On the joint spectra of commuting tuples of operators and a conjugation

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1. Introduction

Let \( \mathcal{H} \) be a complex Hilbert space with the inner product \( \langle \cdot, \cdot \rangle \) and \( B(\mathcal{H}) \) be the set of all bounded linear operators on \( \mathcal{H} \). In [6], S. Jung, E. Ko and Ji Eun Lee showed that if \( C \) is a conjugation and \( T \in B(\mathcal{H}) \), then \( \sigma(C^T) = \sigma(T)' \), \( \sigma_p(C^T) = \sigma_p(T)' \) and \( \sigma_{mp}(C^T) = \sigma_{mp}(T)' \), where \( \sigma(T) \), \( \sigma_p(T) \) and \( \sigma_{mp}(T) \) are the spectrum, the approximate point spectrum and the joint point spectrum of \( T \), respectively. An antilinear operator \( C \) is said to be a conjugation if \( C \) satisfies \( C^2 = I, \langle Cx, Cy \rangle = \langle y, x \rangle \) for all \( x, y \in \mathcal{H} \), where \( I \) is the identity of \( B(\mathcal{H}) \). We have some results of \( m \)-complex isometric, \([m, C]\)-isometric and others for single operators. Please see [1], [2] and [3] for results and examples of these classes.

In this paper we show that if \( C \) is a conjugation and \( T = (T_1, \ldots, T_n) \in B(\mathcal{H})^n \) is a commuting \( n \)-tuple, then we show \( \sigma(C^T) = \sigma(T)' \), \( \sigma_p(C^T) = \sigma_p(T)' \) and \( \sigma_{mp}(C^T) = \sigma_{mp}(T)' \), where \( C^T = (CT_1 C, \ldots, CT_n C) \), and \( \sigma(T), \sigma_p(T) \) and \( \sigma_{mp}(T) \) are the Taylor spectrum, the joint approximate point spectrum and the joint point spectrum of \( T \), respectively. Finally we characterize joint approximate point spectra of \( m \)-symmetric tuples, \( m \)-complex symmetric tuples, skew \( m \)-complex symmetric tuples, \([m, C]\)-symmetric tuples and skew \([m, C]\)-symmetric tuples.

Abstract. In this paper we show that if \( T = (T_1, \ldots, T_n) \) is a commuting \( n \)-tuple of Hilbert space operators and \( C \) is a conjugation, then \( \sigma(C^T) = \sigma(T)' \), where \( \sigma(C^T) = \{CT_1 C, \ldots, CT_n C\} \), \( \sigma(T) \) is the Taylor spectrum of \( T \) and \( \sigma(T)' = \{z = (z_1, \ldots, z_n) : z = (z_1, \ldots, z_n) \in \sigma(T)\} \). We characterize joint approximate point spectra of \( m \)-symmetric tuples, \( m \)-complex symmetric tuples, skew \( m \)-complex symmetric tuples, \([m, C]\)-symmetric tuples and skew \([m, C]\)-symmetric tuples.

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2. Preparation

For a commuting \( n \)-tuple \( \mathbf{T} = (T_1, ..., T_n) \in B(\mathcal{H})^n \), we explain the Taylor spectrum \( \sigma(\mathbf{T}) \) of \( \mathbf{T} \) shortly. Let \( E^n \) be the exterior algebra on \( n \) generators, that is, \( E^n \) is the complex algebra with identity \( e \) generated by indeterminates \( e_1, ..., e_n \). Let \( E_k^n(\mathcal{H}) = \mathcal{H} \otimes E_k^n \). Define \( d_k^n : E_k^n(\mathcal{H}) \rightarrow E_{k-1}^n(\mathcal{H}) \) by

\[
d_k^n(x \otimes e_j, \wedge \cdots \wedge e_k) := \sum_{j=1}^{k} (-1)^{j-1} T_j x \otimes e_j, \wedge \cdots \wedge \hat{e}_j, \wedge \cdots \wedge e_k,
\]

where \( \hat{e}_j \) means deletion. We denote \( d_k^n \) by \( d_k \) simply. We think Koszul complex \( E(\mathbf{T}) \) of \( \mathbf{T} \) as follows:

\[
(*) \quad E(\mathbf{T}) : 0 \rightarrow E_n^0(\mathcal{H}) \xrightarrow{d_1} E_{n-1}^0(\mathcal{H}) \xrightarrow{d_2} \cdots \xrightarrow{d_{n-1}} E_1^0(\mathcal{H}) \xrightarrow{d_n} E_0^0(\mathcal{H}) \rightarrow 0.
\]

It is easy to see that \( E_k^n(\mathcal{H}) \cong \mathcal{H} \oplus \cdots \oplus \mathcal{H} \) (\( k = 1, ..., n \)).

Definition 2.1. A commuting \( n \)-tuple \( \mathbf{T} = (T_1, ..., T_n) \in B(\mathcal{H})^n \) is said to be nonsingular if and only if the Koszul complex \( E(\mathbf{T}) \) is exact.

Definition 2.2. For a commuting \( n \)-tuple \( \mathbf{T} = (T_1, ..., T_n) \in B(\mathcal{H})^n \), \( z = (z_1, ..., z_n) \notin \sigma(\mathbf{T}) \) (Taylor spectrum) if \( \mathbf{T} - z = (T_1 - z_1, ..., T_n - z_n) \) is nonsingular.

About the definition of the Taylor spectrum, see details J. L. Taylor [7] and [8].

The joint approximate point spectrum of \( \mathbf{T} = (T_1, ..., T_n) \in B(\mathcal{H})^n \) is denoted by \( \sigma_{ja}(\mathbf{T}) \), i.e., \((z_1, ..., z_n) \in \sigma_{ja}(\mathbf{T})\) if and only if there exists a sequence \( \{x_k\} \) of unit vectors such that

\[
(T_j - z_j)x_k \rightarrow 0 \quad \text{as } k \rightarrow \infty \quad \text{for all } j = 1, ..., n.
\]

The joint point spectrum \( \sigma_{pt}(\mathbf{T}) \) of \( \mathbf{T} = (T_1, ..., T_n) \in B(\mathcal{H})^n \) is the set of all \((z_1, ..., z_n) \in \mathbb{C}^n\) which there exists a nonzero vector \( x \) such that \((T_j - z_j)x = 0\) for all \( j = 1, ..., n\).

3. Result

First we need the following result by R. Curto [1].

Theorem 3.1. For a commuting \( n \)-tuple \( \mathbf{T} = (T_1, ..., T_n) \in B(\mathcal{H})^n \), \( 0 = (0, ..., 0) \notin \sigma(\mathbf{T}) \) if and only if

\[
\alpha(\mathbf{T}) := \begin{pmatrix}
d_1 & 0 & \cdots & \cdots \\
d_2 & d_3 & \cdots & \cdots \\
0 & d_4 & \cdots & \cdots \\
\vdots & \vdots & \ddots & \ddots
\end{pmatrix}^{2^n-1}
\]

is invertible on \( \mathcal{H} \oplus \cdots \oplus \mathcal{H} \),

where \( d_k \) is the mapping of \( * \) \( (k = 1, 2, ..., n) \).

For a conjugation \( C \) on \( \mathcal{H} \), let \( CTC = (CT_1C, ..., CT_nC) \). If \( \mathbf{T} = (T_1, ..., T_n) \) is a commuting \( n \)-tuple, then \( CTC \) is also commuting \( n \)-tuple.
Lemma 3.2. For a commuting $n$-tuple $T = (T_1, ..., T_n) \in B(H)^n$ and any conjugation $C$, $0 = (0, ..., 0) \notin \sigma(T)$ if and only if $0 = (0, ..., 0) \notin \sigma(CT_{i}C)$, where $CT_{i}C = (CT_{1}C, ..., CT_{n}C)$.

Proof. It holds $CT_{i}C \cdot CT_{i}C = CT_{i}T_{i}C$ and $(CT_{i}C)^* = CT_{i}C$. Hence we have

$$\alpha(C) = \begin{pmatrix} C & 0 & \cdots & \cdots & 0 \\ 0 & C & \cdots & \cdots & 0 \\ 0 & 0 & C & \cdots & \cdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & C \end{pmatrix} \cdot \alpha(T) \cdot \begin{pmatrix} C & 0 & \cdots & \cdots & 0 \\ 0 & C & \cdots & \cdots & 0 \\ 0 & 0 & C & \cdots & \cdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & C \end{pmatrix}$$

on $\mathcal{H} \oplus \cdots \oplus \mathcal{H}$. Since $\tilde{C} = C \oplus \cdots \oplus C$ is a conjugation on $\mathcal{H} \oplus \cdots \oplus \mathcal{H}$, it holds that $\alpha(T)$ is invertible if and only if $\alpha(C\alpha(T)C)$ is invertible. □

Remark 3.3. By Theorem 1.1 of [5], if $T = (T_1, T_2) \in B(H)^2$ is a commuting pair, then $T$ is nonsingular if and only if

$$\alpha(T) = \begin{pmatrix} T_1 & T_2 \\ -T_2 & T_1 \end{pmatrix}$$

is invertible on $\mathcal{H} \oplus \mathcal{H}$. Let $CT_{i}C = (CT_{1}C, CT_{2}C)$. Then we have

$$\alpha(C\alpha(T)C) = \begin{pmatrix} CT_{1}C & CT_{2}C \\ -(CT_{2}C)^* & (CT_{1}C)^* \end{pmatrix} = \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix} \cdot \alpha(T) \cdot \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix}.$$ 

By this equality, it holds $\alpha(T)$ is invertible if and only if $\alpha(C\alpha(T)C)$ is invertible. So Lemma 3.2 is clear.

Theorem 3.4. For a commuting $n$-tuple $T = (T_1, ..., T_n) \in B(H)^n$ and any conjugation $C$, it holds $\sigma(C\alpha(T)C) = \sigma(T)$, $\sigma_p(C\alpha(T)C) = \sigma_p(T)$ and $\sigma_p(C\alpha(T)C) = \sigma_p(T)$ for any conjugation $C$, where $E^* = \{z = (z_1, ..., z_n) : z \in \mathbb{C}^n\} \subset \mathbb{C}^n$.

Proof. It holds that $(C(T_1 - z_1)C, ..., C(T_n - z_n)C) = (CT_1C - \overline{z_1}, ..., CT_nC - \overline{z_n}) = CT_1C - \overline{z},$ where $z = (z_1, ..., z_n).$ Hence proof follows from Lemma 3.2. □

4. Properties of joint approximate point spectra of commuting tuples

Definition 4.1. For a commuting $n$-tuple $T = (T_1, ..., T_n) \in B(H)^n$, we define $\mathcal{P}_m(T)$ by

$$\mathcal{P}_m(T) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} \left( \sum_{|I|=k} \frac{k!}{I!} \cdot T^{I^\perp} \cdot T^I \right).$$

$T = (T_1, ..., T_n)$ is said to be an $m$-isometric tuple if $\mathcal{P}_m(T) = 0$.

Then in [5] J. Gleason and S. Richter proved the following result.

Proposition 4.1. (Lemma 3.2, [5])
Let $T = (T_1, ..., T_n) \in B(H)^n$ be an $m$-isometric tuple. If $z = (z_1, ..., z_n) \in \sigma_p(T)$, then $|z|^2 = |z_1|^2 + \cdots + |z_n|^2 = 1.$
We introduce $m$-symmetric tuples as follows.

**Definition 4.2.** Let, for commuting $n$-tuple $T = (T_1, \ldots, T_n) \in B(H)^n$ and $A \in B(H)$,

$$S_T(A) := (T_1 + \cdots + T_n)^*A - A(T_1 + \cdots + T_n).$$

An $n$-tuple $T = (T_1, \ldots, T_n) \in B(H)^n$ is said to be an $m$-symmetric tuple if

$$S_T^m(I) = 0.$$ 

Then it holds

$$S_T^m(I) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} (T_1^* + \cdots + T_n^*)^m - (T_1 + \cdots + T_n)^j.$$

**Theorem 4.2.** Let $T = (T_1, \ldots, T_n) \in B(H)^n$ be an $m$-symmetric commuting tuple of operators. If $(z_1, \ldots, z_n) \in \sigma_m(T)$, then $z_1 + \cdots + z_n$ is a real number.

**Proof.** Let $(x_k)$ be a sequence of unit vectors such that

$$(T_j - z_j)x_k \rightarrow 0 \text{ as } k \rightarrow \infty \text{ for all } j = 1, \ldots, n.$$ 

Since then $T$ is $m$-symmetric, it holds

$$0 = \left( \sum_{j=0}^{m} (-1)^j \binom{m}{j} (T_1^* + \cdots + T_n^*)^m - (T_1 + \cdots + T_n)^j \right) x_k, x_k$$

$$\rightarrow \left( (z_1 + \cdots + z_n) - (z_1 + \cdots + z_n) \right)^m \text{ as } k \rightarrow \infty.$$ 

Hence $z_1 + \cdots + z_n$ is a real number. \(\square\)

We define a $m$-complex symmetric tuple and skew $m$-complex symmetric tuple as follows:

**Definition 4.3.** For a commuting $n$-tuple $T = (T_1, \ldots, T_n) \in B(H)^n$ and a conjugation $C$, we define $r_m(T; C)$ and $R_m(T; C)$ by

$$r_m(T; C) := \sum_{j=0}^{m} (-1)^j \binom{m}{j} (T_1^* + \cdots + T_n^*)^m - (CT_1C + \cdots + CT_nC)^j$$

and

$$R_m(T; C) := \sum_{j=0}^{m} \binom{m}{j} (T_1^* + \cdots + T_n^*)^m - (CT_1C + \cdots + CT_nC)^j.$$ 

A commuting $n$-tuple $T = (T_1, \ldots, T_n)$ is said to be a $m$-complex symmetric tuple and a skew $m$-complex symmetric tuple with a conjugation $C$ if $r_m(T; C) = 0$ and $R_m(T; C) = 0$, respectively.
Theorem 4.3. Let $T = (T_1, ..., T_n)$ be a commuting $n$-tuple.

(1) If $T$ is an $m$-complex symmetric tuple with a conjugation $C$ and $(z_1, ..., z_n) \in \sigma_{p^m}(T)$, then $(z_1^m + \cdots + z_n^m)$ belongs to the approximate point spectrum of $T_1^m + \cdots + T_n^m$. Hence if $(z_1, ..., z_n) \in \sigma_p(T)$, then $(z_1^m + \cdots + z_n^m) \in \sigma_p(T_1^m + \cdots + T_n^m)$.

(2) If $T$ is a skew $m$-complex symmetric tuple with a conjugation $C$ and $(z_1, ..., z_n) \in \sigma_{p^m}(T)$, then $-(z_1^m + \cdots + z_n^m)$ belongs to the approximate point spectrum of $T_1^m + \cdots + T_n^m$. Hence if $(z_1, ..., z_n) \in \sigma_p(T)$, then $-(z_1^m + \cdots + z_n^m) \in \sigma_p(T_1^m + \cdots + T_n^m)$.

Proof. Let $\{x_k\}$ be a sequence of unit vectors such that $(T_j - z_j)x_k \to 0$ as $k \to \infty$ for all $j = 1, ..., n$.

(1) If $T$ is an $m$-complex symmetric tuple with a conjugation $C$, we have

$$0 = \lim_{k \to \infty} \left\| \sum_{j=0}^{m-1} (-1)^j \binom{m}{j} (T_1^m + \cdots + T_n^m)^{m-j} \overline{C} \right\| C x_k$$

$$= \lim_{k \to \infty} \left\| \left( T_1^m + \cdots + T_n^m \right) - \left( \overline{z_1}^m + \cdots + \overline{z_n}^m \right) \right\| C x_k.$$  

Since $\{C x_k\}$ is a sequence of unit vectors, $\overline{z_1}^m + \cdots + \overline{z_n}^m$ belongs to the approximate point spectrum of $T_1^m + \cdots + T_n^m$. In the case of the joint point spectrum, it is clear.

(2) If $T$ is a skew $m$-complex symmetric tuple with a conjugation $C$, it holds

$$0 = \lim_{k \to \infty} \left\| \sum_{j=0}^{m-1} \binom{m}{j} (T_1^m + \cdots + T_n^m)^{m-j} \overline{C} \right\| C x_k$$

$$= \lim_{k \to \infty} \left\| \left( T_1^m + \cdots + T_n^m \right) + \left( \overline{z_1}^m + \cdots + \overline{z_n}^m \right) \right\| C x_k.$$  

Similarly, we have $-(\overline{z_1}^m + \cdots + \overline{z_n}^m)$ belongs to the approximate point spectrum of $T_1^m + \cdots + T_n^m$. It is clear for eigenvalue case. □

Next we define an $[m, C]$-symmetric tuple and a skew $[m, C]$-symmetric tuple as follows:

**Definition 4.4.** For a commuting $n$-tuple $T = (T_1, ..., T_n) \in B(H)^n$ and a conjugation $C$, we define $w_m(T; C)$ and $W_m(T; C)$ by

$$w_m(T; C) = \sum_{j=0}^{m-1} (-1)^j \binom{m}{j} (T_1 + \cdots + T_n)^{m-j}(T_1 + \cdots + T_n)^j$$

and

$$W_m(T; C) = \sum_{j=0}^{m-1} \binom{m}{j} (T_1 + \cdots + T_n)^{m-j}(T_1 + \cdots + T_n)^j.$$  

A commuting $n$-tuple $T = (T_1, ..., T_n)$ is said to be an $[m, C]$-symmetric tuple and a skew $[m, C]$-symmetric tuple with a conjugation $C$ if $w_m(T; C) = 0$ and $W_m(T; C) = 0$, respectively.

Theorem 4.4. Let $T = (T_1, ..., T_n) \in B(H)^n$ be a commuting $n$-tuple.

(1) If $T$ is an $[m, C]$-symmetric tuple with a conjugation $C$ and $(z_1, ..., z_n) \in \sigma_{p^m}(T)$, then $-(z_1^m + \cdots + z_n^m)$ belongs to the approximate point spectrum of $T_1^m + \cdots + T_n^m$. Hence, if $(z_1, ..., z_n) \in \sigma_p(T)$, then $-(z_1^m + \cdots + z_n^m) \in \sigma_p(T_1^m + \cdots + T_n^m)$.

(2) If $T$ is a skew $[m, C]$-symmetric tuple with a conjugation $C$ and $(z_1, ..., z_n) \in \sigma_{p^m}(T)$, then $-(z_1^m + \cdots + z_n^m)$ belongs to the approximate point spectrum of $T_1^m + \cdots + T_n^m$. Hence, if $(z_1, ..., z_n) \in \sigma_p(T)$, then $-(z_1^m + \cdots + z_n^m) \in \sigma_p(T_1^m + \cdots + T_n^m)$. 


Proof. Let \( \{x_k\} \) be a sequence of unit vectors such that
\[
(T_j - z_j)x_k \rightarrow 0 \quad \text{as} \quad k \to \infty \quad \text{for all} \quad j = 1, \ldots, n.
\]

(1) If \( T \) is an \([m, C]\)-symmetric tuple with a conjugation \( C \), we have
\[
0 = \lim_{k \to \infty} \left\| \left( \sum_{j=0}^{m} (-1)^j \binom{m}{j} (CT_1C + \cdots + CT_n C)^{m-j} (T_1 + \cdots + T_n)^j \right) x_k \right\|
\]
\[
= \lim_{k \to \infty} \left\| \left( CT_1C + \cdots + CT_n C - (z_1 + \cdots + z_n) \right)^m x_k \right\|.
\]

Hence \( z_1 + \cdots + z_n \) belongs to the approximate point spectrum of \( CT_1C + \cdots + CT_n C = C(T_1 + \cdots + T_n)C \) and therefore, by Lemma 3.21 of [6], we have \( \overline{z_1} + \cdots + \overline{z_n} \in \sigma_{a}(T_1 + \cdots + T_n) \). In the case of the joint point spectrum, it is clear.

(2) If \( T \) is skew \([m, C]\)-symmetric with a conjugation \( C \), it holds
\[
0 = \lim_{k \to \infty} \left\| \left( \sum_{j=0}^{m} (-1)^j \binom{m}{j} (CT_1C + \cdots + CT_n C)^{m-j} (T_1 + \cdots + T_n)^j \right) x_k \right\|
\]
\[
= \lim_{k \to \infty} \left\| \left( CT_1C + \cdots + CT_n C + (z_1 + \cdots + z_n) \right)^m x_k \right\|.
\]

Therefore we have \( -(z_1 + \cdots + z_n) \in \sigma_{a}(CT_1C + \cdots + CT_n C) = \sigma_{a}(C(T_1 + \cdots + T_n)C) \). By Lemma 3.21 of [6], we have \( -\overline{(z_1 + \cdots + z_n)} \in \sigma_{a}(T_1 + \cdots + T_n) \). It is clear in the eigenvalue case. □

References