2t_1} |T|^{s_1}\right)^{\frac{s_1 p}{s_1+t_1}}.$$

Similarly, we have

$$(|T^*|^{t_1}|T|^{2s_1}|T^*|^{t_1})^{\frac{t_1 p}{s_1+t_1}} \geq |T^*|^{2t_1 p}.$$

Hence T is a p-$wA(s_1, t_1)$ operator.

□

At first, we investigate quasinormality of class p-$wA(s,t)$ operator. Let $T = U|T|$ be the polar decomposition. We say T is quasinormal if $U|T| = |T|U$. It is known that if T is a class $A(s,t)$ operator with $0 < s,t$ and $T(s,t)$ is quasinormal, then T is also quasinormal by [13]. We extend this resullt as follows.

Theorem 2.2. *Let $T = U|T|$ be a class p-$wA(s,t)$ operator with $0 < s,t$ and $0 < p \leq 1$. If $T(s,t) = |T|^s U|T|^t$ is quasinormal, then T is also quasinormal. Hence T coincides with its Aluthge transform $T(s,t) = |T|^s U|T|^t$ if $s+t=1$.*

Proof. Since T is a class p-$wA(s,t)$ operator,

$$|T(s,t)|^{\frac{2rp}{s+t}} \geq |T|^{2rp} \geq |T(s,t)^*|^{\frac{2rp}{s+t}} \tag{3}$$

for all $r \in (0, \min\{s,t\}]$. Then Douglas's theorem [5] implies that

$$\operatorname{ran}|T(s,t)|^{\frac{rp}{s+t}} \supset \operatorname{ran}|T|^{rp} \supset \operatorname{ran}|T(s,t)^*|^{\frac{rp}{s+t}}.$$

Hence

$$[\operatorname{ran}|T(s,t)|] \supset [\operatorname{ran}|T|] \supset [\operatorname{ran}|T(s,t)^*|] = [\operatorname{ran} T(s,t)]$$

where $[\mathcal{M}]$ denotes the norm closure of $\mathcal{M} \subset \mathcal{H}$. Since $\ker|T| \subset \ker(|T|^s U|T|^t) = \ker T(s,t)$, we have

$$\begin{aligned}[\operatorname{ran}|T|] &= (\ker|T|)^{\perp} \supset (\ker T(s,t))^{\perp} \\ &= (\ker|T(s,t)|)^{\perp} = [\operatorname{ran}|T(s,t)|].\end{aligned}$$

Hence

$$[\operatorname{ran}|T(s,t)|] = [\operatorname{ran}|T|].$$

Let $T(s,t) = W|T(s,t)|$ be the polar decomposition of $T(s,t)$. Then

$$\begin{aligned}E := W^*W &= U^*U \\ &= \text{ the orthogonal projection onto } [\operatorname{ran}|T|] \\ &\geq \text{ the orthogonal projection onto } [\operatorname{ran} T(s,t)] = WW^* =: F.\end{aligned}$$

Put

$$|T(s,t)^*|^{\frac{1}{s+t}} = \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix}$$

on $\mathcal{H} = [\operatorname{ran} T(s,t)] \oplus \ker T(s,t)^*$. Then X is injective and has a dense range. Since $W \subset [\operatorname{ran} T(s,t)]$, we have

$$W = \begin{pmatrix} W_1 & W_2 \\ 0 & 0 \end{pmatrix}.$$

Since $T(s,t)$ is quasinormal, W commutes with $|T(s,t)|$ and

$$\begin{aligned}|T(s,t)|^{\frac{2rp}{s+t}} &= W^*W|T(s,t)|^{\frac{2rp}{s+t}} = W^*|T(s,t)|^{\frac{2rp}{s+t}}W \\ &\geq W^*|T|^{2rp}W \geq W^*|T(s,t)^*|^{\frac{2rp}{s+t}}W = |T(s,t)|^{\frac{2rp}{s+t}}.\end{aligned}$$

Hence

$$\begin{aligned}|T(s,t)|^{\frac{2rp}{s+t}} &= W^*|T(s,t)|^{\frac{2rp}{s+t}}W \\ &= W^*|T(s,t)^*|^{\frac{2rp}{s+t}}W = W^*|T|^{2rp}W\end{aligned}$$

and

$$\begin{pmatrix} X^{2rp} & 0 \\ 0 & 0 \end{pmatrix} = |T(s,t)^*|^{\frac{2rp}{s+t}} = W|T(s,t)|^{\frac{2rp}{s+t}}W^* \tag{4}$$
$$= WW^*|T(s,t)|^{\frac{2rp}{s+t}}WW^* = WW^*|T|^{2rp}WW^*.$$

Since $WW^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, (4) implies that $|T(s,t)|^{\frac{2rp}{s+t}}$ and $|T|^{2rp}$ are of the forms

$$|T(s,t)|^{\frac{2rp}{s+t}} = \begin{pmatrix} X^{2rp} & 0 \\ 0 & Y^{2rp} \end{pmatrix} \geq |T|^{2rp} = \begin{pmatrix} X^{2rp} & 0 \\ 0 & Z^{2rp} \end{pmatrix} \tag{5}$$

where $Y, Z \geq 0$. Since X is injective and has a dense range and $[\operatorname{ran}|T(s,t)|] = [\operatorname{ran}|T|]$, we have

$$[\operatorname{ran} Y] = [\operatorname{ran} Z] = [\operatorname{ran}|T|] \ominus [\operatorname{ran} T(s,t)] = \ker T(s,t)^* \ominus \ker T.$$

Since W commutes with $|T(s,t)|$ and $|T(s,t)|^{\frac{1}{s+t}}$, we have

$$\begin{pmatrix} W_1 & W_2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} = \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}\begin{pmatrix} W_1 & W_2 \\ 0 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} W_1X & W_2Y \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} XW_1 & XW_2 \\ 0 & 0 \end{pmatrix}.$$

So $W_1X = XW_1$ and $W_2Y = XW_2$, and hence $[\operatorname{ran} W_1]$ and $[\operatorname{ran} W_2]$ are reducing subspaces of X. Since $W^*W|T(s,t)| = |T(s,t)|$, we have $W^*W|T(s,t)|^{\frac{1}{s+t}} = |T(s,t)|^{\frac{1}{s+t}}$. Then

$$\begin{pmatrix} W_1^*W_1X & W_1^*W_2Y \\ W_2^*W_1X & W_2^*W_2Y \end{pmatrix} = \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}.$$

Hence $W_1^*W_1 = 1, W_2^*W_2Y = Y$ and

$$X^k = W_1^*W_1X^k = W_1^*X^kW_1,$$
$$Y^k = W_2^*W_2Y^k = W_2^*X^kW_2$$

for all $k = 1, 2, \cdots$. Put $U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$. Then $T(s,t) = |T|^sU|T|^t = W|T(s,t)|$ implies

$$\begin{pmatrix} X^s & 0 \\ 0 & Z^s \end{pmatrix}\begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}\begin{pmatrix} X^t & 0 \\ 0 & Z^t \end{pmatrix} = \begin{pmatrix} W_1 & W_2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} X^{s+t} & 0 \\ 0 & Y^{s+t} \end{pmatrix}$$

and

$$\begin{pmatrix} X^sU_{11}X^t & X_sU_{12}Z^t \\ Z^sU_{21}X^t & Z_sU_{22}Z^t \end{pmatrix} = \begin{pmatrix} W_1X^{s+t} & W_2Y^{s+t} \\ 0 & 0 \end{pmatrix}.$$

Then

$$X^sU_{11}X^t = W_1X^{s+t} = X^sW_1X^t,$$
$$X^sU_{12}Z^t = W_2Y^{s+t} = X^{s+t}W_2$$

and

$$X^s(U_{11} - W_1)X^t = 0,$$
$$X^s(U_{12}Z^t - X^tW_2) = 0.$$

Since X is injective and has a dense range, we have $U_{11} = W_1$ and $U_{12}Z^t = X^tW_2$. Hence $U_{11}^*U_{11} = W_1^*W_1 = 1$. Since U^*U is the orthogonal projection onto $[\operatorname{ran}|T|] \supset [\operatorname{ran} T(s,t)]$ and

$$U^*U = \begin{pmatrix} 1 + U_{21}^*U_{21} & U_{11}^*U_{12} + U_{21}^*U_{22} \\ U_{12}^*U_{11} + U_{22}^*U_{21} & U_{12}^*U_{12} + U_{22}^*U_{22} \end{pmatrix} \le \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

on $\mathcal{H} = [\operatorname{ran} T(s,t)] \oplus \ker T(s,t)^*$, we have $U_{21} = 0$, $U_{12}^*U_{11} = 0$ and

$$U^*U = \begin{pmatrix} 1 & 0 \\ 0 & U_{12}^*U_{12} + U_{22}^*U_{22} \end{pmatrix} \le \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Since $U_{12}Z^t = X^tW_2$, we have

$$Z^{2t} \ge Z^tU_{12}^*U_{12}Z^t = W_2^*X^{2t}W_2 = Y^{2t}.$$

Since $0 < \dfrac{rp}{t} \le 1$, we have

$$\begin{aligned} Z^{2rp} &\ge (Z^tU_{12}^*U_{12}Z^t)^{\frac{rp}{t}} \\ &= \left(W_2^*X^{2t}W_2\right)^{\frac{rp}{t}} = Y^{2rp} \ge Z^{2rp} \end{aligned}$$

by Lowner-Heinz's inequality and (5). Hence

$$(Z^tU_{12}^*U_{12}Z^t)^{\frac{rp}{t}} = Z^{2rp} = Y^{2rp},$$

so $Z = Y$ and

$$|T(s,t)| = |T|^{s+t}.$$

Since

$$\begin{aligned} Z^{2t} &= Z^tU_{12}^*U_{12}Z^t \le Z^tU_{12}^*U_{12}Z^t + Z^tU_{22}^*U_{22}Z^t \\ &= Z^t\left(U_{12}^*U_{12} + U_{22}^*U_{22}\right)Z^t \le Z^{2t}, \end{aligned}$$

we have $Z^tU_{22}^*U_{22}Z^t = 0$ and $Z^tU_{22}^* = 0$. This implies that $[\operatorname{ran} U_{22}^*] \subset \ker Z$. On the other hand $U^* = U^*UU^*$ implies

$$\begin{aligned} \begin{pmatrix} U_{11}^* & 0 \\ U_{12}^* & U_{22}^* \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & U_{12}^*U_{12} + U_{22}^*U_{22} \end{pmatrix} \begin{pmatrix} U_{11}^* & 0 \\ U_{12}^* & U_{22}^* \end{pmatrix} \\ &= \begin{pmatrix} U_{11}^* & 0 \\ (U_{12}^*U_{12} + U_{22}^*U_{22})U_{12}^* & (U_{12}^*U_{12} + U_{22}^*U_{22})U_{22}^* \end{pmatrix}. \end{aligned}$$

Hence $U_{22}^* = (U_{12}^*U_{12} + U_{22}^*U_{22})U_{22}^*$ and

$$\begin{aligned} \operatorname{ran} U_{22}^* &\subset [\operatorname{ran}(U_{12}^*U_{12} + U_{22}^*U_{22})] \\ &= [\operatorname{ran} U^*U] \ominus [\operatorname{ran} T(s,t)] \\ &= [\operatorname{ran}|T|] \ominus [\operatorname{ran} T(s,t)] = [\operatorname{ran} Z]. \end{aligned}$$

Hence

$$\operatorname{ran} U_{22}^* \subset \ker Z \cap [\operatorname{ran} Z] = \{0\}.$$

Hence $U_{22} = 0$. Then $U = \begin{pmatrix} U_{11} & U_{12} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} W_1 & U_{12} \\ 0 & 0 \end{pmatrix}$ and

$$\operatorname{ran} U \subset [\operatorname{ran} T(s,t)] \subset [\operatorname{ran}|T|] = \operatorname{ran} E.$$

Hence $EU = U$. Since W commutes with $|T(s,t)| = |T|^{s+t}$ and $|T|$, we have

$$|T|^s(W-U)|T|^t = W|T|^{s+t} - |T|^s U|T|^t = W|T(s,t)| - T(s,t) = 0.$$

Hence $E(W-U)E = EWE - EUE = 0$. Since $E = U^*U = W^*W$ and

$$[\operatorname{ran} W] \subset [\operatorname{ran} T(s,t)] \subset [\operatorname{ran} |T|] = \operatorname{ran} E,$$

we have $EW = W$. Then

$$\begin{aligned} U &= UU^*U = UE = EUE \\ &= EWE = WE = WW^*W = W. \end{aligned}$$

Thus $U = W$. Since W commutes with $|T(s,t)|$, we have U commutes with $|T|$. Therfore T is quasinormal. □

Corollary 2.3. *Let $T = U|T|$ be a class p-wA(s,t) operator with $0 < s,t$ and $0 < p \leq 1$. If $T(s,t) = |T|^s U|T|^t$ is normal, then T is also normal.*

Proof. T is quasinormal by the above theorem. Hence $T(s,t) = U|T|^{s+t}$ and $T(s,t)^* = |T|^{s+t}U^*$. Thus

$$|T|^{2(s+t)} = |T(s,t)|^2 = |T(s,t)^*|^2 = |T^*|^{2(s+t)}.$$

This implies that $|T| = |T^*|$ and therefore T is normal. □

Next, we investigate subscalarity of class p-$wA(s,t)$ operator. Let $\mathcal{X}$ be a complex Banach space and $\mathcal{U} \subset \mathbb{C}$ be an open subset. Let $\mathcal{O}(\mathcal{U},\mathcal{X})$ denote the Fréchet space of all analytic $\mathcal{X}$-valued functions on $\mathcal{U}$ with the topology of uniform convergence on compact subsets of $\mathcal{U}$. Also, Let $\mathcal{E}(\mathcal{U},\mathcal{X})$ denote the Fréchet space of all infinitely differentiable $\mathcal{X}$-valued functions on $\mathcal{U}$ with the topology of uniform convergence of all derivatives on compact subsets of $\mathcal{U}$. We say that T satisfy Bishop's property (β) if

$$(T-z)f_n(z) \to 0 \text{ in } \mathcal{O}(\mathcal{U},\mathcal{X}) \Longrightarrow f_n(z) \to 0 \text{ in } \mathcal{O}(\mathcal{U},\mathcal{X})$$

for every open set $\mathcal{U} \subset \mathbb{C}$ and $f_n(z) \in \mathcal{O}(\mathcal{U},\mathcal{X})$. E. Albrecht and J. Eschmeier [2] proved that $T \in B(\mathcal{X})$ satisfies Beshop's property (β) if and only if T is subdecomposable, i.e., T is a restriction of a decomposable operator.

We say that T satisfy Eschmeier-Putinar-Bishop's property $(\beta)_\epsilon$ if

$$(T-z)f_n(z) \to 0 \text{ in } \mathcal{E}(\mathcal{U},\mathcal{X}) \Longrightarrow f_n(z) \to 0 \text{ in } \mathcal{E}(\mathcal{U},\mathcal{X})$$

for every open set $\mathcal{U} \subset \mathbb{C}$ and $f_n(z) \in \mathcal{E}(\mathcal{U},\mathcal{X})$. J. Eschmeier and M. Putinar [6] proved that $T \in B(\mathcal{X})$ satisfies Eschmeier-Putinar-Bishop's property $(\beta)_\epsilon$ if and only if T is subscalar, i.e., T is a restriction of a scalar operator.

Theorem 2.4. *If T is p-wA(s,t) with $0 < s+t \leq 1$ and $0 < p \leq 1$, then T satisfies Bishop's property (β) and Eschmeier-Putinar-Bishop's property $(\beta)_\epsilon$. Hence T is subscalar.*

Proof. We may assume $s+t = 1$ by Lema 2.1. Then $T(s,t)$ is $\frac{\min(sp,tp)}{2}$-hyponormal by [14]. Hence $T(s,t)$ satisfies Bishop's property (β) and Eschmeier-Putinar-Bishop's property $(\beta)_\epsilon$ by [4, 12]. Then T satisfies Bishop's property (β) and Eschmeier-Putinar-Bishop's property $(\beta)_\epsilon$ by Theorem 2.1 of [3]. □

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