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Fredholm integral representation of the three-dimensional Helmholtz equation on a cube

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Abstract. This paper is devoted to study of a three-dimensional Helmholtz equation on a cube that can be represented as a system of the second kind Fredholm integral equations with weakly singular kernels.

1. Introduction

Many problems in nature related to steady state oscillations (mechanical, acoustical, thermal, electromagnetic) including Helmholtz equation which is of elliptic type. Using of fundamental solutions is a method for reducing the boundary value problem to the second kind Fredholm integral equations with weakly singular kernels, i .e, Fredholm property. This method can be used for direct or inverse problem when the fundamental solution of the governing equation of the problem in question is known [1-4].
In this paper, the three-dimensional Helmholtz boundary problem is represented as a system of second kind Fredholm integral equations with weakly singular kernels using their foundamental solution and Dirac delta function properties.

2. The Problem

Let $\Omega = \{(x_1, x_2, x_3)|\, x_i \in [-0.5, 0.5];\ i = \overline{1,3}\}$ be an unit cube with boundary $\Gamma = \Gamma^+ \cup \Gamma^-$ where $\Gamma^\pm = \sum_{l=1}^{3} \Gamma_l^\pm$ with $\Gamma_l^\pm = \{(x_1, x_2, x_3)|\, x_l = \pm 0.5,\ x_i \in (-0.5, 0.5);\ i, l = \overline{1,3},\ i \neq l\}$.Consider the three-dimensional Helmholtz boundary value problem

$$\Delta u(x) + k^2 u(x) = 0 \qquad in \quad \Omega, \tag{1}$$

with

$$u(x) = 0 \quad on \ \ \Gamma, \tag{2}$$

where Δ denotes the Laplace operator in $\mathbb{R}^3$.

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3. The system of Fredholm integral equations

Suppose that $U(x-\xi)$ is the fundamental solution of (1) which is given as follows [5],

$$U(x-\xi) = -\frac{1}{4\pi|x-\xi|}e^{ik|x-\xi|}. \tag{3}$$

Then

$$\frac{\partial U(x-\xi)}{\partial x_j} = -\frac{1}{4\pi}\left(\frac{ik}{|x-\xi|} - \frac{1}{|x-\xi|^2}\right)e^{ik|x-\xi|}\cos(x-\xi, x_j), \tag{4}$$

where $\cos(x-\xi, x_j) = \frac{x_j-\xi_j}{|x-\xi|};\ j=\overline{1,3}$.
Multiplying both sides of (1) by (3)-(4), integrating on Ω by applying the Ostrogradski-Gauss's formula [6] and using Dirac's delta function properties (similar to [1-4]) yields that

$$\begin{aligned}\int_\Gamma u(x)\frac{\partial U(x-\xi)}{\partial n_x}\,dx - \int_\Gamma U(x-\xi)\frac{\partial u(x)}{\partial n_x}\,dx &= \int_\Omega (\Delta + k^2)U(x-\xi)u(x)\,dx \\ &= \int_\Omega \delta(x-\xi)u(x)\,dx = \begin{cases} u(\xi), & \xi\in\Omega, \\ 1/2\,u(\xi), & \xi\in\Gamma, \\ 0, & \xi\notin\bar{\Omega}, \end{cases}\end{aligned} \tag{5}$$

$$\begin{aligned}&\int_\Gamma \frac{\partial u(x)}{\partial x_j}\frac{\partial U(x-\xi)}{\partial n_x}\,dx + \int_\Gamma \frac{\partial U(x-\xi)}{\partial x_j}\frac{\partial u(x)}{\partial n_x}\,dx + k^2\int_\Gamma u(x)U(x-\xi)\cos(n_x,x_j)dx \\ &- \int_\Gamma \cos(n_x,x_j)\nabla u(x)\cdot\nabla U(x-\xi)\,dx = \int_\Omega \delta(x-\xi)\frac{\partial u(x)}{\partial x_j}\,dx = \begin{cases} \frac{\partial u(\xi)}{\partial \xi_j}, & \xi\in\Omega, \\ 1/2\,\frac{\partial u(\xi)}{\partial \xi_j}, & \xi\in\Gamma, \\ 0, & \xi\notin\bar{\Omega}, \end{cases}\end{aligned} \tag{6}$$

where (n_x, x_j) is the angles between outward unit normal vector and coordinate axis $x_j, (j=\overline{1,3})$. Also $''\cdot''$ and ∇ denote inner product and gradient operators, respectively. Letting (2)-(3) in (5) yield that

$$\begin{aligned}u(\xi) &= -\int_\Gamma U(x-\xi)\frac{\partial u(x)}{\partial n}\,dx \\ &= \sum_{l=1}^{3}\int_{\Gamma_l^-} U(x-\xi)\frac{\partial u(x)}{\partial x_l}\,dx - \sum_{l=1}^{3}\int_{\Gamma_l^+} U(x-\xi)\frac{\partial u(x)}{\partial x_l}\,dx \\ &= -\frac{1}{4\pi}\sum_{l=1}^{3}\int_{-0.5}^{0.5}\int_{-0.5}^{0.5}\frac{1}{|x-\xi|}e^{ik|x-\xi|}\frac{\partial u(x)}{\partial x_l}|_{x_l=-0.5}\,dx_p dx_q \\ &\quad + \frac{1}{4\pi}\sum_{l=1}^{3}\int_{-0.5}^{0.5}\int_{-0.5}^{0.5}\frac{1}{|x-\xi|}e^{ik|x-\xi|}\frac{\partial u(x)}{\partial x_l}|_{x_l=0.5}\,dx_p dx_q; \\ &\qquad p,q=\overline{1,3},\ \ p<q,\ p,q\neq l,\ \xi\in\Omega.\end{aligned} \tag{7}$$

In a similar way, putting (2)-(4) into (6) we obtain

$$\frac{\partial u(\xi)}{\partial \xi_j} = 2\sum_{\substack{l=1\\ l\neq j}}^{3}\int_{\Gamma_l^+}(\frac{\partial u(x)}{\partial x_j}\times\frac{\partial U(x-\xi)}{\partial x_l}+\frac{\partial u(x)}{\partial x_l}\times\frac{\partial U(x-\xi)}{\partial x_j})\,dx$$
$$-2\sum_{\substack{l=1\\ l\neq j}}^{3}\int_{\Gamma_l^-}(\frac{\partial u(x)}{\partial x_j}\times\frac{\partial U(x-\xi)}{\partial x_l}+\frac{\partial u(x)}{\partial x_l}\times\frac{\partial U(x-\xi)}{\partial x_j})\,dx$$
$$+2\int_{\Gamma_j^+}(\frac{\partial u(x)}{\partial x_j}\times\frac{\partial U(x-\xi)}{\partial x_j}-\sum_{\substack{i=1\\ i\neq j}}^{3}\frac{\partial u(x)}{\partial x_i}\times\frac{\partial U(x-\xi)}{\partial x_i})\,dx$$
$$-2\int_{\Gamma_j^-}(\frac{\partial u(x)}{\partial x_j}\times\frac{\partial U(x-\xi)}{\partial x_j}-\sum_{\substack{i=1\\ i\neq j}}^{3}\frac{\partial u(x)}{\partial x_i}\times\frac{\partial U(x-\xi)}{\partial x_i})\,dx, \quad \xi\in\Gamma.$$

Therefore,

$$\frac{\partial u(\xi)}{\partial \xi_j}|_{\Gamma_j^{\pm}} = -\frac{1}{2\pi}\sum_{\substack{l=1\\ l\neq j}}^{3}\int_{-0.5}^{0.5}\int_{-0.5}^{0.5}(\frac{ik}{|x-\xi|}-\frac{1}{|x-\xi|^2})e^{ik|x-\xi|}H_{jl}(x,\xi)|_{\substack{x\in\Gamma_l^+\\ \xi\in\Gamma_j^{\pm}}}\,dx_p dx_q$$
$$+\frac{1}{2\pi}\sum_{\substack{l=1\\ l\neq j}}^{3}\int_{-0.5}^{0.5}\int_{-0.5}^{0.5}(\frac{ik}{|x-\xi|}-\frac{1}{|x-\xi|^2})e^{ik|x-\xi|}H_{jl}(x,\xi)|_{\substack{x\in\Gamma_l^-\\ \xi\in\Gamma_j^{\pm}}}\,dx_p dx_q$$
$$+\frac{1}{2\pi}\sum_{\substack{l=1\\ l\neq j}}^{3}\int_{-0.5}^{0.5}\int_{-0.5}^{0.5}(\frac{ik}{|x-\xi|}-\frac{1}{|x-\xi|^2})e^{ik|x-\xi|}H_{ll}(x,\xi)|_{\substack{x\in\Gamma_j^+\\ \xi\in\Gamma_j^{\pm}}}\,dx_p dx_q$$
$$-\frac{1}{2\pi}\sum_{\substack{l=1\\ l\neq j}}^{3}\int_{-0.5}^{0.5}\int_{-0.5}^{0.5}(\frac{ik}{|x-\xi|}-\frac{1}{|x-\xi|^2})e^{ik|x-\xi|}H_{ll}(x,\xi)|_{\substack{x\in\Gamma_j^-\\ \xi\in\Gamma_j^{\pm}}}\,dx_p dx_q$$
$$+\frac{1}{2\pi}\int_{-0.5}^{0.5}\int_{-0.5}^{0.5}(\frac{ik}{|x-\xi|}-\frac{1}{|x-\xi|^2})e^{ik|x-\xi|}H_{jj}(x,\xi)|_{\substack{x\in\Gamma_j^-\\ \xi\in\Gamma_j^{\pm}}}\,dx_p dx_q;$$
$$p,q=\overline{1,3},\ \ p<q\ \ p,q\neq l. \tag{8}$$

where for any $j,l=\overline{1,3}$,

$$H_{jl}(x,\xi) := \frac{\partial u(x)}{\partial x_j}\cos(x-\xi,x_l)+\frac{\partial u(x)}{\partial x_l}\cos(x-\xi,x_j); \quad j\neq l,$$
$$H_{jj}(x,\xi) := \frac{\partial u(x)}{\partial x_j}\cos(x-\xi,x_j)\ \ ,\ \ H_{ll}(x,\xi) := \frac{\partial u(x)}{\partial x_l}\cos(x-\xi,x_l)$$

On the other hand,

$$
\begin{aligned}
\frac{\partial u(\xi)}{\partial \xi_j}\Big|_{\substack{\Gamma_\omega^\pm \\ j\neq\omega}} = &-\frac{1}{2\pi}\sum_{\substack{l=1\\ l\neq j}}^{3}\int_{-0.5}^{0.5}\int_{-0.5}^{0.5}\Big(\frac{ik}{|x-\xi|}-\frac{1}{|x-\xi|^2}\Big)e^{ik|x-\xi|}H_{jl}(x,\xi)\Big|_{\substack{x\in\Gamma_l^+\\ \xi\in\Gamma_\omega^\pm}} dx_p dx_q \\
&+\frac{1}{2\pi}\sum_{\substack{l=1\\ l\neq j}}^{3}\int_{-0.5}^{0.5}\int_{-0.5}^{0.5}\Big(\frac{ik}{|x-\xi|}-\frac{1}{|x-\xi|^2}\Big)e^{ik|x-\xi|}H_{jl}(x,\xi)\Big|_{\substack{x\in\Gamma_l^-\\ \xi\in\Gamma_\omega^\pm}} dx_p dx_q \\
&-\frac{1}{2\pi}\int_{-0.5}^{0.5}\int_{-0.5}^{0.5}\Big(\frac{ik}{|x-\xi|}-\frac{1}{|x-\xi|^2}\Big)e^{ik|x-\xi|} \\
&\qquad\qquad\times\Big(H_{jj}(x,\xi)-\sum_{\substack{l=1\\ l\neq j}}^{3}H_{ll}(x,\xi)\Big)\Big|_{\substack{x\in\Gamma_j^+\\ \xi\in\Gamma_\omega^\pm}} dx_p dx_q \\
&+\frac{1}{2\pi}\int_{-0.5}^{0.5}\int_{-0.5}^{0.5}\Big(\frac{ik}{|x-\xi|}-\frac{1}{|x-\xi|^2}\Big)e^{ik|x-\xi|} \\
&\qquad\qquad\times\Big(H_{jj}(x,\xi)-\sum_{\substack{l=1\\ l\neq j}}^{3}H_{ll}(x,\xi)\Big)\Big|_{\substack{x\in\Gamma_j^-\\ \xi\in\Gamma_\omega^\pm}} dx_p dx_q; \\
&\qquad\qquad p,q=\overline{1,3},\ \ p<q\ \ p,q\neq l.
\end{aligned} \tag{9}
$$

4. Main result

The relations (8)-(9) construct a system of second kind Fredholm integral equations with weakly singular kernels which can be solved by one of the numerical methods. Finally, putting the obtained solutions from the system into (7), the solution of (1)-(2) is presented as integral expansion.

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