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Asymptotically Bloch-periodic solutions of abstract fractional nonlinear differential inclusions with piecewise constant argument

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Abstract. The differential equations with piecewise constant arguments (EPCAs, for short) are subjects of a great number of recent scientific researches. The main aim of this paper is to analyze the existence and uniqueness of asymptotically Bloch-periodic solutions of abstract fractional nonlinear differential inclusions with piecewise constant argument in Banach spaces.

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1. Introduction and preliminaries

It is well known that nonlinear differential equations with piecewise constant argument combine the properties of nonlinear differential equations and nonlinear difference equations. The equations of this kind are prototypes of certain sequential continuous models appearing in disease dynamics, and they also have invaluable importance in mathematical economy.

As mentioned in the abstract, the main aim of this paper is to analyze asymptotical Bloch-periodicity of solutions for certain classes of abstract fractional nonlinear differential inclusions with piecewise constant argument. In such a way, we continue a series of our recent research studies whose results are collected in the monograph [20], as well as the research studies carried out by W. Dimbour, V. Valmorin [9] and M. F. Hasler, G. M. N'Guérékata [15] (cf. also [3]-[6], [11], [13]-[14], [16], [21], [25] and [27] for some other results obtained in this field).

The organization of paper is briefly described as follows. In Section 2, we remind ourselves of some known facts and definitions from the theory of multivalued linear operators in Banach spaces. Section 3 is devoted to the study of Bloch-periodic functions and asymptotically Bloch-periodic functions in Banach spaces. Our main contributions are given in Section 4, where we point out that the results of W. Dimbour and V. Valmorin [9] can be generalized for certain classes of abstract nonlinear fractional differential inclusions with multivalued linear operators $\mathcal{A}$ satisfying the following condition:

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(P) There exist finite constants $c,\ M > 0$ and $\beta \in (0,1]$ such that

$$\Psi := \Big\{\lambda \in \mathbb{C} : \Re\lambda \geq -c\big(|\Im\lambda| + 1\big)\Big\} \subseteq \rho(\mathcal{A})$$

and

$$\|R(\lambda : \mathcal{A})\| \leq M\big(1 + |\lambda|\big)^{-\beta}, \quad \lambda \in \Psi.$$

More precisely, in this section, we consider the fractional differential Cauchy inclusion with piecewise constant argument:

$$\mathbf{D}_t^\gamma u(t) \in \mathcal{A}u(t) + A_0 u\big(\lfloor t\rfloor\big) + g\big(t, u\big(\lfloor t\rfloor\big)\big),\ t > 0; \quad u(0) = u_0, \tag{1.1}$$

where $A_0 \in L(X)$, $\lfloor\cdot\rfloor$ is the largest integer function, $g : [0,\infty) \times X \to X$ is a given function, and $\mathbf{D}_t^\gamma u(t)$ denotes the Caputo fractional derivative or order γ. The main aim of Section 5 is to provide the basic information about asymptotically Bloch-periodic solutions of inclusion (1.1), with $\mathcal{A} = A$ being the single-valued generator of an exponentially decaying C-regularized semigroup of operators. Section 6 is reserved for some conclusions and final remarks about the topic examined.

We use the standard notation throughout the paper. By $(X, \|\cdot\|)$ we denote a complex Banach space; $L(X)$ designates the space of all continuous linear mappings from X into X. By I we denote the identity operator on X. If A is a closed linear operator acting on X, then the domain, kernel space and range of A will be denoted by $D(A)$, $N(A)$ and $R(A)$, respectively. The symbol $C_b([0,\infty) : X)$ denotes the Banach space consisting of all bounded continuous functions from $[0,\infty)$ into X, endowed with the norm $\|f\|_\infty := \sup_{t\geq 0}\|f(t)\|$; $C_0([0,\infty) : X)$ denotes the closed subspace of $C_b([0,\infty) : X)$ consisting of functions vanishing at infinity. We define the space $C_b(\mathbb{R} : X)$ similarly. Henceforward, we always assume that $\gamma \in (0,1]$ as well as that the operator $C \in L(X)$ is injective.

Given $s \in \mathbb{R}$ in advance, put $\lfloor s\rfloor := \sup\{l \in \mathbb{Z} : s \geq l\}$ and $\lceil s\rceil := \inf\{l \in \mathbb{Z} : s \leq l\}$. The Gamma function is denoted by $\Gamma(\cdot)$ and the principal branch is always used to take the powers. Set $g_\zeta(t) := t^{\zeta-1}/\Gamma(\zeta)$ and $g_0(t) :=$ δ-distribution ($\zeta > 0$, $t > 0$).

Fractional calculus started more than three centuries ago, probably with some works of Leibnitz. It is a rapidly growing field of research due to its wide and invaluable applications in various fields of science. For basic information on fractional calculus and fractional differential equations, the reader may consult [2], [18]-[19], [23]-[24] and references cited therein. In this paper, we use the Wright functions, for which it is well known that play an important role in fractional caclulus. Let us recall that the Wright function $\Phi_\gamma(\cdot)$ is defined by

$$\Phi_\gamma(z) := \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma(1-\gamma-n\gamma)}, \quad z \in \mathbb{C} \ \ (\gamma \in (0,1)).$$

We refer the reader to [2] and [18, Section 1.3] for more details about the Mittag-Leffler and Wright functions.

Let $0 < \tau \leq \infty$, $m \in \mathbb{N}$ and $I = (0,\tau)$. Then the Sobolev space $W^{m,1}(I : X)$ is defined in the following way (see e.g. [2, p. 7]):

$$W^{m,1}(I : X) := \Bigg\{ f \mid \exists\varphi \in L^1(I : X)\ \exists c_k \in \mathbb{C}\ (0 \leq k \leq m-1)$$
$$f(t) = \sum_{k=0}^{m-1} c_k g_{k+1}(t) + \big(g_m * \varphi\big)(t) \text{ for a.e. } t \in (0,\tau) \Bigg\}.$$

Then we have $\varphi(t) = f^{(m)}(t)$ in distributional sense, and $c_k = f^{(k)}(0)$ $(0 \leq k \leq m-1)$.

In this paper, we use the Caputo fractional derivatives of order $\gamma \in (0,1]$. Let us recall that the Caputo fractional derivative $\mathbf{D}_t^\gamma u(t)$ is defined for those continuous functions $u : [0,\infty) \to X$ satisfying that for each $T > 0$ one has $g_{1-\gamma} * (u(\cdot) - u(0)) \in W^{1,1}((0,T) : X)$, by

$$\mathbf{D}_t^\gamma u(t) := \frac{d}{dt}\Bigg[g_{1-\gamma} * \Big(u(\cdot) - u(0)\Big)\Bigg](t), \quad t \in (0,T].$$

If $\gamma = 1$, then it is worth observing that the Caputo fractional derivative $\mathbf{D}_t^{\gamma}u(t)$ is well-defined for any locally absolutely continuous function $u : [0,\infty) \to X$ (see [1] for the notion).

2. Multivalued linear operators

The main purpose of this short section is to remind the reader of elementary definitions and results from the theory of multivalued linear operators. For more details about subject, we refer to the monographs by R. Cross [7], A. Favini-A. Yagi [12] and M. Kostić [19].

A multivalued map $\mathcal{A} : X \to P(X)$ is said to be a multivalued linear operator (MLO in X, or simply, MLO) iff the following holds:

(i) $D(\mathcal{A}) := \{x \in X : \mathcal{A}x \neq \emptyset\}$ is a linear subspace of X;

(ii) $\mathcal{A}x + \mathcal{A}y \subseteq \mathcal{A}(x+y)$, $x,\ y \in D(\mathcal{A})$ and $\lambda\mathcal{A}x \subseteq \mathcal{A}(\lambda x)$, $\lambda \in \mathbb{C}$, $x \in D(\mathcal{A})$.

It is well known that, for every $x,\ y \in D(\mathcal{A})$ and for every $\lambda,\ \eta \in \mathbb{C}$ with $|\lambda| + |\eta| \neq 0$, we have $\lambda\mathcal{A}x + \eta\mathcal{A}y = \mathcal{A}(\lambda x + \eta y)$. Furthermore, $\mathcal{A}0$ is a linear subspace of X and $\mathcal{A}x = f + \mathcal{A}0$ for any $x \in D(\mathcal{A})$ and $f \in \mathcal{A}x$. Set $R(\mathcal{A}) := \{\mathcal{A}x : x \in D(\mathcal{A})\}$. The set $\mathcal{A}^{-1}0 := N(\mathcal{A}) := \{x \in D(\mathcal{A}) : 0 \in \mathcal{A}x\}$ is called the kernel of $\mathcal{A}$. The inverse $\mathcal{A}^{-1}$ is defined by $D(\mathcal{A}^{-1}) := R(\mathcal{A})$ and $\mathcal{A}^{-1}y := \{x \in D(\mathcal{A}) : y \in \mathcal{A}x\}$. It is checked at once that $\mathcal{A}^{-1}$ is an MLO in E, as well as that $N(\mathcal{A}^{-1}) = \mathcal{A}0$ and $(\mathcal{A}^{-1})^{-1} = \mathcal{A}$. If $N(\mathcal{A}) = \{0\}$, i.e., if $\mathcal{A}^{-1}$ is single-valued, then $\mathcal{A}$ is called injective.

Let $\mathcal{A}$ and $\mathcal{B}$ be two given MLOs in X. Then we define its sum $\mathcal{A}+\mathcal{B}$ by $D(\mathcal{A}+\mathcal{B}) := D(\mathcal{A}) \cap D(\mathcal{B})$ and $(\mathcal{A}+\mathcal{B})x := \mathcal{A}x + \mathcal{B}x$, $x \in D(\mathcal{A}+\mathcal{B})$. It is easily verified that $\mathcal{A}+\mathcal{B}$ is likewise an MLO in X. The product of $\mathcal{A}$ and $\mathcal{B}$ is defined by $D(\mathcal{B}\mathcal{A}) := \{x \in D(\mathcal{A}) : D(\mathcal{B}) \cap \mathcal{A}x \neq \emptyset\}$ and $\mathcal{B}\mathcal{A}x := \mathcal{B}(D(\mathcal{B}) \cap \mathcal{A}x)$. Then $\mathcal{B}\mathcal{A}$ is again an MLO in X and $(\mathcal{B}\mathcal{A})^{-1} = \mathcal{A}^{-1}\mathcal{B}^{-1}$. The inclusion $\mathcal{A} \subseteq \mathcal{B}$ means that $D(\mathcal{A}) \subseteq D(\mathcal{B})$ and $\mathcal{A}x \subseteq \mathcal{B}x$ for all $x \in D(\mathcal{A})$. The scalar multiplication of an MLO $\mathcal{A}$ with the number $z \in \mathbb{C}$ is defined as usually.

We say that an MLO operator $\mathcal{A}$ is closed iff for any sequence (x_n) in $D(\mathcal{A})$ and (y_n) in X such that $y_n \in \mathcal{A}x_n$ for all $n \in \mathbb{N}$ we have that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$ imply $x \in D(\mathcal{A})$ and $y \in \mathcal{A}x$. If the MLO $\mathcal{A}$ is not closed, then we can simply prove that its closure $\mathcal{A}$, defined as usual, is a closed MLO in X. Any multivalued linear operator considered in this paper will be closed.

Suppose that $\mathcal{A}$ is an MLO in X, as well as that $C \in L(X)$ is injective and $C\mathcal{A} \subseteq \mathcal{A}C$. Then the C-resolvent set of $\mathcal{A}$, $\rho_C(\mathcal{A})$ for short, is defined as the union of those complex numbers $\lambda \in \mathbb{C}$ for which

(i) $R(C) \subseteq R(\lambda - \mathcal{A})$;

(ii) $(\lambda - \mathcal{A})^{-1}C$ is a single-valued linear continuous operator on X.

The operator $\lambda \mapsto (\lambda - \mathcal{A})^{-1}C$ is said to be the C-resolvent of $\mathcal{A}$ ($\lambda \in \rho_C(\mathcal{A})$); the resolvent set of $\mathcal{A}$ is defined by $\rho(\mathcal{A}) := \rho_I(\mathcal{A})$, $R(\lambda : \mathcal{A}) \equiv (\lambda - \mathcal{A})^{-1}$ ($\lambda \in \rho(\mathcal{A})$). For more details about C-resolvent sets of multivalued linear operators, we refer the reader to [12] and [19].

Concerning fractional powers of multivalued linear operators and interpolation spaces used below, the reader may consult [12] and [19].

3. Bloch-periodic functions and asymptotically Bloch-periodic functions in Banach spaces

Let $I = [0,\infty)$ or $I = \mathbb{R}$, let $p \geq 0$, and let $k \in \mathbb{R}$. A bounded continuous function $f : I \to X$ is said to be (Bloch) (p,k)-periodic, or Bloch-periodic with period p and Bloch wave vector or Floquet exponent k iff

$$f(x+p) = e^{ikp}f(x), \quad x \in I.$$

The space of all functions $f : I \to X$ that are (p,k)-periodic will be denoted by $\mathcal{P}_{p,k}(I : X)$.

Remark 3.1. Note that the Bloch-periodicity is much more general than the notions of periodicity and anti-periodicity. Indeed, if $f(\cdot)$ is (p,k)-periodic, such that $kp = 2\pi$, then the function $f(\cdot)$ is periodic. If $kp = \pi$, then $f(\cdot)$ is anti-periodic.

Let $f,\ f_1,\ f_2 \in \mathcal{P}_{p,k}(I : X)$. Then we have the following:

(a) $f_1 + f_2(\cdot)$ is (p,k)-periodic;

(b) $cf(\cdot)$ is (p,k)-periodic for $c \in \mathbb{C}$;

(c) $f_t(\cdot) := f(\cdot + t)$ is (p,k)-periodic for $t \in I$;

(d) If $D : X \to X$ is a linear continuous operator, then $Df(\cdot)$ is (p,k)-periodic;

(e) The function $F_a(\cdot) := \int_a^{\cdot+a} f(s)\,ds$ is (p,k)-periodic for $a \in I$.

It is well known that the space $\mathcal{P}_{p,k}(I : X)$, endowed with the sup-norm, is a closed subspace of $C_b(I : X)$.

Now we will repeat the following well-known definition of an asymptotically (p,k)-Bloch-periodic function (see e.g. [15]).

Definition 3.2. Let $p \geq 0$ and let $k \in \mathbb{R}$. A bounded continuous function $f : [0,\infty) \to X$ is called asymptotically (p,k)-Bloch-periodic iff there are a Bloch (p,k)-periodic function $g : \mathbb{R} \to X$ and a function $h \in C_0([0,\infty) : X)$ such that

$$f(x) = g(x) + h(x), \quad x \geq 0. \tag{3.1}$$

The functions $g(\cdot)$ and $h(\cdot)$ are called principal and corrective parts of $f(\cdot)$, respectively. The space of all asymptotically (p,k)-Bloch-periodic X-valued functions will be denoted by $\mathcal{A}P_{p,k}([0,\infty) : X)$.

Notice, finally, that the decomposition of an asymptotically (p,k)-Bloch-periodic X-valued function into its principal and corrective part is unique (cf. (3.1)).

4. Asymptotically Bloch-periodic solutions for nonlinear fractional differential inclusions with piecewise constant argument

In this section, we suppose that $\gamma \in (0,1]$ and $\mathcal{A}$ is an MLO in X satisfying the condition (P).

Definition 4.1. We say that a function $u \in C([0,\infty) : X)$ is a classical solution of the fractional differential inclusion (1.1) iff the following is satisfied:

(i) The derivative $\mathbf{D}_t^\gamma u(t)$ is well-defined;

(ii) We have that $u(0) = u_0$, $g(\cdot, u(\lfloor\cdot\rfloor)) \in L^1_{loc}([0,\infty) : X)$ and (1.1) is satisfied for a.e. $t > 0$.

In contrast to [9, Definition 3.1], where the case $\gamma = 1$ has been considered, we allow the possibility for the non-existence of one-sided derivatives at non-negative integer points (this, clearly, has a certain mathematical sense). Furhtermore, in our analyses, we do not require a priori that the function $g : [0,\infty) \times X \to X$ is jointly continuous.

Example. Consider the case in which $X = \mathbb{C}$, $0 < \epsilon < \gamma \leq 1$, $\mathcal{A} = A_0 = 0$, $u_0 = 0$, $u(t) = g_{1+\epsilon}(t)$, $t \geq 0$ and $g(t,z) := g_{1+\epsilon-\gamma}(t)$, $t > 0$, $z \in \mathbb{C}$. Then $u(\cdot)$ is a classical solution of (1.1) in the sense of Definition 4.1, $\mathbf{D}_t^\gamma u(t) = g_{1+\epsilon-\gamma}(t)$ is not continuous for $t = 0$, $u(\cdot)$ is not a classical solution of (1.1), with $\gamma = 1$, in the sense of [9, Definition 3.1] because the right derivative of $u(\cdot)$ does not exist for $t = 0$, and $g(\cdot,\cdot)$ is not continuous on $[0,\infty) \times X$.

In order to define the notion of mild solution of (1.1), we need some preliminaries from [19]. Assume first that $\gamma \in (0,1)$. Let the contour $\Gamma := \{\lambda = -c(|\eta| + 1) + i\eta : \eta \in \mathbb{R}\}$ be oriented so that $\Im\lambda$ increases along Γ. Set $T(0) := I$ and

$$T(t)x := \frac{1}{2\pi i}\int_\Gamma e^{t\lambda}\big(\lambda - \mathcal{A}\big)^{-1}x\,d\lambda, \quad t > 0,\ x \in X.$$

Then $(T(t))_{t\geq 0} \subseteq L(X)$ is a semigroup on X,

$$\|T(t)\| = O\Big(e^{-ct}t^{\beta-1}\Big), \quad t > 0 \tag{4.1}$$

and $T(t)x \to x$, $t \to 0+$ for any $x \in X$ belonging to the space $D((-\mathcal{A})^{\theta})$ with $\theta > 1-\beta$ ($x \in X^{\theta}_{\mathcal{A}}$ with $1 > \theta > 1-\beta$); see [12] for the notion. Define, for every $\nu > -\beta$,

$$T_{\gamma,\nu}(t)x := t^{-\gamma}\int_0^{\infty} s^{\nu}\Phi_{\gamma}\big(st^{-\gamma}\big)T(s)x\,ds, \quad t > 0,\ x \in X \text{ and } T_{\gamma,0}(0) := I.$$

Then we know that the integral which defines the operator $T_{\gamma,\nu}(t)$ is absolutely convergent as well as that $\|T_{\gamma,\nu}(t)\| = O(t^{\gamma(\nu+\beta-1)})$, $t > 0$.

Following E. Bazhlekova [2] and R.-N. Wang, D.-H. Chen and T.-J. Xiao [26], we define

$$S_{\gamma}(t) := T_{\gamma,0}(t),\ P_{\gamma}(t) := \gamma T_{\gamma,1}(t)/t^{\gamma} \text{ and } R_{\gamma}(t) := t^{\gamma-1}P_{\gamma}(t), \quad t > 0.$$

Then we know (see [20]) that $S_{\gamma}(t)x \to x$, $t \to 0+$ for any $x \in X$ such that $T(t)x \to x$, $t \to 0+$, as well as that there exists a finite constant $M > 0$ such that

$$\big\|S_{\gamma}(t)\big\| \leq Mt^{\gamma(\beta-1)}, \quad t > 0, \quad \big\|R_{\gamma}(t)\big\| \leq Mt^{\gamma\beta-1}, \quad t > 0, \tag{4.2}$$

$$\big\|S_{\gamma}(t)\big\| \leq Mt^{-\gamma},\ t \geq 1 \quad \text{and} \quad \big\|R_{\gamma}(t)\big\| \leq M_2t^{-\gamma-1},\ t \geq 1. \tag{4.3}$$

If $\gamma = 1$, then we define $S_{\gamma}(\cdot) := R_{\gamma}(\cdot) := T(\cdot)$.

Definition 4.2. Let $\gamma \in (0,1]$, and let $u_0 \in X$ be such that $S_{\gamma}(t)u_0 \to u_0$, $t \to 0+$. A function $u \in C([0,\infty) : X)$ is said to be a mild solution of the fractional differential inclusion (1.1) iff $g(\cdot, u(\lfloor\cdot\rfloor)) \in L^1_{loc}([0,\infty) : X)$ and

$$u(t) = S_{\gamma}(t)u_0 + \int_0^t R_{\gamma}(t-s)\Big[A_0u\big(\lfloor s\rfloor\big) + g\big(s, u\big(\lfloor s\rfloor\big)\big)\Big]\,ds, \quad t > 0. \tag{4.4}$$

In the case that $\gamma = 1$ and the function $g : [0,\infty) \times X \to X$ is jointly continuous, the notion of mild solution of (1.1) is equivalent with that one introduced in [9, Definition 3.2]. In the case that $\gamma \in (0,1)$, the argumentation used in the proof of [26, Lemma 4.1] shows that any classical solution of (1.1) has to satisfy the equation (4.4), provided that the mapping $g(\cdot, u(\lfloor\cdot\rfloor))$ is Laplace transformable (cf. [1, Section 1.4]); here, we would like to observe that it is not clear how we can prove the above fact in the case that the mapping $g(\cdot, u(\lfloor\cdot\rfloor))$ does not satisfy this condition.

The following two lemmae and Theorem 4.5 below can be deduced by using the growth rates of operator families $S_{\gamma}(\cdot)$ and $R_{\gamma}(\cdot)$ (cf. (4.1) for the case $\gamma = 1$, and (4.2)-(4.3) for the case $\gamma \in (0,1)$) and the arguments contained in the proofs of [9, Lemma 3.1, Lemma 3.2, Theorem 3.3]; the only thing worth noting is that the semigroup property of $T(\cdot)$ has not been used in the proofs of afore-mentioned results. For the sake of completeness, we will present here only the most important details of the proof of Lemma 4.3:

Lemma 4.3. *Let $A_0 \in L(X)$, let $p \in \mathbb{N}$, and let $k \in \mathbb{R}$. Define the nonlinear operator Λ_1 on $\mathcal{AP}_{p,k}([0,\infty) : X)$ through*

$$\big(\Lambda_1\varphi\big)(t) := \int_0^t R_{\gamma}(t-s)A_0\varphi\big(\lfloor s\rfloor\big)\,ds, \quad t \geq 0,$$

for any $\varphi \in \mathcal{AP}_{p,k}([0,\infty) : X)$. Then Λ_1 is well-defined and maps $\mathcal{AP}_{p,k}([0,\infty) : X)$ into itself.

Proof. The function $\varphi \in \mathcal{A}P_{p,k}([0,\infty) : X)$ can be written as $\varphi = \psi + \theta$, where $\psi \in \mathcal{P}_{p,k}(\mathbb{R} : X)$ and $\theta \in C_0([0,\infty) : X)$. Further on, we can write $\big(\Lambda_1\varphi\big)(t) = \Psi(t) + \Theta(t)$, $t \geq 0$, where

$$\Psi(t) := \int\limits_{-\infty}^{t} R_\gamma(t-s)A_0\psi\big(\lfloor s\rfloor\big)\,ds, \quad t \in \mathbb{R}$$

and

$$\Theta(t) := \int\limits_{0}^{t} R_\gamma(t-s)A_0\theta\big(\lfloor s\rfloor\big)\,ds - \int\limits_{-\infty}^{0} R_\gamma(t-s)A_0\psi\big(\lfloor s\rfloor\big)\,ds, \quad t \geq 0.$$

Using the proof of [9, Lemma 3.1] and the estimates (4.1)-(4.3), it can be easily checked that $\Theta \in C_0([0,\infty) : X)$. Hence, it is enough to prove that $\Psi \in \mathcal{P}_{p,k}(\mathbb{R} : X)$. It can be simply proved that $\Psi \in C_b(\mathbb{R} : X)$ [20], so that the final conclusion follows by using the fact that $p \in \mathbb{N}$ and the following simple computation:

$$\begin{aligned}\Psi(t+p) &= \int\limits_{-\infty}^{t+p} R_\gamma(t+p-s)A_0\psi\big(\lfloor s\rfloor\big)\,ds = \int\limits_{-\infty}^{t} R_\gamma(t-s)A_0\psi\big(\lfloor s+p\rfloor\big)\,ds \\ &= \int\limits_{-\infty}^{t} R_\gamma(t-s)A_0\psi\big(\lfloor s\rfloor + p\big)\,ds = e^{ikp}\int\limits_{-\infty}^{t} R_\gamma(t-s)A_0\psi\big(\lfloor s\rfloor\big)\,ds = e^{ikp}\Psi(t), \quad t \in \mathbb{R}.\end{aligned}$$

□

Lemma 4.4. *Let $p \in \mathbb{N}$, let $k \in \mathbb{R}$, and let $g : \mathbb{R} \times X \to X$ be a bounded mapping such that:*

(i) For all $(t,x) \in \mathbb{R} \times X$, we have $g(t+p, e^{ikp}x) = e^{ikp}g(t,x)$;

(ii) There exists $L > 0$ such that for all $(t,x) \in \mathbb{R} \times X$, we have $\|g(t,x) - g(t,y)\| \leq L\|x-y\|$;

(iii) The mapping $t \mapsto g(t, \psi(\lfloor t\rfloor))$, $t \in \mathbb{R}$ is measurable for all $\psi \in \mathcal{P}_{p,k}([0,\infty) : X)$, and the mapping $t \mapsto g(t, \theta(\lfloor t\rfloor))$, $t \geq 0$ is measurable for all $\theta \in \mathcal{A}P_{p,k}([0,\infty) : X)$.

Define

$$\big(\Lambda_2\varphi\big)(t) := \int\limits_{0}^{t} R_\gamma(t-s)g\big(s, \varphi\big(\lfloor s\rfloor\big)\big)\,ds, \quad t \geq 0,$$

for any $\varphi \in \mathcal{A}P_{p,k}([0,\infty) : X)$. Then the nonlinear operator Λ_2 is well-defined and maps $\mathcal{A}P_{p,k}([0,\infty) : X)$ into itself.

Theorem 4.5. *Let $p \in \mathbb{N}$, let $k \in \mathbb{R}$, and let $g : \mathbb{R} \times X \to X$ be a bounded mapping such that the conditions (i)-(iii) of Lemma 4.4 hold. If $A_0 \in L(X)$ and*

$$\big(\|A_0\| + L\big)\int_0^\infty R_\gamma(t)\,dt < 1,$$

then the equation (1.1) has a unique asymptotically Bloch-periodic solution.

Therefore, we are in a position to analyze the existence and uniqueness of asymptotically Bloch-periodic solutions of nonlinear Poisson heat equations with piecewise constant argument in L^p-spaces ([12], [19]) and nonlinear fractional differential equations with piecewise constant argument in Hölder spaces ([28]).

5. Applications of C-regularized semigroups

Throughout this section, we assume that $\gamma \in (0,1]$ and a closed linear operator $\mathcal{A} = A$ generates an exponentially decaying C-regularized semigroup $(T(t))_{t\geq 0}$ (with the meaning clear and a tolarated abuse of notation). We introduce the operator families $T_{\gamma,\nu}(\cdot)$, $S_\gamma(\cdot)$, $P_\gamma(\cdot)$ and $R_\gamma(\cdot)$ as in the previous section ($\nu > -1$)). Then there exists a number $c > 0$ such that $\{z \in \mathbb{C} : \Re z > -c\} \subseteq \rho_C(A)$ and the estimates (4.1)-(4.3) hold with $\beta = 1$.

We define the classical solution of the fractional differential inclusion (1.1) as in Definition 4.1. The notion of a mild solution of (1.1) is introduced in the following definition.

Definition 5.1. Let $\gamma \in (0,1]$, and let $u_0 \in R(C)$. A function $u \in C([0,\infty) : X)$ is said to be a mild solution of the fractional differential inclusion (1.1) iff $g(\cdot, u(\lfloor \cdot \rfloor)) \in L^1_{loc}([0,\infty) : X)$,

$$\int_0^t R_\gamma(t-s)\Big[A_0 u\big(\lfloor s\rfloor\big) + g\big(s, u\big(\lfloor s\rfloor\big)\big)\Big]\,ds \in R(C), \quad t > 0 \tag{5.1}$$

and

$$u(t) = S_\gamma(t)C^{-1}u_0 + C^{-1}\int_0^t R_\gamma(t-s)\Big[A_0 u\big(\lfloor s\rfloor\big) + g\big(s, u\big(\lfloor s\rfloor\big)\big)\Big]\,ds, \quad t > 0. \tag{5.2}$$

In the case that $\gamma \in (0,1)$, we can use again the argumentation contained in the proof of [26, Lemma 4.1] to see that any classical solution of (1.1) has to satisfy the equation (4.4), provided that the mapping $g(\cdot, u(\lfloor \cdot \rfloor))$ is Laplace transformable and $u_0 \in R(C)$. In the case that $\gamma = 1$, we can prove the following proposition:

Proposition 5.2. *Suppose that $u_0 \in R(C)$ and $g(\cdot, u(\lfloor \cdot \rfloor)) \in L^1_{loc}([0,\infty) : X)$. If $u(\cdot)$ is a classical solution of (1.1), then $u(\cdot)$ is a mild solution of (1.1).*

Proof. Let $t > 0$ be fixed. Set $v(s) := T(t-s)u(s)$, $s \in [0,t]$. The function $v(\cdot)$ is differentiable for a.e. $s \in [0,t]$ due to the fact that $u(\cdot)$ is a classical solution of (1.1); furthermore, since A generates $(T(t))_{t\geq 0}$, we have

$$\begin{aligned}\frac{dv(s)}{ds} &= -AT(t-s)u(s) + T(t-s)u'(s) = -AT(t-s)u(s) + T(t-s)Au(s)\\ &\quad + T(t-s)A_0 u\big(\lfloor s\rfloor\big) + T(t-s)g\big(s, u\big(\lfloor s\rfloor\big)\big),\end{aligned}$$

for a.e. $s \in [0,t]$; see also [9, p. 1728]. Hence,

$$\frac{dv(s)}{ds} = T(t-s)A_0 u\big(\lfloor s\rfloor\big) + T(t-s)g\big(s, u\big(\lfloor s\rfloor\big)\big), \text{ for a.e. } s \in [0,t]. \tag{5.3}$$

Integrating (5.3) over $[0,t]$, we obtain

$$Cu(t) - T(t)u(0) = \int_0^t T(t-s)\Big[A_0 u\big(\lfloor s\rfloor\big) + g\big(s, u\big(\lfloor s\rfloor\big)\big)\Big]\,ds.$$

Since $u(0) = u_0 \in R(C)$, this immediately implies (5.1)-(5.2), finishing the proof. □

Moreover, the following analogues of Lemma 4.3, Lemma 4.4 and Theorem 4.5 hold good:

Lemma 5.3. *Let $C^{-1}A_0 \in L(X)$, let $p \in \mathbb{N}$, and let $k \in \mathbb{R}$. Define the nonlinear operator Λ_1 on $\mathcal{A}P_{p,k}([0,\infty) : X)$ through*

$$\big(\Lambda_1\varphi\big)(t) := \int_0^t R_\gamma(t-s)C^{-1}A_0\varphi\big(\lfloor s\rfloor\big)\,ds, \quad t \geq 0,$$

for any $\varphi \in \mathcal{A}P_{p,k}([0,\infty) : X)$. Then Λ_1 is well-defined and maps $\mathcal{A}P_{p,k}([0,\infty) : X)$ into itself.

Lemma 5.4. *Let $p \in \mathbb{N}$, let $k \in \mathbb{R}$, and let $C^{-1}g : \mathbb{R} \times X \to X$ be a bounded mapping such that:*

(i) For all $(t,x) \in \mathbb{R} \times X$, we have $C^{-1}g(t+p, e^{ikp}x) = e^{ikp}C^{-1}g(t,x)$;

(ii) There exists $L > 0$ such that for all $(t,x) \in \mathbb{R} \times X$, we have $\|C^{-1}g(t,x) - C^{-1}g(t,y)\| \leq L\|x-y\|$;

(iii) The mapping $t \mapsto C^{-1}g(t, \psi(\lfloor t \rfloor))$, $t \in \mathbb{R}$ is measurable for all $\psi \in \mathcal{P}_{p,k}([0,\infty) : X)$, and the mapping $t \mapsto C^{-1}g(t, \theta(\lfloor t \rfloor))$, $t \geq 0$ is measurable for all $\theta \in \mathcal{AP}_{p,k}([0,\infty) : X)$.

Define

$$\big(\Lambda_2 \varphi\big)(t) := \int_0^t R_\gamma(t-s) C^{-1} g\big(s, \varphi\big(\lfloor s \rfloor\big)\big)\, ds, \quad t \geq 0,$$

for any $\varphi \in \mathcal{AP}_{p,k}([0,\infty) : X)$. Then the nonlinear operator Λ_2 is well-defined and maps $\mathcal{AP}_{p,k}([0,\infty) : X)$ into itself.

Theorem 5.5. *Let $p \in \mathbb{N}$, let $k \in \mathbb{R}$, and let $C^{-1}g : \mathbb{R} \times X \to X$ be a bounded mapping such that the conditions (i)-(iii) of Lemma 5.4 hold. If $C^{-1}A_0 \in L(X)$ and*

$$\big(\|C^{-1}A_0\| + L\big) \int_0^\infty R_\gamma(t)\, dt < 1,$$

then the equation (1.1) has a unique asymptotically Bloch-periodic solution.

Remark 5.6. The boundedness of mapping $C^{-1}g(\cdot,\cdot)$ is taken only for the sake of convenience. In Theorem 5.5, we can assume that this mapping is Stepanov-like bounded, in a certain sense; see e.g. [20, Proposition 2.7.5]. A similar statement holds in the case of consideration of Theorem 4.5.

Remark 5.7. The condition (iii) of Lemma 5.4 automatically holds if $C^{-1}g(\cdot,\cdot)$ is continuous. A similar statement holds in the case of consideration of Lemma 4.4.

The results of this section clearly apply in the analysis of existence and uniqueness of asymptotically Bloch-periodic solutions of nonlinear differential equations with piecewise constant argument in $L^p(\mathbb{R}^n)$ spaces (see e.g. [18] and references cited therein).

6. Conclusions and final remarks

The Weyl-Liouville fractional derivative $D_{t,+}^\gamma u(t)$ of order $\gamma \in (0,1)$ is defined for those continuous functions $u : \mathbb{R} \to X$ such that $t \mapsto \int_{-\infty}^t g_{1-\gamma}(t-s)u(s)\, ds$, $t \in \mathbb{R}$ is a well-defined continuously differentiable mapping, by

$$D_{t,+}^\gamma u(t) := \frac{d}{dt} \int_{-\infty}^t g_{1-\gamma}(t-s)u(s)\, ds, \quad t \in \mathbb{R};$$

see e.g. the paper [22] by J. Mu, Y. Zhoa and L. Peng for further information in this direction. Set $D_{t,+}^1 u(t) := -(d/dt)u(t)$. Using the computation carried out in [22, Lemma 6], it seems very reasonable to define the notion of a mild solution of the following nonlinear fractional Cauchy inclusion with piecewise constant argument

$$D_{t,+}^\gamma u(t) \in -\mathcal{A}u(t) + A_0 u\big(\lfloor t \rfloor\big) + g\big(t, u\big(\lfloor t \rfloor\big)\big),\ t \in \mathbb{R}, \tag{6.1}$$

where $\mathcal{A}$ is an MLO satisfying the condition (P), $A_0 \in L(X)$, $\lfloor \cdot \rfloor$ is the largest integer function, and $g : [0,\infty) \times X \to X$ is a given function, by

$$u(t) := \int_{-\infty}^t R_\gamma(t-s)\Big[A_0 u\big(\lfloor s \rfloor\big) + g\big(s, u\big(\lfloor s \rfloor\big)\big)\Big]\, ds, \quad t \in \mathbb{R}.$$

Therefore, the results of W. Dimbour, V. Valmorin [9] and M. F. Hasler, G. M. N'Guérékata [15] enable one to analyze the existence and uniqueness of Bloch-periodic solutions of nonlinear fractional Cauchy inclusion (6.1), as well.

In [15], M. F. Hasler and G. M. N'Guérékata have investigated the existence and uniqueness of (asymptotically) Bloch-periodic solutions of some semilinear integro-differential equations in Banach spaces. Applications have been given in the study of the following semilinear integro-differential equation

$$u'(t) = Au(t) + \int_{-\infty}^{t} a(t-s)Au(s)\,ds + f(t, Du(t)), \quad t \in \mathbb{R}, \tag{6.2}$$

where A is a closed linear operator satisfying certain conditions, $D \in L(X)$, $a \in L^1_{loc}([0,\infty))$ and $f(\cdot,\cdot)$ obeys some properties. We would like to note that the assertions of [15, Lemma 4.1, Theorem 4.2] hold even if the considered operator families are not strongly continuous at the point $t = 0$. We can simply apply this observation in the analysis of existence and uniqueness of Bloch-periodic solutions of the following Cauchy inclusion

$$u(t) \in \mathcal{A} \int_{-\infty}^{t} a(t-s)u(s)\,ds + f(t),\ t \in \mathbb{R} \tag{6.3}$$

and its semilinear analogue

$$u(t) \in \mathcal{A} \int_{-\infty}^{t} a(t-s)u(s)\,ds + f(t, u(t)),\ t \in \mathbb{R},$$

where $\mathcal{A}$ is an MLO satisfying the condition (P); see the papers by C. Cuevas, C. Lizama [8] and H. R. Henríquez, C. Lizama [17] for more details on the subject.

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