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On Wavelet Approximation of a function by Legendre Wavelet Methods

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Abstract. In this paper, two new wavelet estimates for a function f having bounded second derivative and bounded M^{th} derivative are obtained by Legendre Wavelet Method.

1. Introduction

At present, the approximation of a function by Fourier series method is at common places of analysis. Wavelet approximation method is a new tool as well as recent trend to detect and analyze abrupt change in seismic signal processing. The wavelet approximations of certain function by Haar wavelet have been determined by several researcher like DeVore [1], Debnath [2], Meyer [3], Morlet [4, 5] and Lal and Kumar [6]. But till now no work seems to have been done for wavelet approximation of a function by Legendre wavelet methods. In an attempt to make an advance study in this direction,in this paper, the wavelet approximation of a function f with $0 \leq \sup_{x\in[0,1]} \left|f^{(2)}(x)\right| \leq A < \infty$ and a new Legendre wavelet estimate for a function f with $0 \leq \sup_{x\in[0,1]} \left|f^{(M)}(x)\right| \leq B < \infty$, where M is the positive integer, have been obtained. It is important to note that estimate of a function is better and sharper than the estimate having less bounded derivative, so the comparison of estimated approximations has significant importance in wavelet analysis.

2. Definitions

2.1. *Legendre Wavelets*

In recent years, wavelets have found their ways into many different fields of science and engineering. Wavelets constitute a family of functions constructed from dilation and translation of a single function called mother wavelet. If $\psi \in L^2(R)$ satisfies the 'admissibility condition'

$$C_\psi = \int_{-\infty}^{\infty} \frac{\left|\hat{\psi}(\omega)\right|^2}{|\omega|} d\omega < \infty \tag{1}$$

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then ψ is called basic wavelet. The Integral Wavelet Transform of (IWT) on $L^2(R)$ is defined by

$$W_\psi(f) = |a|^{-\frac{1}{2}} \int_{-\infty}^{\infty} f(t)\overline{\psi\left(\frac{t-b}{a}\right)}dt, \quad f \in L^2(R) \tag{2}$$

where $a, b \in R$ with $a \neq 0$. If in addition, both ψ and $\hat{\psi}$ satisfy $t\psi(t) \in L^2(R), \omega\hat{\psi}(\omega) \in L^2(R)$ then basic wavelet ψ provides a time-frequency window with finite area given by $4\Delta\psi\Delta\hat{\psi}$. In addition, under this additional assumption, it follows that $\hat{\psi}$ is a continuous function so that the finiteness of C_ψ in (1) implies $\hat{\psi}(0) = 0$ or equivalently $\int_{-\infty}^{\infty} \psi(t)dt = 0$. This is the reason that ψ is called a Wavelet. We note that the admissibility condition (1) is needed in obtaining the inverse of the IWT.
By setting,

$$\psi_{b,a}(t) = |a|^{\frac{-1}{2}}\, \psi\left(\frac{t-b}{a}\right) \tag{3}$$

the IWT defined in (2) can be written as

$$W_\psi f(b, a) = \langle f, \psi_{b,a} \rangle.$$

In this paper, Legendre Wavelet $\psi_{n,m}(t)$ have four argument $(k, \hat{n}, m, t)$, where $k = 1, 2, \ldots$ $\hat{n} = 2n - 1$, m is the order of Legendre polynomials and t is the normalized time. Legendre Wavelet defined on the interval $[0, 1)$ by

$$\psi_{n,m}(t) = \begin{cases} \sqrt{m + \frac{1}{2}}\, 2^{\frac{k}{2}}\, L_m(2^k t - \hat{n}), & \frac{\hat{n}-1}{2^k} \leq t < \frac{\hat{n}+1}{2^k}; \\ 0, & otherwise, \end{cases}$$

It is mentionable that $L_m(t)$ are well known Legendre polynomials of order m which are orthonormal with respect to the weight function $w(t) = 1$ and satisfy the following recursive formula ,
(i) $L_0(t) = 1$ (ii) $L_1(t) = t$ and (iii) $(m + 1)L_{m+1}(t) = (2m + 1)tL_m(t) - mL_{m-1}(t)$, where $m = 1, 2, \ldots$.
The set of Legendre Wavelets are an orthonormal set.

2.2. *Function Approximation*

A function $f \in L^2(R)$ defined over $[0, 1)$ is expanded as Legendre wavelet series in the form of

$$f(t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}(t), \text{ where } c_{n,m} = \langle f, \psi_{n,m} \rangle \tag{4}$$

and $\langle ., . \rangle$ denotes the inner product.
If the infinite series in (4) is truncated then it can be written as

$$S_{2^{k-1},M}(t) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(t) = C^T\Psi(t) \tag{5}$$

where C and $\Psi(t)$ are $2^{k-1}M \times 1$ matrices given by
$C = \left[c_{1,0}, c_{1,1}, \ldots, c_{1,M-1}, c_{2,0}, c_{2,1}, \ldots, c_{2,M-1}, \ldots, c_{2^{k-1},0}, c_{2^{k-1},1}, \ldots, c_{2^{k-1},M-1}\right]$ *and*
$\Psi(t) = \left[\psi_{1,0}(t), \psi_{1,1}(t), \ldots, \psi_{1,M-1}(t), \psi_{2,0}(t), \psi_{2,1}(t), \ldots, \psi_{2,M-1}(t), \ldots, \psi_{2^{k-1},0}(t), \psi_{2^{k-1},1}(t), \ldots, \psi_{2^{k-1},M-1}(t)\right]$.

2.3. *Projection $P_n f$*

Let $P_n f$ be the orthogonal projection of $L^2(R)$ onto V_n. Then

$$P_n(f) = \sum_{k=-\infty}^{\infty} a_{n,k}\phi_{n,k}, \quad \text{where } a_{n,k} = \langle f, \phi_{n,k} \rangle \phi_{n,k}, \quad n = 1, 2, 3, \ldots.$$

Thus

$$P_n(f) = \sum_{k=-\infty}^{\infty} \langle f, \phi_{n,k} \rangle \phi_{n,k}, \qquad \text{(Sweldens and Piessen[7])}$$

2.4. *Wavelet Approximation*

The Wavelet Approximation under supremum norm is defined by

$$E_n(f) = \|f - P_n f\|_\infty = Sup|f(x) - P_n f(x)|, (Zygmund[1], pp.114)$$

We define

$$\|f\|_p = \left\{ \int_0^1 |f(x)|^p dx \right\}^{\frac{1}{p}}, \quad 1 \le p < \infty.$$

The degree of wavelet approximation $E_n(f)$ of f by $P_n(f)$ under the norm $\|.\|_p$is given by

$$E_n(f) = \min_{P_n f} \|f - P_n f\|_p.$$

If $E_n(f) \to 0$ as $n \to \infty$ then $E_n(f)$ is called the best wavelet approximation of f of order n. (Zygmund [1], pp. 115)

3. Theorems

In this paper, we prove the following theorems.

Theorem 3.1. *If a function $f \in L^2(R)$ is defined over $[0,1)$ such that its second derivative is bounded i.e $\sup_{t\in[0,1]} \left|f''(t)\right| \le A < \infty$ and is expanded as Legendre Wavelet series*

$$f(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{n,m} \psi_{n,m}(t), \quad where \;\; c_{n,m} = \langle f, \psi_{n,m} \rangle. \tag{6}$$

Then Legendre Wavelet Approximation $E_{2^k,M}(f)$ of f by $(2^{k-1}, M)^{th}$ partial sums $S_{2^{k-1},M} = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(t)$ of its Legendre Wavelet series (6) *in $L^2[0,1]$ is given by*

$$E_{2^k,M}(f) = \left\| f - \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(t) \right\|_2 = O\left(\frac{1}{2^{2k}\,(2M-3)^{\frac{3}{2}}} \right), \;\; M \ge 2.$$

Theorem 3.2. *Let a function $f \in L^2(R)$ be a function whose M^{th} derivative is bounded i.e $\sup_{t\in[0,1]} \left|f^M(t)\right| < \infty$ then Legendre Wavelet Approximation of f by $S_{2^{k-1},M} = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(t)$, $(2^{k-1}, M)^{th}$ partial sums of its Legendre Wavelet series is given by*

$$E_{2^k,M}(f) = \left\| f - S_{2^{k-1},M} \right\|_2 = \left\| f - \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(t) \right\|_2 = O\left(\frac{1}{M!\, 2^{Mk}} \right).$$

4. Proofs

4.1. *Proof of the Theorem 3.1*

Legendre Wavelet series of $f \in L^2[0,1]$ is given by

$$\begin{aligned} f &= \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\psi_{n,m} \\ &= \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{n,m}\psi_{n,m} + \sum_{n=1}^{2^{k-1}}\sum_{m=M}^{\infty} c_{n,m}\psi_{n,m} + \sum_{n=2^{k-1}+1}^{\infty}\sum_{m=0}^{M-1} c_{n,m}\psi_{n,m} + \sum_{n=2^{k-1}+1}^{\infty}\sum_{m=M}^{\infty} c_{n,m}\psi_{n,m} \\ &= S_{2^{k-1},M} + \sum_{n=1}^{2^{k-1}}\sum_{m=M}^{\infty} c_{n,m}\psi_{n,m} + \sum_{n=2^{k-1}+1}^{\infty}\sum_{m=0}^{M-1} c_{n,m}\psi_{n,m} + \sum_{n=2^{k-1}+1}^{\infty}\sum_{m=M}^{\infty} c_{n,m}\psi_{n,m}. \end{aligned} \tag{7}$$

By definition of $\psi_{n,m}$,

$$\psi_{n,m}(t) = \begin{cases} \sqrt{m+\frac{1}{2}}\, 2^{\frac{k}{2}}\, L_m(2^k t - \hat{n}), & \frac{\hat{n}-1}{2^k} \le t < \frac{\hat{n}+1}{2^k}; \\ 0, & otherwise, \end{cases}$$

We know that for Legendre Wavelet,

$$\frac{\hat{n}-1}{2^k} \le t < \frac{\hat{n}+1}{2^k},\ \frac{2n-2}{2^k} \le t < \frac{2n}{2^k}.$$

If we take $n = 2^{k-1}+1$, then

$$\frac{2(2^{k-1}+1)-2}{2^k} \le t < \frac{2(2^{k-1}+1)}{2^k},\ \frac{2^k}{2^k} \le t < \frac{2^k+1}{2^k}\ \Rightarrow 1 \le t < 1 + \frac{1}{2^k}\forall\, k.$$

Since $\psi_{n,m}$ vanishes outside the interval $[0,1)$, therefore the third and fourth terms in (7) become zero. In this way,

$$f = S_{2^{k-1},M} + \sum_{n=1}^{2^{k-1}}\sum_{m=M}^{\infty} c_{n,m}\psi_{n,m}.$$

Then

$$\begin{aligned} \left\|f - S_{2^{k-1},M}\right\|_2^2 &= \left\|\sum_{n=1}^{2^{k-1}}\sum_{m=M}^{\infty} c_{n,m}\psi_{n,m}\right\|_2^2 \\ &= \left\langle \sum_{n=1}^{2^{k-1}}\sum_{m=M}^{\infty} c_{n,m}\psi_{n,m}, \sum_{n=1}^{2^{k-1}}\sum_{m=M}^{\infty} c_{n,m}\psi_{n,m}\right\rangle \\ &= \sum_{n=1}^{2^{k-1}}\sum_{m=M}^{\infty} \left|c_{n,m}\right|^2 \left\|\psi_{n,m}\right\|_2^2,\ \textit{other terms vanish due to orthonormality of } \psi_{n,m}. \end{aligned} \tag{8}$$

Here ,

$$\begin{aligned}
\left\|\psi_{n,m}\right\|_2^2 &= \int_{-\infty}^{\infty} \psi_{n,m}(t)\overline{\psi_{n,m}(t)}dt \\
&= \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} \left\{\left(m+\frac{1}{2}\right)^{\frac{1}{2}} \; 2^{\frac{k}{2}}\right\}^2 \; L_m(2^k t-\hat{n}) \; \overline{L_m(2^k t-\hat{n})} \; dt \\
&= \left(m+\frac{1}{2}\right) 2^k \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} L_m(2^k t-\hat{n}) \; \overline{L_m(2^k t-\hat{n})} \; dt \\
&= \frac{2m+1}{2} \, 2^k \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} \left|L_m(2^k t-\hat{n})\right|^2 \; dt \\
&= \frac{2m+1}{2} \, 2^k \int_{-1}^{1} |L_m(u)|^2 \; \frac{du}{2^k}, \; \textit{taking } 2^k t-\hat{n}=u \\
&= \frac{2m+1}{2} \int_{-1}^{1} |L_m(u)|^2 \; du
\end{aligned}$$

$$= \quad 1, \; \textit{by orthogonal property of Legendre polynomial and } \int_{-1}^{1} (L_m(u))^2 \; du = \frac{2}{2m+1}.$$

Thus

$$\left\|\psi_{n,m}\right\|_2^2 = 1 \tag{9}$$

Using equation (8) and (9), we have

$$\left\|f - S_{2^{k-1},M}\right\|_2^2 = \sum_{n=1}^{2^{k-1}} \sum_{m=M}^{\infty} \left|c_{n,m}\right|^2 \tag{10}$$

Next,

$$\begin{aligned}
c_{n,m} &= \int_0^1 f(x) \; \psi_{n,m} \; dx \\
&= \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} f(x) \; \left(m+\frac{1}{2}\right)^{\frac{1}{2}} \; 2^{\frac{k}{2}} \; L_m(2^k x-\hat{n}) \; dx \\
&= \int_{-1}^{1} f\left(\frac{\hat{n}+t}{2^k}\right) \left(\frac{2m+1}{2}\right)^{\frac{1}{2}} \; 2^{\frac{k}{2}} \; L_m(t) \; \frac{dt}{2^k}, \;\; \textit{taking } 2^k x-\hat{n}=t \\
&= \left(\frac{2m+1}{2^{k+1}}\right)^{\frac{1}{2}} \int_{-1}^{1} f\left(\frac{\hat{n}+t}{2^k}\right) \; \frac{1}{2m+1} \; \frac{d}{dt}(L_{m+1}(t)-L_{m-1}(t)) \; dt, \;\; L_m(t) = \frac{L'_{m+1}(t)-L'_{m-1}(t)}{2m+1}, \; m \geq 1
\end{aligned}$$

$$= \left(\frac{1}{2^{k+1}(2m+1)}\right)^{\frac{1}{2}} \int_{-1}^{1} f\left(\frac{\hat{n}+t}{2^k}\right) \frac{d}{dt}(L_{m+1}(t) - L_{m-1}(t))\, dt$$

$$= \left(\frac{1}{2^{k+1}(2m+1)}\right)^{\frac{1}{2}} \left\{ \left(f\left(\frac{\hat{n}+t}{2^k}\right) (L_{m+1}(t) - L_{m-1}(t))\right)_{-1}^{1} - \int_{-1}^{1} f'\left(\frac{\hat{n}+t}{2^k}\right) \frac{(L_{m+1}(t) - L_{m-1}(t))}{2^k}\, dt \right\},$$

, intergrating by parts

$$= -\left(\frac{1}{2^{3k+1}(2m+1)}\right)^{\frac{1}{2}} \int_{-1}^{1} f'\left(\frac{\hat{n}+t}{2^k}\right) (L_{m+1}(t) - L_{m-1}(t))\, dt$$

$$= -\left(\frac{1}{2^{3k+1}(2m+1)}\right)^{\frac{1}{2}} \int_{-1}^{1} f'\left(\frac{\hat{n}+t}{2^k}\right) L_{m+1}(t)\, dt + \left(\frac{1}{2^{3k+1}(2m+1)}\right)^{\frac{1}{2}} \int_{-1}^{1} f'\left(\frac{\hat{n}+t}{2^k}\right) L_{m-1}(t)\, dt$$

$$= I_1 + I_2, \text{ say.} \tag{11}$$

$$I_1 = -\left(\frac{1}{2^{3k+1}(2m+1)}\right)^{\frac{1}{2}} \int_{-1}^{1} f'\left(\frac{\hat{n}+t}{2^k}\right) \frac{L'_{m+2}(t) - L'_m(t)}{2m+3}\, dt, \; L_{m+1}(t) = \frac{L'_{m+2}(t) - L'_m(t)}{2m+3}$$

$$= -\left(\frac{1}{2^{3k+1}(2m+1)}\right)^{\frac{1}{2}} \int_{-1}^{1} f'\left(\frac{\hat{n}+t}{2^k}\right) \frac{d}{dt}\left(\frac{L_{m+2}(t) - L_m(t)}{2m+3}\right) dt$$

$$= -\left(\frac{1}{2^{3k+1}(2m+1)}\right)^{\frac{1}{2}} \frac{1}{2m+3} \left\{ \left(f'\left(\frac{\hat{n}+t}{2^k}\right)(L_{m+2}(t) - L_m(t))\right)_{-1}^{1} - \int_{-1}^{1} f''\left(\frac{\hat{n}+t}{2^k}\right) \frac{1}{2^k} (L_{m+2}(t) - L_m(t))\, dt \right\},$$

, integrating by parts

$$= -\left(\frac{1}{2^{3k+1}(2m+1)}\right)^{\frac{1}{2}} \frac{1}{2m+3} \left\{ 0 - \int_{-1}^{1} f''\left(\frac{\hat{n}+t}{2^k}\right) \frac{1}{2^k} \; (L_{m+2}(t) - L_m(t))\; dt \right\}$$

$$= \left(\frac{1}{2^{5k+1}(2m+1)}\right)^{\frac{1}{2}} \int_{-1}^{1} f''\left(\frac{\hat{n}+t}{2^k}\right) \frac{(L_{m+2}(t) - L_m(t))}{2m+3}\, dt \tag{12}$$

Similarly,

$$I_2 = \left(\frac{1}{2^{3k+1}(2m+1)}\right)^{\frac{1}{2}} \int_{-1}^{1} f'\left(\frac{\hat{n}+t}{2^k}\right) \frac{L'_m(t) - L'_{m-2}(t)}{2m-1}\, dt, \; L_{m-1}(t) = \frac{L'_m(t) - L'_{m-2}(t)}{2m-1}, \; m \geq 2$$

$$= \left(\frac{1}{2^{3k+1}(2m+1)}\right)^{\frac{1}{2}} \int_{-1}^{1} f'\left(\frac{\hat{n}+t}{2^k}\right) \frac{d}{dt}\left(\frac{L_m(t) - L_{m-2}(t)}{2m-1}\right) dt$$

$$= \left(\frac{1}{2^{3k+1}(2m+1)}\right)^{\frac{1}{2}} \frac{1}{2m-1} \left\{ \left(f'\left(\frac{\hat{n}+t}{2^k}\right)(L_m(t) - L_{m-2}(t))\right)_{-1}^{1} - \int_{-1}^{1} f''\left(\frac{\hat{n}+t}{2^k}\right) \frac{1}{2^k} (L_m(t) - L_{m-2}(t))\, dt \right\},$$

, integrating by parts

$$= \left(\frac{1}{2^{3k+1}(2m+1)}\right)^{\frac{1}{2}} \frac{1}{2m-1} \left\{ 0 - \int_{-1}^{1} f''\left(\frac{\hat{n}+t}{2^k}\right) \frac{1}{2^k} \; (L_m(t) - L_{m-2}(t))\; dt \right\}$$

$$= -\left(\frac{1}{2^{5k+1}(2m+1)}\right)^{\frac{1}{2}} \int_{-1}^{1} f''\left(\frac{\hat{n}+t}{2^k}\right) \frac{(L_m(t) - L_{m-2}(t))}{2m-1}\, dt. \tag{13}$$

By equation (11), (12) and (13), we have

$$c_{n,m} = \left(\frac{1}{2^{5k+1}(2m+1)}\right)^{\frac{1}{2}} \int_{-1}^{1} f''\left(\frac{\hat{n}+t}{2^k}\right) \left\{\frac{(L_{m+2}(t) - L_m(t))}{2m+3} - \frac{(L_m(t) - L_{m-2}(t))}{2m-1}\right\} dt. \tag{14}$$

Thus,

$$\begin{aligned}
|c_{n,m}|^2 &= \left|\left(\frac{1}{2^{5k+1}(2m+1)}\right)^{\frac{1}{2}} \int_{-1}^{1} f''\left(\frac{\hat{n}+t}{2^k}\right) \left\{\frac{(L_{m+2}(t) - L_m(t))}{2m+3} - \frac{(L_m(t) - L_{m-2}(t))}{2m-1}\right\} dt\right|^2 \\
&= \left(\frac{1}{2^{5k+1}(2m+1)}\right)\left|\int_{-1}^{1} f''\left(\frac{\hat{n}+t}{2^k}\right) \left\{\frac{(L_{m+2}(t) - L_m(t))}{2m+3} - \frac{(L_m(t) - L_{m-2}(t))}{2m-1}\right\} dt\right|^2 \\
&= \left(\frac{1}{2^{5k+1}(2m+1)}\right)\left|\int_{-1}^{1} f''\left(\frac{\hat{n}+t}{2^k}\right) \left\{\frac{(2m-1)L_{m+2}(t) - (4m+2)L_m(t) + (2m+3)L_{m-2}(t)}{(2m+3)(2m-1)}\right\} dt\right|^2 \\
&\leq \left(\frac{1}{2^{5k+1}(2m+1)}\right)\int_{-1}^{1} \left|f''\left(\frac{\hat{n}+t}{2^k}\right)\right|^2 dt \int_{-1}^{1} \left|\left\{\frac{(2m-1)L_{m+2}(t) - (4m+2)L_m(t) + (2m+3)L_{m-2}(t)}{(2m+3)(2m-1)}\right\}\right|^2 dt \\
&\leq \left(\frac{1}{2^{5k+1}(2m+1)}\right)\int_{-1}^{1} A^2 dt \int_{-1}^{1} \left|\left\{\frac{(2m-1)L_{m+2}(t) - (4m+2)L_m(t) + (2m+3)L_{m-2}(t)}{(2m+3)(2m-1)}\right\}\right|^2 dt \\
&\leq \left(\frac{2A^2}{2^{5k+1}(2m+1)}\right)\int_{-1}^{1} \left|\left\{\frac{(2m-1)L_{m+2}(t) - (4m+2)L_m(t) + (2m+3)L_{m-2}(t)}{(2m+3)(2m-1)}\right\}\right|^2 dt \\
&\leq \left(\frac{2A^2}{2^{5k+1}(2m+1)}\right)\int_{-1}^{1} \left\{\frac{|(2m-1)L_{m+2}(t) - (4m+2)L_m(t) + (2m+3)L_{m-2}(t)|^2}{(2m+3)^2(2m-1)^2}\right\} dt \\
&\leq \left(\frac{2A^2}{2^{5k+1}(2m+1)}\right)\int_{-1}^{1} \left\{\frac{(2m-1)^2L^2_{m+2}(t) + (4m+2)^2L^2_m(t) + (2m+3)^2L^2_{m-2}(t)}{(2m+3)^2(2m-1)^2}\right\} dt \\
&\quad \textit{, other terms vanish due to orthogonal property of Legendre polynomial} \\
&\leq \left(\frac{2A^2}{2^{5k+1}(2m+1)}\right)\left(\frac{1}{(2m+3)^2(2m-1)^2}\right)\left[(2m-1)^2\int_{-1}^{1} L^2_{m+2}(t)dt + (4m+2)^2\int_{-1}^{1} L^2_m(t)dt + (2m+3)^2\int_{-1}^{1} L^2_{m-2}(t)dt\right] \\
&\leq \left(\frac{2A^2}{2^{5k+1}(2m+1)}\right)\left(\frac{1}{(2m+3)^2(2m-1)^2}\right)\left[(2m-1)^2\frac{2}{2m+5} + (4m+2)^2\frac{2}{2m+1} + (2m+3)^2\frac{2}{2m-3}\right] \\
&\leq \left(\frac{2A^2}{2^{5k+1}(2m+1)}\right)\left(\frac{1}{(2m+3)^2(2m-1)^2}\right)\left[(2m-1)^2\frac{2}{2m-3} + (4m+2)^2\frac{2}{2m-3} + (2m+3)^2\frac{2}{2m-3}\right] \\
&\leq \left(\frac{2A^2}{2^{5k+1}(2m+1)}\right)\left(\frac{1}{(2m+3)^2(2m-1)^2}\right)\left[\frac{12(2m+3)^2}{(2m-3)}\right] \\
&\leq \left(\frac{24A^2}{2^{5k+1}(2m+1)(2m-1)^2(2m-3)}\right)
\end{aligned}$$

Hence,

$$\begin{aligned}\left|c_{n,m}\right|^2 &\leq \frac{24A^2}{2^{5k+1}(2m+1)(2m-1)^2(2m-3)} \\ &\leq \frac{12\,A^2}{2^{5k}(2m-3)^4},\quad m \geq 2.\end{aligned} \tag{15}$$

By equation (9) and (15), we have

$$\begin{aligned}\left\|f - S_{2^{k-1},M}\right\|_2^2 &\leq \sum_{n=1}^{2^{k-1}}\sum_{m=M}^{\infty}\frac{12A^2}{(2m-3)^4}\frac{1}{2^{5k}} \\ &\leq \frac{12A^2}{2^{4k+1}(2M-3)^3},\quad M \geq 2.\end{aligned} \tag{16}$$

Hence,

$$\left\|f - S_{2^{k-1},M}\right\|_2 = O\left(\frac{1}{(2M-3)^{\frac{3}{2}}2^{2k}}\right),\quad M \geq 2$$

Thus, this theorem (3.1) is completely established.

*4.2. **Proof of the Theorem*** (3.2)

A function f is M times differentiable therefore by Taylor's expansion, we have

$$f(a+h) = f_{M+1} = f(a) + \frac{h}{1!}f'(a) + \ldots + \frac{h^{M-1}}{(M-1)!}f^{(M-1)}(a) + \frac{h^M}{M!}f^{(M)}(a+\theta h)$$

$$f_{M+1} = f_M + \frac{h^M}{M!}f^M(a+\theta h),\ where\ 0<\theta<1\ \ and\ \ f_M = f(a) + \frac{h}{1!}f'(a) + \ldots + \frac{h^{M-1}}{(M-1)!}f^{(M-1)}(a)$$

Then,

$$f_{M+1} - f_M = \frac{h^M}{M!}f^M(a+\theta h),\ where\ 0<\theta<1.$$

Using this and dividing the interval $[0,1]$ in $\left[\frac{l}{2^k},\frac{l+1}{2^k}\right]$ subintervals, we have,

$$\begin{aligned}\left\|f - S_{2^{k-1},M}\right\|_2^2 &= \int_0^1\left|f(x) - \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}c_{n,m}\psi_{n,m}\right|^2 dx \\ &= \sum_{l=0}^{2^k-1}\int_{\frac{l}{2^k}}^{\frac{l+1}{2^k}}\left|f(x) - \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}c_{n,m}\psi_{n,m}\right|^2 dx \\ &\leq \sum_{l=0}^{2^k-1}\int_{\frac{l}{2^k}}^{\frac{l+1}{2^k}}\left(\frac{1}{M!}\left(\frac{1}{2^k}\right)^M \sup_{x\in[0,1]}\left|f^{(M)}(x)\right|\right)^2 dx \\ &= \int_0^1\left(\frac{1}{M!}\left(\frac{1}{2^k}\right)^M \sup_{x\in[0,1]}\left|f^{(M)}(x)\right|\right)^2 dx \\ &= \left(\frac{1}{M!}\right)^2\left(\frac{1}{2^{Mk}}\right)^2 \sup_{x\in[0,1]}\left|f^{(M)}(x)\right|^2\end{aligned}$$

$$\left\|f - S_{2^{k-1},M}\right\|_2^2 \leq \left(\frac{1}{M!}\frac{1}{2^{Mk}}\right)^2 \sup_{x\in[0,1]} \left|f^{(M)}(x)\right|^2$$

Hence,

$$\left\|f - S_{2^{k-1},M}\right\|_2 \leq \left(\frac{1}{M!}\frac{1}{2^{Mk}}\right) \sup_{x\in[0,1]} \left|f^{(M)}(x)\right|$$

Therefore,

$$E_{2^k,M}(f) = \left\|f - S_{2^{k-1},M}\right\|_2 \leq \left(\frac{1}{M!}\frac{1}{2^{Mk}}\right) \sup_{x\in[0,1]} \left|f^{(M)}(x)\right| = O\left(\frac{1}{M!2^{Mk}}\right).$$

Hence, this has been proved.

5. Conclusions

Since $M!\,2^{Mk} \geq (2M-3)^{\frac{3}{2}}\,2^{2k}, M \geq 2$. Therefore $\frac{1}{M!\,2^{Mk}} \leq \frac{1}{(2M-3)^{\frac{3}{2}}\,2^{2k}}$, $M \geq 2$. Thus, estimate of a function having more bounded derivative is better and sharper than the function of less bounded derivative.

6. Remarks

In the Theorem(3.1),

$$E_{2^k,M}(f) = O\left(\frac{1}{2^{2k}\,(2M-3)^{\frac{3}{2}}}\right) = \frac{C_1}{2^{2k}\,(2M-3)^{\frac{3}{2}}} \to 0 \; as \; k \to \infty,\ M \to \infty$$

and also in Theorem(3.2),

$$E_{2^k,M}(f) = O\left(\frac{1}{M!\,2^{Mk}}\right) = \frac{C_2}{M!\,2^{Mk}} \to 0 \; as \; k \to \infty,\ M \to \infty,$$

where C_1 and C_2 are positive constants.
Therefore, Legendre Wavelet approximation estimated is best possible in each of the Theorems (3.1) and (3.2).

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