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On the joint spectra of commuting tuples of operators and a conjugation

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Abstract. In this paper we show that if $\mathbf{T} = (T_1, ..., T_n)$ is a commuting n-tuple of Hilbert space operators and C is a conjugation, then $\sigma(C\mathbf{T}C) = \sigma(\mathbf{T})^*$, where $\sigma(C\mathbf{T}C) = (CT_1C, ..., CT_nC)$, $\sigma(\mathbf{T})$ is the Taylor spectrum of $\mathbf{T}$ and $\sigma(\mathbf{T})^* = \{\bar{z} = (\overline{z_1}, ..., \overline{z_n}) : z = (z_1, ..., z_n) \in \sigma(\mathbf{T})\}$. We characterize joint approximate point spectra of m-symmetric tuples, m-complex symmetric tuples, skew m-complex symmetric tuples, $[m, C]$-symmetric tuples and skew $[m, C]$-symmetric tuples.

1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space with the inner product $\langle \ , \ \rangle$ and $B(\mathcal{H})$ be the set of all bounded linear operators on $\mathcal{H}$. In [6], S. Jung, E. Ko and Ji Eun Lee showed that if C is a conjugation and $T \in B(\mathcal{H})$, then $\sigma(CTC) = \sigma(T)^*, \sigma_a(CTC) = \sigma_a(T)^*$ and $\sigma_p(CTC) = \sigma_p(T)^*$, where $\sigma(T), \sigma_a(T)$ and $\sigma_p(T)$ are the spectrum, the approximate point spectrum and the point spectrum of T, respectively. An antilinear operator C is said to be a conjugation if C satisfies $C^2 = I, \langle Cx , Cy \rangle = \langle y , x \rangle$ for all $x, y \in \mathcal{H}$, where I is the identity of $B(\mathcal{H})$. We have some results of m-complex isometric, $[m, C]$-isometric and others for single operators. Please see [1], [2] and [3] for results and examples of these classes.

In this paper we show that if C is a conjugation and $\mathbf{T} = (T_1, ..., T_n) \in B(\mathcal{H})^n$ is a commuting n-tuple, then we show $\sigma(C\mathbf{T}C) = \sigma(\mathbf{T})^*, \sigma_{ja}(C\mathbf{T}C) = \sigma_{ja}(\mathbf{T})^*$ and $\sigma_{jp}(C\mathbf{T}C) = \sigma_{jp}(\mathbf{T})^*$, where $C\mathbf{T}C = (CT_1C, ..., CT_nC)$, and $\sigma(\mathbf{T}), \sigma_{ja}(\mathbf{T})$ and $\sigma_{jp}(\mathbf{T})$ are the Taylor spectrum, the joint approximate point spectrum and the joint point spectrum of $\mathbf{T}$, respectively. Finally we characterize joint approximate point spectra of m-symmetric tuples, m-complex symmetric tuples, skew m-complex symmetric tuples, $[m, C]$-symmetric tuples and skew $[m, C]$-symmetric tuples.

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2. Preparation

For a commuting n-tuple $\mathbf{T} = (T_1, ..., T_n) \in B(\mathcal{H})^n$, we explain the Taylor spectrum $\sigma(\mathbf{T})$ of $\mathbf{T}$ shortly. Let E^n be the exterior algebra on n generators, that is, E^n is the complex algebra with identity e generated by indeterminates $e_1, ..., e_n$. Let $E_k^n(\mathcal{H}) = \mathcal{H} \otimes E_k^n$. Define $d_k^n : E_k^n(\mathcal{H}) \longrightarrow E_{k-1}^n(\mathcal{H})$ by

$$d_k^n(x \otimes e_{j_1} \wedge \cdots \wedge e_{j_k}) := \sum_{i=1}^{k} (-1)^{i-1} T_{j_i} x \otimes e_{j_1} \wedge \cdots \wedge \check{e}_{j_i} \wedge \cdots \wedge e_{j_k},$$

where $\check{e}_{j_i}$ means deletion. We denote d_k^n by d_k simply. We think Koszul complex $E(\mathbf{T})$ of $\mathbf{T}$ as follows:

$$(*) \quad E(\mathbf{T}) \, : \, 0 \longrightarrow E_n^n(\mathcal{H}) \xrightarrow{d_n} E_{n-1}^n(\mathcal{H}) \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} E_1^n(\mathcal{H}) \xrightarrow{d_1} E_0^n(\mathcal{H}) \longrightarrow 0.$$

It is easy to see that $E_k^n(\mathcal{H}) \cong \overbrace{\mathcal{H} \oplus \cdots \oplus \mathcal{H}}^{\frac{n!}{(n-k)!k!}}$ $(k = 1, ..., n)$.

Definition 2.1. *A commuting n-tuple $\mathbf{T} = (T_1, ..., T_n) \in B(\mathcal{H})^n$ is said to be nonsingular if and only if the Koszul complex $E(\mathbf{T})$ is exact.*

Definition 2.2. *For a commuting n-tuple $\mathbf{T} = (T_1, ..., T_n) \in B(\mathcal{H})^n$, $z = (z_1, ..., z_n) \notin \sigma(\mathbf{T})$ (Taylor spectrum) if $\mathbf{T} - z = (T_1 - z_1, ..., T_n - z_n)$ is nonsingular.*

About the definition of the Taylor spectrum, see details J. L. Taylor [7] and [8].

The joint approximate point spectrum of $\mathbf{T} = (T_1, ..., T_n) \in B(\mathcal{H})^n$ is denoted by $\sigma_{ja}(\mathbf{T})$, i.e., $(z_1, ..., z_n) \in \sigma_{ja}(\mathbf{T})$ if and only if there exists a sequence $\{x_k\}$ of unit vectors such that

$$(T_j - z_j)x_k \longrightarrow 0 \text{ as } k \to \infty \text{ for all } j = 1, ..., n.$$

The joint point spectrum $\sigma_{jp}(\mathbf{T})$ of $\mathbf{T} = (T_1, ..., T_n) \in B(\mathcal{H})^n$ is the set of all $(z_1, ..., z_n) \in \mathbb{C}^n$ which there exists a nonzero vector x such that $(T_j - z_j)x = 0$ for all $j = 1, ..., n$.

3. Result

First we need the following result by R. Curto [1].

Theorem 3.1. *For a commuting n-tuple $\mathbf{T} = (T_1, ..., T_n) \in B(\mathcal{H})^n$, $0 = (0, ..., 0) \notin \sigma(\mathbf{T})$ if and only if*

$$\alpha(\mathbf{T}) := \begin{pmatrix} d_1 & 0 & \cdots & \cdots \\ d_2^* & d_3 & \cdots & \cdots \\ 0 & d_4^* & \cdots & \cdots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix} \text{ is invertible on } \overbrace{\mathcal{H} \oplus \cdots \oplus \mathcal{H}}^{2^{n-1}},$$

where d_k is the mapping of $(*)$ $(k = 1, 2, ..., n)$.

For a conjugation C on $\mathcal{H}$, let $C\mathbf{T}C = (CT_1C, ..., CT_nC)$. If $\mathbf{T} = (T_1, ..., T_n)$ is a commuting n-tuple, then $C\mathbf{T}C$ is also commuting n-tuple.

Lemma 3.2. *For a commuting n-tuple* $\mathbf{T} = (T_1, ..., T_n) \in B(\mathcal{H})^n$ *and any conjugation* C, $0 = (0, ..., 0) \notin \sigma(\mathbf{T})$ *if and only if* $0 = (0, ..., 0) \notin \sigma(C\mathbf{T}C)$, *where* $C\mathbf{T}C = (CT_1C, ..., CT_nC)$.

Proof. It holds $CT_iC \cdot CT_jC = CT_iT_jC$ and $(CT_iC)^* = CT_i^*C$. Hence we have

$$\alpha(C\mathbf{T}C) = \begin{pmatrix} C & 0 & \cdots & \cdots & 0 \\ 0 & C & \cdots & \cdots & 0 \\ 0 & 0 & C & \cdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & C \end{pmatrix} \cdot \alpha(\mathbf{T}) \cdot \begin{pmatrix} C & 0 & \cdots & \cdots & 0 \\ 0 & C & \cdots & \cdots & 0 \\ 0 & 0 & C & \cdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & C \end{pmatrix}$$

on $\overbrace{\mathcal{H} \oplus \cdots \oplus \mathcal{H}}^{2^{n-1}}$. Since $\tilde{C} = \overbrace{C \oplus \cdots \oplus C}^{2^{n-1}}$ is a conjugation on $\overbrace{\mathcal{H} \oplus \cdots \oplus \mathcal{H}}^{2^{n-1}}$, it holds that $\alpha(\mathbf{T})$ is invertible if and only if $\alpha(C\mathbf{T}C)$ is invertible. □

Remark 3.3. *By Theorem* 1.1 *of* [5], *if* $\mathbf{T} = (T_1, T_2) \in B(\mathcal{H})^2$ *is a commuting pair, then* $\mathbf{T}$ *is nonsingular if and only if*

$$\alpha(\mathbf{T}) = \begin{pmatrix} T_1 & T_2 \\ -T_2^* & T_1^* \end{pmatrix}$$

is invertible on $\mathcal{H} \oplus \mathcal{H}$. *Let* $C\mathbf{T}C = (CT_1C, CT_2C)$. *Then we have*

$$\alpha(C\mathbf{T}C) = \begin{pmatrix} CT_1C & CT_2C \\ -(CT_2C)^* & (CT_1C)^* \end{pmatrix} = \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix} \cdot \alpha(\mathbf{T}) \cdot \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix}.$$

By this equality, it holds $\alpha(\mathbf{T})$ *is invertible if and only if* $\alpha(C\mathbf{T}C)$ *is invertible. So Lemma* 3.2 *is clear.*

Theorem 3.4. *For a commuting n-tuple* $\mathbf{T} = (T_1, ..., T_n) \in B(\mathcal{H})^n$ *and any conjugation* C, *it holds* $\sigma(C\mathbf{T}C) = \sigma(\mathbf{T})^*$, $\sigma_{ja}(C\mathbf{T}C) = \sigma_{ja}(\mathbf{T})^*$ *and* $\sigma_{jp}(C\mathbf{T}C) = \sigma_{jp}(\mathbf{T})^*$ *for any conjugation* C, *where* $E^* = \{\overline{z} = (\overline{z_1}, ..., \overline{z_n}) : z \in E\} \subset \mathbb{C}^n$.

Proof. It holds that $(C(T_1 - z_1)C, ..., C(T_n - z_n)C) = (CT_1C - \overline{z_1}, ..., CT_nC - \overline{z_n}) = C\mathbf{T}C - \overline{z}$, where $z = (z_1, ..., z_n)$. Hence proof follows from Lemma 3.2. □

4. Properties of joint approximate point spectra of commuting tuples

Definition 4.1. *For a commuting n-tuple* $\mathbf{T} = (T_1, ..., T_n) \in B(\mathcal{H})^n$, *we define* $\mathcal{P}_m(\mathbf{T})$ *by*

$$\mathcal{P}_m(\mathbf{T}) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} \Big(\sum_{|j|=k} \frac{k!}{j!} \mathbf{T}^{*j} \cdot \mathbf{T}^j \Big).$$

$\mathbf{T} = (T_1, ..., T_n)$ *is said to be an m-isometric tuple if* $\mathcal{P}_m(\mathbf{T}) = 0$.

Then in [5] J. Gleason and S. Richter proved the following result.

Proposition 4.1. (Lemma 3.2, [5])
Let $\mathbf{T} = (T_1, ..., T_n) \in B(\mathcal{H})^n$ *be an m-isometric tuple. If* $z = (z_1, ..., z_n) \in \sigma_{ja}(\mathbf{T})$, *then* $|z|^2 = |z_1|^2 + \cdots + |z_n|^2 = 1$.

We introduce m-symmetric tuples as follows.

Definition 4.2. *Let, for commuting n-tuple $\mathbf{T} = (T_1, ..., T_n) \in B(\mathcal{H})^n$ and $A \in B(\mathcal{H})$,*

$$\mathcal{S}_{\mathbf{T}}(A) := (T_1 + \cdots + T_n)^* A - A(T_1 + \cdots + T_n).$$

An n-tuple $\mathbf{T} = (T_1, ..., T_n) \in B(\mathcal{H})^n$ is said to be an m-symmetric tuple if

$$\mathcal{S}^m_{\mathbf{T}}(I) = 0.$$

Then it holds

$$\mathcal{S}^m_{\mathbf{T}}(I) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} (T_1^* + \cdots + T_n^*)^{m-j} (T_1 + \cdots + T_n)^j.$$

Theorem 4.2. *Let $\mathbf{T} = (T_1, ..., T_n) \in B(\mathcal{H})^n$ be an m-symmetric commuting tuple of operators. If $(z_1, ..., z_n) \in \sigma_{ja}(\mathbf{T})$, then $z_1 + \cdots + z_n$ is a real number.*

Proof. Let $\{x_k\}$ be a sequence of unit vectors such that

$$(T_j - z_j)x_k \longrightarrow 0 \text{ as } k \to \infty \text{ for all } j = 1, ..., n.$$

Since then $\mathbf{T}$ is m-symmetric, it holds

$$0 = \langle \Big(\sum_{j=0}^{m} (-1)^j \binom{m}{j} (T_1^* + \cdots + T_n^*)^{m-j} (T_1 + \cdots + T_n)^j \Big) x_k,\ x_k \rangle$$

$$\longrightarrow \Big(\overline{(z_1 + \cdots + z_n)} - (z_1 + \cdots + z_n) \Big)^m \text{ as } k \to \infty.$$

Hence $z_1 + \cdots + z_n$ is a real number. □

We define a m-complex symmetric tuple and skew m-complex symmetric tuple as follows:

Definition 4.3. *For a commuting n-tuple $\mathbf{T} = (T_1, ..., T_n) \in B(\mathcal{H})^n$ and a conjugation C, we define $r_m(\mathbf{T}; C)$ and $\mathcal{R}_m(\mathbf{T}; C)$ by*

$$r_m(\mathbf{T}; C) := \sum_{j=0}^{m} (-1)^j \binom{m}{j} (T_1^* + \cdots + T_n^*)^{m-j} (CT_1C + \cdots + CT_nC)^j$$

and

$$\mathcal{R}_m(\mathbf{T}; C) = \sum_{j=0}^{m} \binom{m}{j} (T_1^* + \cdots + T_n^*)^{m-j} (CT_1C + \cdots + CT_nC)^j.$$

A commuting n-tuple $\mathbf{T} = (T_1, ..., T_n)$ is said to be a m-complex symmetric tuple and a skew m-complex symmetric tuple with a conjugation C if $r_m(\mathbf{T}; C) = 0$ and $\mathcal{R}_m(\mathbf{T}; C) = 0$, respectively.

Theorem 4.3. *Let* $\mathbf{T} = (T_1, ..., T_n)$ *be a commuting n-tuple.*
(1) If $\mathbf{T}$ *is an m-complex symmetric tuple with a conjugation C and* $(z_1, ..., z_n) \in \sigma_{ja}(\mathbf{T})$*, then* $(\overline{z_1} + \cdots + \overline{z_n})$ *belongs to the approximate point spectrum of* $T_1^* + \cdots + T_n^*$*. Hence if* $(z_1, ..., z_n) \in \sigma_{jp}(\mathbf{T})$*, then* $(\overline{z_1} + \cdots + \overline{z_n}) \in \sigma_p(T_1^* + \cdots + T_n^*)$*.*
(2) If $\mathbf{T}$ *is a skew m-complex symmetric tuple with a conjugation C and* $(z_1, ..., z_n) \in \sigma_{ja}(\mathbf{T})$*, then* $-(\overline{z_1} + \cdots + \overline{z_n})$ *belongs to the approximate point spectrum of* $T_1^* + \cdots + T_n^*$*. Hence if* $(z_1, ..., z_n) \in \sigma_{jp}(\mathbf{T})$*, then* $-(\overline{z_1} + \cdots + \overline{z_n}) \in \sigma_p(T_1^* + \cdots + T_n^*)$*.*

Proof. Let $\{x_k\}$ be a sequence of unit vectors such that

$$(T_j - z_j)x_k \longrightarrow 0 \text{ as } k \to \infty \text{ for all } j = 1, ..., n.$$

(1) If $\mathbf{T}$ is an m-complex symmetric tuple with a conjugation C, we have

$$0 = \lim_{k\to\infty} \Big\|\Big(\sum_{j=0}^{m} (-1)^j \binom{m}{j} (T_1^* + \cdots + T_n^*)^{m-j} (CT_1C + \cdots + CT_nC)^j \Big) Cx_k\Big\|$$

$$= \lim_{k\to\infty} \Big\|\Big((T_1^* + \cdots + T_n^*) - (\overline{z_1} + \cdots + \overline{z_n}) \Big)^m Cx_k\Big\|.$$

Since $\{Cx_k\}$ is a sequence of unit vectors, $\overline{z_1} + \cdots + \overline{z_n}$ belongs to the approximate point spectrum of $T_1^* + \cdots + T_n^*$. In the case of the joint point spectrum, it is clear.
(2) If $\mathbf{T}$ is a skew m-complex symmetric tuple with a conjugation C, it holds

$$0 = \lim_{k\to\infty} \Big\|\Big(\sum_{j=0}^{m} \binom{m}{j} (T_1^* + \cdots + T_n^*)^{m-j} (CT_1C + \cdots + CT_nC)^j \Big) Cx_k\Big\|$$

$$= \lim_{k\to\infty} \Big\|\Big((T_1^* + \cdots + T_n^*) + (\overline{z_1} + \cdots + \overline{z_n}) \Big)^m Cx_k\Big\|.$$

Similarly, we have $-(\overline{z_1} + \cdots + \overline{z_n})$ belongs to the approximate point spectrum of $T_1^* + \cdots + T_n^*$. It is clear for eigenvalue case. □

Next we define an $[m, C]$-symmetric tuple and a skew $[m, C]$-symmetric tuple as follows:

Definition 4.4. *For a commuting n-tuple* $\mathbf{T} = (T_1, ..., T_n) \in B(\mathcal{H})^n$ *and a conjugation C, we define* $w_m(\mathbf{T}; C)$ *and* $\mathcal{W}_m(\mathbf{T}; C)$ *by*

$$w_m(\mathbf{T}; C) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} (CT_1C + \cdots + CT_nC)^{m-j} (T_1 + \cdots + T_n)^j$$

and

$$\mathcal{W}_m(\mathbf{T}; C) = \sum_{j=0}^{m} \binom{m}{j} (CT_1C + \cdots + CT_nC)^{m-j} (T_1 + \cdots + T_n)^j.$$

A commuting n-tuple $\mathbf{T} = (T_1, ..., T_n)$ *is said to be an* $[m, C]$*-symmetric tuple and a skew* $[m, C]$*-symmetric tuple with a conjugation C if* $w_m(\mathbf{T}; C) = 0$ *and* $\mathcal{W}_m(\mathbf{T}; C) = 0$*, respectively.*

Theorem 4.4. *Let* $\mathbf{T} = (T_1, ..., T_n) \in B(\mathcal{H})^n$ *be a commuting n-tuple.*
(1) If $\mathbf{T}$ *is an* $[m, C]$*-symmetric tuple with a conjugation C and* $(z_1, ..., z_n) \in \sigma_{ja}(\mathbf{T})$*, then* $(\overline{z_1} + \cdots + \overline{z_n})$ *belongs to the approximate point spectrum of* $T_1 + \cdots + T_n$*. Hence, if* $(z_1, ..., z_n) \in \sigma_{jp}(\mathbf{T})$*, then* $(\overline{z_1} + \cdots + \overline{z_n}) \in \sigma_p(T_1 + \cdots + T_n)$*.*
(2) If $\mathbf{T}$ *is a skew* $[m, C]$*-symmetric tuple with a conjugation C and* $(z_1, ..., z_n) \in \sigma_{ja}(\mathbf{T})$*, then* $-(\overline{z_1} + \cdots + \overline{z_n})$ *belongs to the approximate point spectrum of* $T_1 + \cdots + T_n$*. Hence, if* $(z_1, ..., z_n) \in \sigma_{jp}(\mathbf{T})$*, then* $-(\overline{z_1} + \cdots + \overline{z_n}) \in \sigma_p(T_1 + \cdots + T_n)$*.*

Proof. Let $\{x_k\}$ be a sequence of unit vectors such that

$$(T_j - z_j)x_k \longrightarrow 0 \text{ as } k \to \infty \text{ for all } j = 1, ..., n.$$

(1) If **T** is an $[m, C]$-symmetric tuple with a conjugation C, we have

$$0 = \lim_{k\to\infty} \|\Big(\sum_{j=0}^{m} (-1)^j \binom{m}{j} (CT_1C + \cdots + CT_nC)^{m-j}(T_1 + \cdots + T_n)^j \Big) x_k\|$$

$$= \lim_{k\to\infty} \|\Big((CT_1C + \cdots + CT_nC) - (z_1 + \cdots + z_n) \Big)^m x_k\|.$$

Hence $z_1 + \cdots + z_n$ belongs to the approximate point spectrum of $CT_1C + \cdots + CT_nC = C(T_1 + \cdots + T_n)C$ and therefore, by Lemma 3.21 of [6], we have $\overline{z_1} + \cdots + \overline{z_n} \in \sigma_a(T_1 + \cdots + T_n)$. In the case of the joint point spectrum, it is clear.

(2) If **T** is skew $[m, C]$-symmetric with a conjugation C, it holds

$$0 = \lim_{k\to\infty} \|\Big(\sum_{j=0}^{m} \binom{m}{j} (CT_1C + \cdots + CT_nC)^{m-j}(T_1 + \cdots + T_n)^j \Big) x_k\|$$

$$= \lim_{k\to\infty} \|\Big((CT_1C + \cdots + CT_nC) + (z_1 + \cdots + z_n) \Big)^m x_k\|.$$

Therefore we have $-(z_1 + \cdots + z_n) \in \sigma_a(CT_1C + \cdots + CT_nC) = \sigma_a(C(T_1 + \cdots + T_n)C)$. By Lemma 3.21 of [6], we have $-(\overline{z_1} + \cdots + \overline{z_n}) \in \sigma_a(T_1 + \cdots + T_n)$. It is clear in the eigenvalue case. □

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