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Perturbed Browder, Weyl theorems and their variations: Equivalences

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To Sara D. Lovegrove on her birthday

Abstract. The presence, or the lack of, SVEP on the holes of the Weyl (resp., a-Weyl) spectrum of a Banach space operator characterizes Browder and generalized Browder (resp., a-Browder and generalized a-Browder) theorems for the operator. The isolated points of the Weyl spectrum (resp., the a-Weyl spectrum, the B-Weyl spectrum and the upper B-Weyl spectrum) play a similar role in determining Weyl's (resp., a-Weyl's, generalized Weyl's and generalized a-Weyl's) theorem for the operator. This paper establishes the role played by the isolated points of these Weyl spectra in establishing equivalences between Browder, Weyl type theorems, their (recently considered) avatars and perturbations by commuting Riesz operators.

1. Introduction

Let $B(\mathcal{X})$ (resp., $B(\mathcal{H})$) denote the algebra of operators, equivalently bounded linear transformations, on a complex infinite dimensional Banach space $\mathcal{X}$ (resp., Hilbert space $\mathcal{H}$) into itself. Given $A \in B(\mathcal{X})$, let $\sigma(A)$, $\sigma_a(A)$, $\sigma_w(A)$, $\sigma_{aw}(A)$ and $\sigma_{ab}(A)$ denote, respectively, the spectrum, the approximate point spectrum, the Weyl spectrum, the approximate Weyl (equivalently, a-Weyl) and approximate Browder (equivalently, a-Browder) spectrum of A; let $\Pi_0(A)$, $\Pi_0^a(A)$, $E_0(A)$ and $E_0^a(A)$ denote, respectively, the set of finite rank poles (of the resolvent) of A, the set of finite rank left poles of A, the set of finite multiplicity eigenvalues which are isolated points of $\sigma(A)$ and the set of finite multiplicity eigenvalues which are isolated points of $\sigma_a(A)$. Recall, [1], that $A \in B(\mathcal{X})$ satisfies *Browder's theorem* (*a-Browder's theorem*) , $A \in (Bt)$ (resp., $A \in (a-Bt)$), if $\sigma(A)\setminus\sigma_w(A) = \Pi_0(A)$ (resp., if $\sigma_a(A)\setminus\sigma_{aw}(A) = \Pi_0^a(A)$), and A satisfies *Weyl's theorem* (*a-Weyl's theorem*), $A \in (Wt)$ (resp.,$A \in (a-Wt)$), if $\sigma(A)\setminus\sigma_w(A) = E_0(A)$ (resp., $\sigma_a(A)\setminus\sigma_{aw}(A) = E_0^a(A)$). Browder and Weyl theorems have been considered in the recent past by a number of authors and there exists in the current literature a large body of information on Browder and Weyl thoerems, their generalized

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extensions and their variations (see [1–4, 7–12, 14–17, 19–21, 31–33] for further references). It is well known that $A \in (Bt)$ if and only if A has SVEP, *the single-valued extension property*, on the complement $\sigma_w(A)^{\mathcal{C}}$ of $\sigma_w(A)$ (in $\mathcal{C}$), $A \in (a-Bt)$ if and only if A has SVEP on the complement $\sigma_{aw}(A)^{\mathcal{C}}$ of $\sigma_{aw}(A)$, $A \in (Wt)$ if and only if $A \in (Bt)$ and $E_0(A) = \Pi_0(A)$ and $A \in (a-Wt)$ if and only if $A \in (a-Bt)$ and $E_0^a(A) = \Pi_0^a(A)$ [17].

If we let $\eta' S$ denote the union of *the holes* (i.e., of the bounded components of the complement $S^{\mathcal{C}}$ of S in $\mathcal{C}$ [24]), then a characterization of operators satisfying (Bt) or (gBt) (resp., $(a-Bt)$ or $(a-gBt)$) is obtained as the set of $A \in B(\mathcal{X})$ which have SVEP on $\eta'\sigma_w(A)$ (resp., which have SVEP on $\eta'\sigma_{aw}(A)$) [18, Theorem 4.1 and Lemma 4.5]. Similarly, a characterization of operators satisfying (Wt) (resp., (gWt)) is obtained as the set of operators $A \in B(\mathcal{X})$ such that $A \in (Bt)$ and $E_0(A) \cap \mathrm{iso}\sigma_{\mathrm{w}}(\mathrm{A}) = \emptyset$ (resp., $A \in (Bt)$ and $E(A) \cap \mathrm{iso}\sigma_{\mathrm{Bw}}(\mathrm{A}) = \emptyset$), and a characterization of operators satisfying $(a-Wt)$ (resp., $(a-gWt)$) is obtained as the set of operators $A \in B(\mathcal{X})$ such that $A \in (a-Bt)$ and $E_0^a(A) \cap \mathrm{iso}\sigma_{\mathrm{aw}}(\mathrm{A}) = \emptyset$ (resp., $A \in (a-Bt)$ and $E^a(A) \cap \mathrm{iso}\sigma_{\mathrm{uBw}}(\mathrm{A}) = \emptyset$). (Here $\sigma_{Bw}(A)$ and $\sigma_{uBw}(A)$ denote, respectively, the B–Weyl and upper B–Weyl spectrum of A.) Isolated points $\mathrm{iso}\sigma_{\mathrm{w}}(\mathrm{A})$, $\mathrm{iso}\sigma_{\mathrm{aw}}(\mathrm{A})$ etc. play a crucial role also in defining relationships between different variations of Weyl type theorems and their perturbations by commuting Riesz operators. Thus, if we let $A \in (b) = \{A \in B(\mathcal{X}) : \sigma_a(A) \cap \mathrm{iso}\sigma_{\mathrm{aw}}(\mathrm{A}) = \Pi_0(\mathrm{A})\}$ [5], $A \in (w) = \{A \in B(\mathcal{X}) : \sigma_a(A) \cap \mathrm{iso}\sigma_{\mathrm{aw}}(\mathrm{A}) = \mathrm{E}_0(\mathrm{A})\}$ [2], $A \in (gb) = \{A \in B(\mathcal{X}) : \sigma_a(A) \cap \mathrm{iso}\sigma_{\mathrm{uBw}}(\mathrm{A}) = \Pi(\mathrm{A})\}$ [11] and $A \in (gw) = \{A \in B(\mathcal{X}) : \sigma_a(A) \cap \mathrm{iso}\sigma_{\mathrm{uBw}}(\mathrm{A}) = \mathrm{E}(\mathrm{A})\}$ [13], then $A \in (w) \iff A \in (b)$ if and only if $E_0(A) \cap \mathrm{iso}\sigma_{\mathrm{aw}}(\mathrm{A}) = \emptyset$ [18, Theorem 5.1] and $A \in (gw) \iff A \in (gb)$ if and only if $E(A) \cap \mathrm{iso}\sigma_{\mathrm{uBw}}(\mathrm{A}) = \emptyset$ [18, Corollary 5.2]. Again, if $R \in B(\mathcal{X})$ is a Riesz operator which commutes with A, then $A \in (w)$ implies $A + R \in (w)$ if and only if $E_0(A+R) \cap \mathrm{iso}\sigma_{\mathrm{aw}}(\mathrm{A}) = \emptyset$ and A^* has SVEP on $\mathrm{iso}\sigma_a(A+R) \cap \sigma_{aw}(A)^{\mathcal{C}}$ [18, Theorem 6.1].

This paper, which continues the work started in [18], further explores the important role played by the isolated points of various Weyl spectra in determining equivalences between Browder, Weyl theorems, their variants and their perturbations by commuting Riesz operators. Using at times what are essentially algebraic arguments, we prove, amongst other results, that if $A \in B(\mathcal{X})$, then:

$$\{A \in (Wt) \iff A \in (gWt)\} \iff E(A) \cap \mathrm{iso}\sigma_{\mathrm{Bw}}(\mathrm{A}) = \emptyset;$$
$$\{A \in (Wt) \iff A \in (a-Wt)\} \iff E_0^a(A) \cap \mathrm{iso}\sigma_{\mathrm{w}}(\mathrm{A}) = \emptyset;$$
$$\{A \in (gWt) \iff A \in (a-gWt)\} \iff E^a(A) \cap \mathrm{iso}\sigma_{\mathrm{Bw}}(\mathrm{A}) = \emptyset;$$
$$\{A \in (a-Wt) \iff A \in (a-gWt)\} \iff E^a(A) \cap \mathrm{iso}\sigma_{\mathrm{uBw}}(\mathrm{A}) = \emptyset;$$
$$\{A \in (w) \iff A \in (a-Wt)\} \iff E_0^a(A) \cap \mathrm{iso}\sigma_{\mathrm{w}}(\mathrm{A}) = \emptyset;$$
$$\{A \in (gw) \iff A \in (a-gWt)\} \iff E^a(A) \cap \mathrm{iso}\sigma_{\mathrm{Bw}}(\mathrm{A}) = \emptyset, \text{ and}$$
$$\{A \in (w) \iff A \in (gw)\} \iff E(A) \cap \mathrm{iso}\sigma_{\mathrm{uBw}}(\mathrm{A}) = \emptyset.$$

Again, if R is a Riesz operator in $B(\mathcal{X})$ which commutes with A, $\mathrm{iso}\sigma_{\mathrm{a}}(\mathrm{A}) = \mathrm{iso}\sigma_{\mathrm{a}}(\mathrm{A}+\mathrm{R})$, and $\Phi_{uBw}^{\mathrm{iso}}(A)$ denotes the set $\{\lambda : \lambda \in \mathrm{iso}\sigma_{\mathrm{aw}}(\mathrm{A}) \cap \sigma_{\mathrm{uBw}}(\mathrm{A})^{\mathcal{C}}\}$, then:

$$\{\sigma(A) \cap \sigma_w(A)^{\mathcal{C}} = E_0^a(A)\} \iff \{\sigma(A+R) \cap \sigma_w(A+R)^{\mathcal{C}} = E_0^a(A+R)\}$$
$$\text{if and only if } E_0^a(A+R) \cap \mathrm{iso}\sigma_{\mathrm{aw}}(\mathrm{A}) = \emptyset;$$
$$\{\sigma(A) \cap \sigma_{Bw}(A)^{\mathcal{C}} = E^a(A)\} \iff \{\sigma(A+R) \cap \sigma_{Bw}(A+R)^{\mathcal{C}} = E^a(A+R)\}$$
$$\text{if and only if } E^a(A+R) \cap \Phi_{uBw}^{\mathrm{iso}}(A) = \emptyset;$$
$$\text{if } \mathrm{iso}\sigma_{\mathrm{aw}}(\mathrm{A}) = \emptyset, \text{ then } \{\mathrm{E}_0(\mathrm{A}) = \Pi_0^{\mathrm{a}}(\mathrm{A})\} \iff \{\mathrm{E}_0(\mathrm{A}+\mathrm{R}) = \Pi_0^{\mathrm{a}}(\mathrm{A}+\mathrm{R})\};$$
$$\text{if } \Phi_{uBw}^{\mathrm{iso}}(A) = \emptyset, \text{ then } \{E(A) = \Pi^a(A)\} \iff \{E(A+R) = \Pi^a(A+R)\};$$
$$\text{if } \mathrm{iso}\sigma_{\mathrm{aw}}(\mathrm{A}) = \emptyset, \text{ then } \{\mathrm{E}_0^{\mathrm{a}}(\mathrm{A}) = \Pi_0(\mathrm{A})\} \iff \{\mathrm{E}_0^{\mathrm{a}}(\mathrm{A}+\mathrm{R}) = \Pi_0(\mathrm{A}+\mathrm{R})\} \text{ and}$$
$$\text{if } \Phi_{uBw}^{\mathrm{iso}}(A) = \emptyset, \text{ then } \{E^a(A) = \Pi(A)\} \iff \{E^a(A+R) = \Pi(A+R)\}.$$

The results of the paper, alongwith proving a large number of new results, subsume a substantial number of extant results. The plan of the paper is as follows. We introduce additional notation and terminology in Section 2, Section 3 consists of some complementary results, Sections 4, 5, 6 and 7 deal with equivalences for

Browder, Weyl type theorems and their variants, and Section 8 considers the preservation of these properties under perturbation by commuting Riesz operators.

2. Notation and terminology

In addition to the notation and terminology already introduced, we shall use the following further notation and terminology. The boundary of a subset S of the set $\mathbb{C}$ of complex numbers will be denoted by ∂S and we shall write $S^{\mathcal{C}}$ for the complement of S in $\mathbb{C}$. We denote the open unit disc by $\mathcal{D}$ and its closure by $\overline{\mathcal{D}}$. An operator $A \in B(\mathcal{X})$ has SVEP, *the single-valued extension property,* at a point $\lambda_0 \in \mathbb{C}$ if for every open disc $\mathcal{D}_{\lambda_0}$ centered at λ_0 the only analytic function $f : \mathcal{D}_{\lambda_0} \longrightarrow \mathcal{X}$ satisfying $(A-\lambda)f(\lambda) = 0$ is the function $f \equiv 0$. (Here, and in the sequel, we have shortened $A - \lambda I$ to $A - \lambda$.) Every $A \in B(\mathcal{X})$ has SVEP at points in the resolvent $\rho(A) = \mathbb{C} \setminus \sigma(A)$ and the boundary $\partial\sigma(A)$ of the spectrum $\sigma(A)$. *We say that T has SVEP if it has SVEP at every $\lambda \in \mathcal{C}$.* The *ascent of* A, asc(A) (resp. *descent of* A, dsc(A)), is the least non-negative integer n such that $A^{-n}(0) = A^{-(n+1)}(0)$ (resp., $A^n(\mathcal{X}) = A^{n+1}(\mathcal{X})$): If no such integer exists, then asc(A) (resp. dsc(A))$= \infty$. It is well known that asc$(A) < \infty$ implies A has SVEP at 0, dsc$(A) < \infty$ implies A^* (= the dual operator) has SVEP at 0, finite ascent and descent for an operator implies their equality, and that a point $\lambda \in \sigma(A)$ is a pole (of the resolvent) of A if and only if asc$(A-\lambda) =$ dsc$(A-\lambda) < \infty$ [1, 23, 25].

$A \in B(\mathcal{X})$ is: *upper semi–Fredholm* at $\lambda \in \mathbb{C}$, $\lambda \in \Phi_{uf}(A)$ (or, $A - \lambda \in \Phi_+(\mathcal{X})$), if $(A-\lambda)(\mathcal{X})$ is closed and the deficiency index $\alpha(A-\lambda) = \dim(\mathrm{A}-\lambda)^{-1}(0) < \infty$; *lower semi–Fredholm at* $\lambda \in \mathcal{C}$, $\lambda \in \Phi_{lf}(A)$ (or, $A - \lambda \in \Phi_-(\mathcal{X})$) , if $\beta(A-\lambda) = \dim(\mathcal{X}/(\mathrm{A}-\lambda)(\mathcal{X})) < \infty$; $A - \lambda$ is semi–Fredholm, $A - \lambda \in \Phi_\pm(\mathcal{X})$, if $A - \lambda$ is either upper or lower semi–Fredholm, and A is Fredholm at $\lambda \in \mathbb{C}$, $\lambda \in \Phi(A)$ or $A - \lambda \in \Phi(\mathcal{X})$, if $A - \lambda$ is both upper and lower semi–Fredholm. The index of a semi–Fredholm operator is the integer $\mathrm{ind}(A) = \alpha(A) - \beta(A)$. Corresponding to these classes of one sided Fredholm operators, we have the following spectra: The *upper semi Fredholm spectrum* $\sigma_{uf}(A) = \{\lambda \in \sigma(A) : A - \lambda \notin \Phi_+(\mathcal{X})\}$, the *lower semi Fredholm spectrum* $\sigma_{lf}(A) = \{\lambda \in \sigma(A) : A - \lambda \notin \Phi_-(\mathcal{X})\}$ and the *Fredholm spectrum* $\sigma_e(A) = \sigma_{uf}(A) \cup \sigma_{lf}(A)$. $A \in B(\mathcal{X})$ is upper Weyl (resp., lower Weyl, (simply) Weyl) at 0 if it is upper semi Fredholm with $\mathrm{ind}(A) \leq 0$ (resp., lower semi Fredholm with $\mathrm{ind}(A) \geq 0$, Fredholm with $\mathrm{ind}(A) = 0$). The upper (or, approximate) Weyl spectrum, the lower (or, surjectivity) Weyl spectrum and the Weyl spectrum of A are respectively the sets $\sigma_{aw}(A) = \{\lambda \in \sigma_a(A) : \lambda \notin \Phi_+(A) \text{ or } \mathrm{ind}(A-\lambda) \not\leq 0\}$, $\sigma_{sw}(A) = \{\lambda \in \sigma_s(A) : \lambda \notin \Phi_-(A) \text{ or } \mathrm{ind}(A-\lambda) \not\geq 0\}$ and $\sigma_w(A) = \sigma_{aw}(A) \cup \sigma_{sw}(A)$. It is well known, [1, Theorems 3.16, 3.17], that a semi- Fredholm operator A (resp., its conjugate operator A^*) has SVEP at a point λ if and only if asc$(A-\lambda) < \infty$ (resp., dsc$(A-\lambda) < \infty$) ; furthermore, if $A - \lambda$ is Weyl (resp., upper Weyl), i.e. if $\lambda \in \Phi(A)$ and $\mathrm{ind}(A-\lambda) = 0$ (resp., $\lambda \in \Phi_+(A)$ and $\mathrm{ind}(A-\lambda) \leq 0$), then A has SVEP at λ implies $\lambda \in \mathrm{iso}\sigma(\mathrm{A})$ with asc$(A-\lambda) =$ dsc$(A-\lambda) < \infty$ (resp., $\lambda \in \mathrm{iso}\sigma_{\mathrm{a}}(\mathrm{A})$ with asc$(A-\lambda) < \infty$). If we let $\sigma_{ab}(A) = \{\lambda \in \sigma_a(A) : \lambda \notin \Phi_+(A) \text{ or } \mathrm{asc}(\mathrm{A}-\lambda) \not< \infty\}$ and $\sigma_{sb}(A) = \{\lambda \in \sigma_s(A) : \lambda \notin \Phi_-(A) \text{ or } \mathrm{des}(\mathrm{A}-\lambda) \not< \infty\}$ denote, respectively, the *upper (or approximate) and the lower (or surjectivity) Browder spectrum* of A, then $\sigma_{sb}(A) = \sigma_{ab}(A^*)$ and $\sigma_b(A) = \sigma_{ab}(A) \cup \sigma_{sb}(A)$ is the *Browder spectrum* of A. (For further information on Fredholm theory, SVEP, and isolated points etc, see [1, 23–25, 30].)

A generalization of Fredholm and Weyl spectrum is obtained as follows. An operator $A \in B(\mathcal{X})$ *is semi B-Fredholm* if there exists an integer $n \geq 1$ such that $A^n(\mathcal{X})$ is closed and the induced operator $A_{[n]} = A|_{A^n(\mathcal{X})}$, $A_{[0]} = A$, is semi Fredholm (in the usual sense). It is seen that if $A_{[n]} \in \Phi_\pm(\mathcal{X})$ for an integer $n \geq 1$, then $A_{[m]} \in \Phi_\pm(\mathcal{X})$ for all integers $m \geq n$, and one may unambiguously define the index of A by $\mathrm{ind}(A) = \alpha(A) - \beta(A)$ (= $\mathrm{ind}(A_{[n]})$) [10]. Upper semi B-Fredholm, lower semi B-Fredholm and B-Fredholm spectra of A are then the sets

$\sigma_{uBf}(A) = \{\lambda \in \sigma(A) : A - \lambda$ is not upper semi B-Fredholm$\}$,

$\sigma_{lBf}(A) = \{\lambda \in \sigma(A) : A - \lambda$ is not lower semi B-Fredholm$\}$, and

$\sigma_{Be}(A) = \sigma_{uBf}(A) \cup \sigma_{lBf}(A)$.

Letting

$$\begin{aligned}
&\sigma_{Bw}(A) = \{\lambda \in \sigma(A) : \lambda \in \sigma_{Be}(A) \text{ or } \operatorname{ind}(A-\lambda) \neq 0\},\\
&\sigma_{aBw}(A) = \{\lambda \in \sigma_a(A) : \lambda \in \sigma_{uBf}(A) \text{ or } \operatorname{ind}(A-\lambda) \not\leq 0\},\\
&\sigma_{sBw}(A) = \{\lambda \in \sigma_s(A) : \lambda \in \sigma_{lBf}(A) \text{ or } \operatorname{ind}(A-\lambda) \not\geq 0\},\\
&\sigma_{Bb}(A) = \{\lambda \in \sigma(A) : \lambda \in \sigma_{Be}(A) \text{ or } \operatorname{asc}(A-\lambda) \neq \operatorname{dsc}(A-\lambda)\},\\
&\sigma_{aBb}(A) = \{\lambda \in \sigma_a(A) : \lambda \in \sigma_{uBf}(A) \text{ or } \operatorname{asc}(A-\lambda) = \infty\}, \text{ and}\\
&\sigma_{sBb}(A) = \{\lambda \in \sigma_s(A) : \lambda \in \sigma_{lBf}(A) \text{ or } \operatorname{dsc}(A-\lambda) = \infty\}
\end{aligned}$$

denote, respectively, the *the B-Weyl, the upper B-Weyl, the lower B-Weyl, the B-Browder, the upper B-Browder* and *the lower B-Browder spectrum* of A, we have $\sigma_{Bw}(A) = \sigma_{aBw}(A) \cup \sigma_{sBw}(A)$, $\sigma_{Bb}(A) = \sigma_{aBb}(A) \cup \sigma_{sBb}(A)$, $\sigma_{aBw}(A) = \sigma_{sBw}(A^*)$ and $\sigma_{aBb}(A) = \sigma_{sBb}(A^*)$.

3. Some complementary results.

The following implications are well known [8, Theorems 2.1 and 2.2]:

$\sigma_w(A) = \sigma_b(A) \Longleftrightarrow \sigma_{Bw}(A) = \sigma_{Bb}(A) \Longleftrightarrow \sigma(A) \setminus \sigma_{Bw}(A) = \Pi(A) \Longleftrightarrow A$ has SVEP at points in $\sigma(A) \setminus \sigma_{Bw}(A)$, and

$\sigma_{aw}(A) = \sigma_{ab}(A) \Longleftrightarrow \sigma_{aBw}(A) = \sigma_{aBb}(A) \Longleftrightarrow \sigma_a(A) \setminus \sigma_{aBw}(A) = \Pi^a(A) \Longleftrightarrow A$ has SVEP at points in $\sigma_a(A) \setminus \sigma_{aBw}(A)$.

Evidently, $\sigma_{aw}(A) \subseteq \sigma_w(A)$ and $\sigma_{aBw}(A) \subseteq \sigma_{Bw}(A)$; hence

$\sigma_{aBw}(A) = \sigma_{aBb}(A) \Longleftrightarrow \sigma_{aw}(A) = \sigma_{ab}(A) \Longrightarrow \sigma_w(A) = \sigma_b(A) \Longleftrightarrow \sigma_{Bw}(A) = \sigma_{Bb}(A)$

(where the one way implications are strict). Following current terminology [1, 8, 10, 16], we say that an operator $A \in B(\mathcal{X})$ satisfies

Browder's theorem , $A \in (Bt)$, if $\sigma_w(A) = \sigma_b(A)$, equivalently $\sigma(A) \cap \sigma_w(A)^{\mathcal{C}} = \Pi_0(A)$;

generalized Browder's theorem, $A \in (gBt)$, if $\sigma_{Bw}(A) = \sigma_{Bb}(A)$, equivalently $\sigma(A) \cap \sigma_{Bw}(A)^{\mathcal{C}} = \Pi(A)$;

a-Browder's theorem, $A \in (a - Bt)$, if $\sigma_{aw}(A) = \sigma_{ab}(A)$, equivalently $\sigma_a(A) \cap \sigma_{aw}(A)^{\mathcal{C}} = \Pi_0^a(A)$;

generalized a-Browder's theorem, or $A \in (a - gBt)$, if $\sigma_{aBw}(A) = \sigma_{aBb}(A)$, equivalently $\sigma_a(A) \cap \sigma_{uBw}(A) = \Pi^a(A)$.

Let $E(A) = \{\lambda \in \mathrm{iso}\sigma(\mathrm{A}) : 0 < \alpha(\mathrm{A}-\lambda)\}$ and $E^a(A) = \{\lambda \in \mathrm{iso}\sigma_{\mathrm{a}}(\mathrm{A}) : 0 < \alpha(\mathrm{A}-\lambda)\}$. We say that the operator $A \in B(\mathcal{X})$ satisfies:

Weyl's theorem , $A \in (Wt)$, if $\sigma(A) \cap \sigma_w(A)^{\mathcal{C}} = E_0(A)$;
generalized Weyl's theorem, $A \in (gWt)$, if $\sigma(A) \cap \sigma_{Bw}(A)^{\mathcal{C}} = E(A)$;
a-Weyl's theorem, $A \in (a - Wt)$, if $\sigma_a(A) \cap \sigma_{aw}(A)^{\mathcal{C}} = E_0^a(A)$;
generalized a-Weyl's theorem, $A \in (a - gWt)$, if $\sigma_a(A) \cap \sigma_{uBw}(A) = E^a(A)$.

The following implications

$$(a - gWt) \Longrightarrow (gWt) \Longrightarrow (Wt), \quad (a - gWt) \Longrightarrow (a - Wt) \Longrightarrow (Wt)$$

hold, but the reverse implications are in general false [1, 10, 15–17]. It is evident that $(Wt) \Longrightarrow (Bt)$, $(a - Wt) \Longrightarrow (a - Bt)$, $(gWt) \Longrightarrow (gBt)$ and $(a - gWt) \Longrightarrow (a - gBt)$. Also, since $\Pi_0^x(A) \subseteq E_0^x(A)$ and $\Pi^x(A) \subseteq E^x(A)$, where $\Pi^x = \Pi$ or Π^a and correspondingly $E^x = E$ or E^a, a necessary and sufficient condition for an $A \in (Bt)$ to satisfy $A \in (Wt)$ is that $E_0(A) \subseteq \Pi_0(A)$ (resp., $A \in (gBt)$ to satisfy $A \in (gWt)$ is that $E(A) \subseteq \Pi(A)$, $A \in (a - Bt)$ to satisfy $A \in (a - Wt)$ is that $E_0^a(A) \subseteq \Pi_0^a(A)$ and $A \in (a - gBt)$ to satisfy $A \in a - (gWt)$ is that $E^a(A) \subseteq \Pi^a(A)$). Since

$$E_0^a(A) = \{E_0^a(A) \cap \sigma_{aw}(A)^{\mathcal{C}}\} \cup \{E_0^a(A) \cap \sigma_{aw}(A)\}$$

and since $E_0^a(A) \cap \sigma_{aw}(A)^{\mathcal{C}} \subseteq \Pi_0^a(A)$, a sufficient condition for $E_0^a(A) \subseteq \Pi_0^a(A)$ is

$$E_0^a(A) \cap \sigma_{aw}(A) = E_0^a(A) \cap \mathrm{iso}\sigma_{aw}(A) = \emptyset.$$

Similarly, a sufficient condition for $E_0(A) \subseteq \Pi_0(A)$ is $E_0(A) \cap \mathrm{iso}\sigma_w(A) = \emptyset$. These conditions are necessary too [18, Theorem 4.8].

$A \in B(\mathcal{X})$ is *polaroid* if $\lambda \in \mathrm{iso}\sigma(\mathrm{A})$ implies $\mathrm{asc}(A-\lambda) = \mathrm{dsc}(A-\lambda)$ (i.e., if the isolated points of the spectrum of A are poles of the resolvent of A), A is *finitely polaroid* if points $\lambda \in \mathrm{iso}\sigma(\mathrm{A})$ are finite rank poles of the resolvent of A, A is *left polaroid* if $\lambda \in \mathrm{iso}\sigma_{\mathrm{a}}(\mathrm{A})$ implies $\mathrm{asc}(A-\lambda) = d < \infty$ for some integer $d > 0$ and $(A-\lambda)^{d+1}(\mathcal{X})$ is closed (i.e., if the isolated points of the approximate point spectrum of A are left poles of the resolvent of A), A is *finitely left polaroid* if points $\lambda \in \mathrm{iso}\sigma_{\mathrm{a}}(\mathrm{A})$ are finite rank left poles of A, and A is *a-polaroid* if points $\lambda \in \mathrm{iso}\sigma_{\mathrm{a}}(\mathrm{A})$ are poles of A . Given $A \in B(\mathcal{X})$, it is clear that a-polaroid operators are polaroid; furthermore

$$\Pi_0(A) \subseteq \Pi_0^a(A) \subseteq \Pi^a(A), \;\; \Pi_0(A) \subseteq \Pi(A) \subseteq \Pi^a(A),$$

where the reverse inclusions generally fail (see [1, 23, 25, 30**?**]). $A \in B(\mathcal{X})$ is *isoloid* (*finitely isoloid*) if points $\lambda \in \mathrm{iso}\sigma(A)$ are eigenvalues (resp., finite multiplicity eigenevalues) of A; A is *a-isoloid* (*finitely a-isoloid*) if points $\lambda \in \mathrm{iso}\sigma_a(A)$ are eigenvalues (resp., finite multiplicity eigenvalues) of A. It is clear that A is polaroid implies A is isoloid and A is left polaroid implies A is a-isoloid (where the reverse implications are, in general, false). Recall from [6] that *perturbation by commuting Riesz operators preserves SVEP at points.* The left polaroid and polaroid properties do not survive perturbation by commuting Riesz operators: The 0 operator is polaroid but its perturbation $A = 0 + R$ by the non-nilpotent quasinilpotent operator $R(x_1, x_2, x_3, ...) = (\frac{x_2}{2}, \frac{x_3}{3}, ...)$ is neither left polaroid nor polaroid. However:

Proposition 3.1. *If a Riesz operator $R \in B(\mathcal{X})$ is such that $[A, R] = 0$ and $iso\sigma_a(A+R) = iso\sigma_a(A)$ for an operator $A \in B(\mathcal{X})$, then $\Pi_0^a(A+R) = \Pi_0^a(A)$ and $\Pi_0(A+R) = \Pi_0(A)$.*

Proof. We have:

$$\begin{aligned} \Pi_0^a(A+R) &= \{\lambda \in \mathrm{iso}\sigma_a(A+R) : \lambda \in \sigma_{aw}(A+R)^{\mathcal{C}}\} \\ &= \{\lambda \in \mathrm{iso}\sigma_a(A) : \lambda \in \sigma_{aw}(A)^{\mathcal{C}}\} = \Pi_0^a(A) \end{aligned}$$

and

$$\begin{aligned} \Pi_0(A+R) &= \{\lambda \in \mathrm{iso}\sigma(A+R) : \lambda \in \sigma_w(A+R)^{\mathcal{C}}\} \\ &= \{\lambda \in \mathrm{iso}\sigma_a(A+R) : \lambda \in \sigma_{aw}(A+R)^{\mathcal{C}}, (A+R)^* \text{ has SVEP at } \lambda\} \\ &= \{\lambda \in \mathrm{iso}\sigma_a(A) : \lambda \in \sigma_{aw}(A)^{\mathcal{C}}, \;\; A^* \text{ has SVEP at } \lambda\} \\ &= \{\lambda \in \sigma(A) : \lambda \in \sigma_w(A)^{\mathcal{C}}\} = \Pi_0(A). \end{aligned}$$

This completes the proof. □

Proposition 3.1 is an improved version of [18, Proposition 3.1].

The hypothesis A is finitely left polaroid (resp., finitely polaroid) implies $\mathrm{iso}\sigma_{\mathrm{a}}(\mathrm{A}) \cap \mathrm{iso}\sigma_{\mathrm{aw}}(\mathrm{A}) = \emptyset$ (resp., $\mathrm{iso}\sigma(\mathrm{A}) \cap \mathrm{iso}\sigma_{\mathrm{w}}(\mathrm{A}) = \emptyset$); hence, if A is finitely left polaroid (resp., finitely polaroid), then $\Pi_0^a(A) \setminus \Pi_0^a(A+R)$ is contained in the resolvent $\rho_a(A+R)$ (resp., $\Pi_0(A) \setminus \Pi_0(A+R)$ is contained in the resolvent $\rho(A+R)$) of $A+R$ (for every Riesz operator R commuting with A). A sufficient condition for A and $A+R$, R a Riesz operator commuting with A, to be finitely left polaroid (resp., finitely polaroid) is that $\mathrm{iso}\sigma_{\mathrm{aw}}(\mathrm{A}) = \emptyset$ (resp., $\mathrm{iso}\sigma_{\mathrm{w}}(\mathrm{A}) = \emptyset$). Indeed a stronger result is possible in the case in which $\mathrm{iso}\sigma_{\mathrm{aw}}(\mathrm{A}) = \emptyset$.

Since $\lambda \in \sigma_{aw}(A) \setminus \sigma_{uBw}(A)$ if and only if $\lambda \in \sigma_{aw}(A)$ and $A-\lambda$ is upper semi B-Fredhom with $\mathrm{ind}(A-\lambda) \leq 0$, there exists an $\epsilon > 0$ such that $A-\mu$ is upper semi-Fredholm of $ind(A-\mu) \leq 0$ for all $0 < |\mu - \lambda| < \epsilon$ [22], i.e., $\mu \in \sigma_{aw}(A)^{\mathcal{C}}$ for all $0 < |\mu - \lambda| < \epsilon$. If we now let $\Phi_{uBw}^{\mathrm{iso}}(A) = \{\lambda \in \mathrm{iso}\sigma_{\mathrm{aw}}(\mathrm{A}) : \lambda \notin \sigma_{\mathrm{uBw}}(\mathrm{A})\}$, then $\sigma_{aw} \setminus \sigma_{uBw}(A) = \Phi_{uBw}^{\mathrm{iso}}(A)$, equivalently

$$\sigma_{aw}(A) = \sigma_{uBw}(A) \cup \Phi_{uBw}^{\mathrm{iso}}(A).$$

(Berkani and Zariouh, [13], have observed that $\sigma_{aw}(A) \setminus \sigma_{uBw}(A) \subseteq \mathrm{iso}\sigma_{\mathrm{aw}}(\mathrm{A})$, equivalently $\sigma_{aw}(A) \subseteq \sigma_{uBw}(A) \cup \mathrm{iso}\sigma_{\mathrm{aw}}(\mathrm{A})$. We note here that the inclusion may be proper, as follows from a consideration of the operator $0 \oplus R$ of the example above.) A similar argument proves that

$$\sigma_w(A) = \sigma_{Bw}(A) \cup \Phi^{\mathrm{iso}}_{Bw}(A),$$

where $\Phi^{\mathrm{iso}}_{Bw}(A) = \{\lambda \in \mathrm{iso}\sigma_{\mathrm{w}}(\mathrm{A}) : \lambda \notin \sigma_{\mathrm{Bw}}(\mathrm{A})\}$.

Proposition 3.2. *If $A, R \in B(\mathcal{X})$, where R is Riesz, $[A, R] = 0$ and $\mathrm{iso}\sigma_{\mathrm{a}}(\mathrm{A}) = \mathrm{iso}\sigma_{\mathrm{a}}(\mathrm{A}+\mathrm{R})$, then a necessary and sufficient condition for $\Pi^a(A+R) = \Pi^a(A)$, and a sufficient condition for $\Pi(A+R) = \Pi(A)$, is that $\Phi^{\mathrm{iso}}_{uBw}(A) = \Phi^{\mathrm{iso}}_{uBw}(A+R)$.*

Proof. Sufficiency. If $\Phi^{\mathrm{iso}}_{uBw}(A) = \Phi^{\mathrm{iso}}_{uBw}(A+R)$, then

$$\begin{aligned}
\Pi^a(A) &= \{\lambda \in \mathrm{iso}\sigma_{\mathrm{a}}(\mathrm{A}) : \lambda \in \Phi_{\mathrm{uBw}}(\mathrm{A})\} \\
&= \{\lambda \in \mathrm{iso}\sigma_{\mathrm{a}}(\mathrm{A}) : \lambda \in \sigma_{\mathrm{aw}}(\mathrm{A})^{\mathcal{C}} \cup \Phi^{\mathrm{iso}}_{\mathrm{uBw}}(\mathrm{A}), \mathrm{A} \text{ has SVEP at } \lambda\} \\
&= \{\lambda \in \mathrm{iso}\sigma_{\mathrm{a}}(\mathrm{A}) : \lambda \in \sigma_{\mathrm{aw}}(\mathrm{A}+\mathrm{R})^{\mathcal{C}} \cup \Phi^{\mathrm{iso}}_{\mathrm{uBw}}(\mathrm{A}), \mathrm{A} \text{ has SVEP at } \lambda\} \\
&= \{\lambda \in \mathrm{iso}\sigma_{\mathrm{a}}(\mathrm{A}+\mathrm{R}) : \lambda \in \sigma_{\mathrm{aw}}(\mathrm{A}+\mathrm{R})^{\mathcal{C}} \cup \Phi^{\mathrm{iso}}_{\mathrm{uBw}}(\mathrm{A}+\mathrm{R}), \mathrm{A}+\mathrm{R} \text{ has SVEP at } \lambda\} \\
&= \{\lambda \in \mathrm{iso}\sigma_{\mathrm{a}}(\mathrm{A}+\mathrm{R}) : \lambda \in \sigma_{\mathrm{uBw}}(\mathrm{A}+\mathrm{R})^{\mathcal{C}}\} \subseteq \Pi^{\mathrm{a}}(\mathrm{A}+\mathrm{R}),
\end{aligned}$$

and (arguing similarly)

$$\begin{aligned}
\Pi^a(A+R) &= \{\lambda \in \mathrm{iso}\sigma_{\mathrm{a}}(\mathrm{A}+\mathrm{R}) : \lambda \in \Phi_{\mathrm{uBw}}(\mathrm{A}+\mathrm{R})\} \\
&= \{\lambda \in \sigma_a(A) : \lambda \in \sigma_{aw}(A)^{\mathcal{C}} \cup \Phi^{\mathrm{iso}}_{uBw}(A), A \text{ has SVEP at } \lambda\} \\
&= \{\lambda \in \mathrm{iso}\sigma_{\mathrm{a}}(\mathrm{A}) : \lambda \in \Phi_{\mathrm{uBw}}(\mathrm{A})\} \subseteq \Pi^{\mathrm{a}}(\mathrm{A}).
\end{aligned}$$

Thus the condition is sufficient for $\Pi^a(A+R) = \Pi^a(A)$. The proof of the sufficiency for $\Pi(A+R) = \Pi(A)$ follows from the following argument:

$$\begin{aligned}
\Pi(A) &= \{\lambda \in \mathrm{iso}\sigma(\mathrm{A}) : \lambda \in \sigma_{\mathrm{Bw}}(\mathrm{A})^{\mathcal{C}}\} \\
&\subseteq \{\lambda \in \mathrm{iso}\sigma(\mathrm{A}) : \lambda \in \Phi_{\mathrm{uBw}}(\mathrm{A}), \mathrm{A} \text{ and } \mathrm{A}^* \text{ have SVEP at } \lambda\} \\
&= \{\lambda \in \mathrm{iso}\sigma_{\mathrm{a}}(\mathrm{A}) : \lambda \in \sigma_{\mathrm{aw}}(\mathrm{A})^{\mathcal{C}} \cup \Phi^{\mathrm{iso}}_{\mathrm{uBw}}(\mathrm{A}), \mathrm{A} \text{ and } \mathrm{A}^* \text{ have SVEP at } \lambda\} \\
&= \{\lambda \in \mathrm{iso}\sigma_{\mathrm{a}}(\mathrm{A}+\mathrm{R}) : \lambda \in \sigma_{\mathrm{aw}}(\mathrm{A}+\mathrm{R})^{\mathcal{C}} \cup \Phi^{\mathrm{iso}}_{\mathrm{uBw}}(\mathrm{A}+\mathrm{R}), \mathrm{A}+\mathrm{R} \text{ and } (\mathrm{A}+\mathrm{R})^* \\
&\qquad \text{have SVEP at } \lambda\} \\
&= \{\lambda \in \mathrm{iso}\sigma(\mathrm{A}+\mathrm{R}) : \lambda \in \Phi_{\mathrm{uBw}}(\mathrm{A}+\mathrm{R}), \mathrm{A}+\mathrm{R} \text{ and } (\mathrm{A}+\mathrm{R})^* \text{ have SVEP at } \lambda\} \\
&= \{\lambda \in \mathrm{iso}\sigma_{\mathrm{a}}(\mathrm{A}+\mathrm{R}) : \lambda \in \Phi_{\mathrm{Bw}}(\mathrm{A}+\mathrm{R})\} \subseteq \Pi(\mathrm{A}+\mathrm{R});
\end{aligned}$$

arguing similarly, $\Pi(A+R) \subseteq \Pi(A)$ (and the proof of the sufficiency is complete).
Necessity. Given $\mathrm{iso}\sigma_{\mathrm{a}}(\mathrm{A}+\mathrm{R}) = \mathrm{iso}\sigma_{\mathrm{a}}(\mathrm{A})$, since $\Pi^a(A) = \Pi^a(A+R)$ if and only if

$$\{\lambda : \lambda \in \Phi_{uBw}(A) \setminus \Phi_{uBw}(A+R), A \text{ (hence also } A+R) \text{ has SVEP at } \lambda\} = \emptyset,$$

we must have

$$\begin{aligned}
\emptyset &= \{\sigma_{aw}(A)^{\mathcal{C}} \cup \Phi^{\mathrm{iso}}_{uBw}(A)\} \cap \{\sigma_{aw}(A+R) \cap \Phi^{\mathrm{iso}}_{uBw}(A+R)^{\mathcal{C}}\} \\
&= \{\sigma_{aw}(A)^{\mathcal{C}} \cup \Phi^{\mathrm{iso}}_{uBw}(A)\} \cap \{\sigma_{aw}(A) \cap \Phi^{\mathrm{iso}}_{uBw}(A+R)^{\mathcal{C}}\} \\
&= \{\Phi^{\mathrm{iso}}_{uBw}(A) \cap \sigma_{aw}(A)\} \cap \Phi^{\mathrm{iso}}_{uBw}(A+R)^{\mathcal{C}}\} \\
&= \Phi^{\mathrm{iso}}_{uBw}(A) \setminus \Phi^{\mathrm{iso}}_{uBw}(A+R),
\end{aligned}$$

i.e., $\Phi^{\mathrm{iso}}_{uBw}(A) = \Phi^{\mathrm{iso}}_{uBw}(A+R)$. $\square$

Recall from [18, Proposition 3.3] that the polaroid and the left polaroid properties for an operator $A \in B(\mathcal{X})$ survive perturbation by commuting finite rank perturbations $F \in B(\mathcal{X})$ such that $\mathrm{iso}\sigma_a(A+F) = \mathrm{iso}\sigma_a(A)$. The following proposition says a bit more.

Proposition 3.3. *Let $A, F \in B(\mathcal{X})$, where $[A, F] = 0$. If F^n is finite rank for some integer $n > 0$ and $iso\sigma_a(A) = iso\sigma_a(A+F)$, then $\Pi^a(A+F) = \Pi^a(A)$ (resp., $\Pi(A+F) = \Pi(A)$).*

Proof. Recall from [9] that the semi B-Fredholm spectrum of an operator is stable under finite rank perturbations. Since $\sigma_a(A) = \sigma_a(A+R)$ and

$$\begin{aligned} & \lambda \in \Pi^a(A+F) \Longleftrightarrow \lambda \in \mathrm{iso}\sigma_a(A+F) \cap \sigma_{uBw}(A+F)^{\mathcal{C}} \\ \Longleftrightarrow\quad & \lambda \in \mathrm{iso}\sigma_a(A+F) \cap \sigma_{uBe}(A+F)^{\mathcal{C}} \Longleftrightarrow \lambda \in \mathrm{iso}\sigma_a(A) \cap \sigma_{uBe}(A)^{\mathcal{C}} \\ \Longleftrightarrow\quad & \lambda \in \mathrm{iso}\sigma_a(A) \cap \sigma_{uBw}(A)^{\mathcal{C}} \Longleftrightarrow \lambda \in \Pi^a(a), \end{aligned}$$

$\Pi^a(A+F) = \Pi^a(A)$. Since $\mathrm{iso}\sigma_a(A) = \mathrm{iso}\sigma_a(A+F)$ implies $\sigma(A+F) = \sigma(A)$, a similar argument proves $\Pi(A+F) = \Pi(A)$. $\square$

The proposition implies in particular that *if $N \in B(\mathcal{X})$ is a nilpotent operator which commutes with $A \in B(\mathcal{X})$, then $\Pi^a(A+N) = \Pi^a(A)$ and $\Pi(A+N) = \Pi(A)$.* We observe here that the Proposition 3.3 does not extend to commuting quasinilpotents [13].

4. Variations on Browder's Theorem: Equivalences

The a-Browder and a-Weyl theorems are obtained from (their classical counterparts) Browder and Weyl theorems $\sigma(A) \cap \sigma_w(A)^{\mathcal{C}} = \Pi_0(A)$ and $\sigma(A) \cap \sigma_w(A)^{\mathcal{C}} = E_0(A)$ by replacing $\sigma(A)$ by $\sigma_a(A)$, $\sigma_w(A)$ by $\sigma_{aw}(A)$, $\Pi_0(A)$ by $\Pi_0^a(A)$ and $E_0(A)$ by $E_0^a(A)$; similarly, the generalized versions of the Browder and Weyl theorems (resp., the a-generalized versions of the Browder and Weyl theorems) are obtained upon replacing $\sigma_w(A)$, $\Pi_0(A)$ and $E_0(A)$ by $\sigma_{Bw}(A)$, $\Pi(A)$ and $E(A)$ (resp., $\sigma_{aw}(A)$, $\Pi_0^a(A)$ and $E_0^a(A)$ by $\sigma_{uBw}(A)$, $\Pi^a(A)$ and $E^a(A)$). A number of further variations, obtained by making other suitably meaningful choices, have been considered in the recent past (see [2, 3, 5, 7, 11, 12, 14, 29] for a flavour of the type of variations considered). Prominent amongst the variations to Browder type theorems that have attracted some attention are the properties (*b*), (*ab*), (*gb*) and (*gab*). We say that an operator $A \in B(\mathcal{X})$ satisfies property:

(*b*) if $\sigma_a(A) \cap \sigma_{aw}(A)^{\mathcal{C}} = \Pi_0(A)$, equivalently $A \in (a - Bt), \Pi_0^a(A) = \Pi_0(A)$;

(*gb*) if $\sigma_a(A) \cap \sigma_{aBw}(A)^{\mathcal{C}} = \Pi(A)$, equivalently $A \in (a - gBt), \Pi^a(A) = \Pi(A)$;

(*ab*) if $\sigma(A) \cap \sigma_w(A)^{\mathcal{C}} = \Pi_0^a(A)$, equivalently $A \in (Bt), \Pi_0(A) = \Pi_0^a(A)$;

(*gab*) if $\sigma(A) \cap \sigma_{Bw}(A)^{\mathcal{C}} = \Pi^a(A)$, equivalently $A \in (gBt), \Pi(A) = \Pi^a(A)$.

It is clear from the definitions above that

$$\begin{aligned} & (gb) \Longrightarrow (b) \Longrightarrow (ab), \;\; (gb) \Longrightarrow (gab) \Longrightarrow (ab) \\ & \{(b) \Longrightarrow (gb)\} \Longleftrightarrow \{\Pi_0^a(A) = \Pi_0(A)\}, \{(ab) \Longrightarrow (gab)\} \Longleftrightarrow \{\Pi^a(A) = \Pi(A)\}. \end{aligned}$$

The operator $A = U \oplus 0 \in B(\mathcal{H} \oplus \mathcal{H})$, where U is the forward unilateral shift, satisfies $A \in (ab) \wedge (b)$ and $A \notin (gab) \vee (gb)$ (for the reason that $\sigma(A) = \sigma_w(A) = \overline{\mathcal{D}}$, $\sigma_a(A) = \sigma_{aw}(A) = \partial\mathcal{D} \cup \{0\}$, $\Pi_0(A) = \Pi_0^a(A) = \emptyset = \Pi(A)$ and $\Pi^a(A) = \{0\}$). Observe that A^* does not have SVEP on $\sigma_{aw}(A)^{\mathcal{C}} \cap \sigma_w(A) = \{0\}$.

For an operator $A \in B(\mathcal{X})$, *let $\Pi_\infty(A)$ (resp., $\Pi_\infty^a(A)$) denote the set $\Pi_\infty(A) = \Pi(A) \setminus \Pi_0(A)$ of infinite rank poles (resp., the set $\Pi_\infty^a(A) = \Pi^a(A) \setminus \Pi_0^a(A)$ of infinite rank left poles) of A; let $E_\infty(A)$ (resp., $E_\infty^a(A)$) denote the set $E_\infty(A) = E(A) \setminus E_0(A)$ of infinite multiplicity eigenvalues of A which are isolated points of $\sigma(A)$ (resp., denote the set $E_\infty^a(A) = E^a(A) \setminus E_0^a(A)$ of infinite multiplicity eigenvalues of A which are isolated points of $\sigma_a(A)$).*

Proposition 4.1. *Given an operator $A \in B(\mathcal{X})$,*

(i) $\{A \in (b) \Longleftrightarrow A \in (ab)\} \vee \{A \in (gb) \Longleftrightarrow A \in (gab)\} \Longleftrightarrow \{A$ has SVEP on $\sigma_{aw}(A)^{\mathcal{C}} \cap \sigma_w(A)\}$.

Furthermore,

(ii) $A \in (ab) \Longrightarrow A \in (gab)$ *if and only if* $\Pi^a_\infty(A) \cap \sigma_{Bw}(A) = \emptyset$*, and* $A \in (b) \Longrightarrow (gb)$ *if and only if* $\Pi^a_\infty(A) \cap \sigma_{uBw}(A) = \emptyset$*.*

Proof. (i). The proof for both the implications being similar, we prove

$$\{A \in (gab) \Longleftrightarrow A \in (gb)\} \Longleftrightarrow \{A \text{ has SVEP on } \sigma_{aw}(A)^{\mathcal{C}} \cap \sigma_w(A)\}.$$

As remarked upon above $(gb) \Longrightarrow (gab)$ (no additional hypotheses required); to complete the proof, we prove

$$\{A \in (gab) \Longrightarrow A \in (gb)\} \Longleftrightarrow \{A \text{ has SVEP on } \sigma_{aw}(A)^{\mathcal{C}} \cap \sigma_w(A)\}.$$

Since $A \in (gab)$ implies $A \in (gBt)$, hence $A \in (Bt)$, and since $A \in (Bt)$ if and only if A has SVEP on $\sigma_w(A)^{\mathcal{C}}$, A has SVEP on $\sigma_{aw}(A)^{\mathcal{C}}$ if and only if A has SVEP on

$$\sigma_{aw}(A)^{\mathcal{C}} \setminus \sigma_w(A)^{\mathcal{C}} = \sigma_{aw}(A)^{\mathcal{C}} \cap \sigma_w(A).$$

Noticing that already $\Pi^a(A) = \Pi(A)$ (given $A \in (gab)$), the proof follows since $A \in (a - Bt)$ (hence $A \in (a - gBt)$) if and only if A has SVEP on $\sigma_{aw}(A)^{\mathcal{C}}$.

(ii). We argue:

$$\begin{aligned} A \in (ab) &\Longleftrightarrow A \in (Bt), \Pi_0(A) = \Pi^a_0(A) \Longleftrightarrow A \in (gBt), \Pi_0(A) = \Pi^a_0(A) \\ &\Longrightarrow A \in (gBt), \Pi(A) = \Pi^a(A) \end{aligned}$$

if and only if

$$\begin{aligned} \Pi^a(A) \setminus \Pi(A) &= \{\Pi^a(A) \cap \sigma(A)^{\mathcal{C}}\} \cup \{\Pi^a(A) \cap \sigma_{Bw}(A)\} = \Pi^a(A) \cap \sigma_{Bw}(A) \\ &= \{\Pi^a_0(A) \cap \sigma_{Bw}(A)\} \cup \{\Pi^a_\infty(A) \cap \sigma_{Bw}(A)\} \\ &= \Pi^a_\infty(A) \cap \sigma_{Bw}(A) \text{ (since } \Pi^a_0(A) \cap \sigma_{Bw}(A) \subseteq \Pi^a_0(A) \cap \sigma_w(A) = \emptyset) \\ &= \emptyset \end{aligned}$$

and

$$\begin{aligned} A \in (b) &\Longleftrightarrow A \in (a - Bt), \Pi_0(A) = \Pi^a_0(A) \Longleftrightarrow A \in a - (gBt), \Pi_0(A) = \Pi^a_0(A) \\ &\Longrightarrow A \in (a - gBt), \Pi(A) = \Pi^a(A) \end{aligned}$$

if and only if

$$\begin{aligned} \Pi^a(A) \setminus \Pi(A) &= \{\Pi^a(A) \cap \sigma(A)^{\mathcal{C}}\} \cup \{\Pi^a(A) \cap \sigma_{Bw}(A)\} \text{ (since } A \in (a - gBt) \Longrightarrow A \in (gBt)) \\ &= \Pi^a(A) \cap \sigma_{Bw}(A) = \{\Pi^a_0(A) \cap \sigma_{Bw}(A)\} \cup \{\Pi^a_\infty(A) \cap \sigma_{Bw}(A)\} \\ &= \Pi^a_\infty(A) \cap \sigma_{Bw}(A) \text{ (since } \Pi^a_0(A) \cap \sigma_{Bw}(A) \subseteq \Pi_0(A) \cap \sigma_{Bw}(A) = \emptyset) \\ &= \emptyset. \end{aligned}$$

This completes the proof. □

It is immediate from Proposition 4.1 that a sufficient condition for $\{A \in (ab) \Longrightarrow A \in (b)\} \vee \{A \in (gab) \Longrightarrow A \in (gb)\}$ is that $\mathrm{iso}\sigma_{\mathrm{a}}(\mathrm{A}) \cap \sigma_{\mathrm{w}}(\mathrm{A}) = \emptyset$, and a sufficient condition for $A \in (ab) \Longrightarrow A \in (gab)$ (resp., $A \in (b) \Longrightarrow (gb)$) is that $\Pi^a(A) \cap \sigma_{Bw}(A) = \emptyset$ (resp., $\Pi^a(A) \cap \sigma_{uBw}(A) = \emptyset$).

Commuting Riesz perturbations preserve Browder's theorems (all four varieties) [18] and the Browder and Weyl spectra (both the regular and the approximate regular varieties) [20, 28]. Thus, if $A, R \in B(\mathcal{X})$, where R is a Riesz operator such that $[A, R] = 0$, then $A \in (b)$ (resp., $A \in (ab)$) implies $A + R \in (a - Bt)$ (resp., $A + R \in (Bt)$). Let $A \in (b)$, and let $\lambda \in \Pi^a_0(A + R) = \mathrm{iso}\sigma_a(A + R) \cap \sigma_{aw}(A + R)^{\mathcal{C}}$. Then either $\lambda \in \sigma_{aw}(A)$ or $\lambda \notin \sigma_{aw}(A)$. (Here we may assume that $\lambda \in \sigma(A)$; for if λ is not in $\sigma(A)$, then A^* has SVEP at λ implies $(A + R)^*$ has SVEP at λ which in turn implies, precisely the implication we are after, i.e. $\lambda \in \Pi_0(A + R)$.) If $\lambda \in \sigma_{aw}(A)$, then $\lambda \in \sigma_{aw}(A + R)$ implies $\lambda \notin \sigma_{aw}(A + R)^{\mathcal{C}}$, which is a contradiction. Hence $\lambda \in \sigma_{aw}(A)^{\mathcal{C}}$. Since $A + R$ has SVEP at λ implies A has SVEP at λ, $\lambda \in \Pi^a_0(A) = \Pi_0(A)$. Hence A^*, so also $(A+R)^*$, has SVEP at λ. Conclusion: $\Pi^a_0(A+R) \subseteq \Pi_0(A+R)$. The reverse inclusion being obvious,

$\Pi_0^a(A+R) = \Pi_0(A+R)$, and $A+R \in (b)$. A similar argument proves that $A \in (ab)$ implies $A+R \in (ab)$. Thus:

Proposition 4.2. *Given $A, R \in B(\mathcal{X})$, where R is a Riesz operator such that $[A,R]=0$, $A \in (b) \Longleftrightarrow A+R \in (b)$ and $A \in (ab) \Longleftrightarrow A+R \in (ab)$.*

The argument above does not extend to operators satisfying properties (gb) and (gab), as the following example shows.

Example 4.3. Let $A = U \oplus I \in B(\ell^2 \oplus \ell^2)$ and $R = 0 \oplus F \in B(\ell^2 \oplus \ell^2)$, where U is the forward unilateral shift and F is the finite rank operator $F(x_1, x_2, x_3, \cdots) = (-\frac{x_1}{2}, 0, 0, \cdots)$. Then

$$\begin{aligned}
&\sigma(A) = \sigma_w(A) = \sigma_{Bw}(A) = \overline{\mathcal{D}}, \sigma_a(A) = \sigma_{aw}(A) = \sigma_{uBw}(A) = \partial\mathcal{D}, \Pi(A) = \Pi^a(A) = \emptyset\\
&\text{and } \ A \in (gb), \sigma(A+R) = \overline{\mathcal{D}}, \sigma_a(A+R) = \partial\mathcal{D} \cup \{\frac{1}{2}\}, \sigma_w(A+R) = \sigma_{Bw}(A+R) = \overline{\mathcal{D}},\\
&\sigma_{aw}(A+R) = \sigma_{uBw}(A+R) = \partial\mathcal{D}, \Pi(A+R) = \emptyset, \Pi^a(A+R) = \{\frac{1}{2}\}, A+R \in (gBt)\\
&\text{and } \ A+R \notin (gb).
\end{aligned}$$

Observe that $\text{iso}\sigma_\text{a}(\text{A}) = \emptyset$, $\sigma_a(A+R)$ has an isolated point, and $\Pi_0^a(A+R) \neq \Pi_0(A+R)$.

Consider $A \in (gb)$. Then $A \in (a-gBt)$ and $\Pi^a(A) = \Pi(A)$. If $R \in B(\mathcal{X})$ is a Riesz operator which commutes with A, then $A+R \in (a-gBt)$ and $(A+R)^*$ has SVEP on $\Pi^a(A)$. Assume now that $\text{iso}\sigma_\text{a}(\text{A}) = \text{iso}\sigma_\text{a}(\text{A}+\text{R})$ and $\Phi_{uBw}^{\text{iso}}(A) = \Phi_{uBw}^{\text{iso}}(A+R)$. Then, see Proposition 3.2, $\Pi^a(A) = \Pi^a(A+R)$, and hence $(A+R)^*$ has SVEP on $\Pi^a(A+R)$. Thus $\Pi^a(A+R) = \Pi(A+R)$ and $A+R \in (gb)$.

Proposition 4.4. *Given operators $A, R \in B(\mathcal{X})$ such that R is a Riesz operator which commutes with A, if $\text{iso}\sigma_\text{a}(\text{A}) = \text{iso}\sigma_\text{a}(\text{A}+\text{R})$ and $\Phi_{uBw}^{\text{iso}}(A) = \Phi_{uBw}^{\text{iso}}(A+R)$, then*

$$A \in (gb) \Longleftrightarrow A+R \in (gb), \ \ \textit{and} \ \ A \in (gab) \Longleftrightarrow A+R \in (gab).$$

Proof. We have already seen that $A \in (gb)$ implies $A+R \in (gb)$. If, instead, $A \in (gab)$, then $A \in (gBt)$ and $\Pi(A) = \Pi^a(A)$. Since the hypotheses imply $\Pi^a(A) = \Pi^a(A+R)$ and $\Pi(A) = \Pi(A+R)$, see Proposition 3.2, $(A+R) \in (gab)$. Since the reverse implication in either of the cases follows by symmetry, the proof is complete. □

A couple of variations of Weyl's theorem which have attracted some attention *vis-a-vis* variations of Browder's theorem are the properties (w) and (gw), where $A \in B(\mathcal{X})$ satisfies

property (w) if $\sigma_a(A) \cap \sigma_{aw}(A)^{\mathcal{C}} = E_0(A)$, equivalently $A \in (a-Bt), \Pi_0(A) = \Pi_0^a(A) = E_0(A)$;

property (gw) if $\sigma_a(A) \cap \sigma_{uBw}(A)^{\mathcal{C}} = E(A)$, equivalently $A \in (a-gBt), \Pi(A) = \Pi^a(A) = E(A)$.

Evidently, property (w) implies both properties (ab) and (b), and property (gw) implies both properties (gab) and (gb). Just as evidently, the reverse implications fail.

Theorem 4.5. *Given $A \in B(\mathcal{X})$:*

(a) $\{A \in (b) \Longleftrightarrow A \in (w)\} \Longleftrightarrow \{E_0(A) \cap \sigma_{aw}(A) = \emptyset\}$.

(b) $\{A \in (gb) \Longleftrightarrow A \in (gw)\} \Longleftrightarrow \{E(A) \cap \sigma_{uBw}(A) = \emptyset\}$.

(c)(i) $A \in (w) \Longrightarrow A \in (ab)$, (ii) $\{A \in (ab) \wedge A^*$ has SVEP on $\sigma_{aw}(A)^{\mathcal{C}} \cap \sigma_w(A)\} \Longrightarrow A \in (b)$, (iii) $\{A \in (b) \wedge (E_0(A) \cap \sigma_{aw}(A) = \emptyset\} \Longrightarrow A \in (w)$.

(d)(i) $A \in (gw) \Longrightarrow A \in (gab)$, (ii) $A \in \{(gab) \wedge A^*$ has SVEP on $\sigma_{aw}(A)^{\mathcal{C}} \cap \sigma_w(A)\} \Longrightarrow A \in (gb)$, (iii) $\{A \in (gb) \wedge (E(A) \cap \sigma_{uBw}(A) = \emptyset)\} \Longrightarrow A \in (gw)$.

Proof. (a). Property (w) implies property (b) (without further additional hypothses). For the reverse implication, we observe from $A \in (b)$ that $A \in (a - Bt)$ and $\Pi_0^a(A) = \Pi_0(A)$. Hence, since $\Pi_0(A) \subseteq E_0(A)$, $A \in (b)$ implies $A \in (w)$ if and only if $E_0(A) \setminus \Pi_0^a(A) = \emptyset$, i.e., if and only if

$$\begin{aligned} \emptyset &= E_0(A) \cap \{\sigma_a(A) \cap \sigma_{aw}(A)^{\mathcal{C}}\}^{\mathcal{C}} \\ &= \{E_0(A) \cap \sigma_a(A)^{\mathcal{C}}\} \cup \{E_0(A) \cap \sigma_{aw}(A)\} \\ &= E_0(A) \cap \sigma_{aw}(A). \end{aligned}$$

(b). The proof of the equivalence here is similar to that for the equivalence of part (a); the equivalence holds if and only if

$$\begin{aligned} \emptyset &= E(A) \setminus \Pi^a(A) = E(A) \cap \{\sigma_a(A) \cap \sigma_{uBw}(A)^{\mathcal{C}}\}^{\mathcal{C}} \\ &= \{E(A) \cap \sigma_a(A)^{\mathcal{C}}\} \cup \{E(A) \cap \sigma_{uBw}(A)\} \\ &= E(A) \cap \sigma_{uBw}(A). \end{aligned}$$

(c). The implication $(w) \Longrightarrow (ab)$ is evident. If $A \in (ab)$, then $A \in (Bt)$ and A^* has SVEP on $\sigma_w(A)^{\mathcal{C}}$. If we now assume that A^* has SVEP on $\sigma_{aw}(A)^{\mathcal{C}} \cap \sigma_w(A)$, then A^* has SVEP on

$$\begin{aligned} \sigma_{aw}(A)^{\mathcal{C}} &= \sigma_w(A)^{\mathcal{C}} \cup \{\sigma_{aw}(A)^{\mathcal{C}} \setminus \sigma_w(A)^{\mathcal{C}}\} \\ &= \sigma_w(A)^{\mathcal{C}} \cup \{\sigma_{aw}(A)^{\mathcal{C}} \cap \sigma_w(A)\}. \end{aligned}$$

Consequently, $\sigma_{aw}(A)^{\mathcal{C}} = \sigma_w(A)^{\mathcal{C}}$. Hence, since

$$\Pi_0^a(A) = \sigma(A) \cap \sigma_w(A)^{\mathcal{C}} = \sigma_a(A) \cap \sigma_w(A)^{\mathcal{C}} = \sigma_a(A) \cap \sigma_{aw}(A)^{\mathcal{C}}$$

and $\Pi_0^a(A) = \Pi_0(A)$, $A \in (b)$. Consider now $A \in (b) \wedge \{E_0(A) \cap \sigma_{aw}(A) = \emptyset\}$. We have:

$$\begin{aligned} E_0(A) \setminus \Pi_0(A) &= E_0(A) \setminus \Pi_0^a(A) = E_0(A) \cap \{\sigma_a(A) \cap \sigma_{aw}(A)^{\mathcal{C}}\}^{\mathcal{C}} \\ &= \{E_0(A) \cap \sigma_a(A)^{\mathcal{C}}\} \cup \{E_0(A) \cap \sigma_{aw}(A)\} \\ &= E_0(A) \cap \sigma_{aw}(A) = \emptyset, \end{aligned}$$

i.e., $\Pi_0(A) = E_0(A)$. Hence $A \in (w)$.

(d). The implication $(gw) \Longrightarrow (gab)$ is evident; the proof of the remaining implications being similar to that of the implications in part (c), we shall be brief. Since

$$A \in (gab) \Longrightarrow A \in (gBt), \Pi^a(A) = \Pi(A) \Longleftrightarrow A \in (Bt), \Pi^a(A) = \Pi(A),$$

the hypothesis A^* has SVEP on $\sigma_{aw}(A)^{\mathcal{C}} \cap \sigma_w(A)$ implies A^* has SVEP on $\sigma_{aw}(A)^{\mathcal{C}}$. Hence, as seen above,

$$A \in (a - Bt), \Pi^a(A) = \Pi(A) \Longleftrightarrow A \in (a - gBt), \Pi^a(A) = \Pi(A) \Longleftrightarrow A \in (gb).$$

The proof now follows since $E(A) = \Pi(A)$ if and only if $E(A) \cap \sigma_{uBw}(A) = \emptyset$. □

5. Weyl's Theorems: Equivalences.

It is well known that if either of A and A^* has SVEP, then A satisfies (all four versions of) Browder's theorem. A necessary and sufficient condition for $A \in (Bt)$ and $A \in (gBt)$ (resp., $A \in (a - Bt)$ and $A \in (a - gBt)$) is that A has SVEP on $\sigma_w(A)^{\mathcal{C}}$ (resp., $\sigma_{aw}(A)^{\mathcal{C}}$) [1, 8, 17].

Let, for an operator $A \in B(\mathcal{X})$, $E_\infty^x(A) = \{\lambda \in \text{iso}\sigma_x(A) : \alpha(A - \lambda) = \infty\}$, $\sigma_x = \sigma$ or σ_a.

Theorem 5.1. *(A). $A \in (gWt) \Longrightarrow A \in (Wt)$ and the reverse implication $A \in (Wt) \Longrightarrow A \in (gWt)$ holds if and only if $E_\infty(A) \cap iso\sigma_{Bw}(A) = \emptyset$.*

(B). $A \in (a - Wt) \Longrightarrow A \in (Wt)$ and the reverse implication $A \in (Wt) \Longrightarrow A \in (a - Wt)$ holds if and only if $E_0^a(A) \cap iso\sigma_w(A) = \emptyset$.

(C). $A \in (a-gWt) \Longrightarrow A \in (gWt)$ *and the reverse implication* $A \in (gWt) \Longrightarrow A \in (a-gWt)$ *holds if and only if* $E^a_\infty(A) \cap iso\sigma_{Bw}(A) = \emptyset$.

(D). $A \in (a-gWt) \Longrightarrow A \in (a-Wt)$ *and the reverse implication* $A \in (a-Wt) \Longrightarrow A \in (a-gWt)$ *holds if and only if* $E^a_\infty(A) \cap iso\sigma_{uBw}(A) = \emptyset$.

Proof. (A). The forward implication (is well known) and follows from

$$\begin{aligned} A \in (gWt) &\iff A \in (gBt), E(A) \cap \sigma_{Bw}(A) = \emptyset \\ &\iff A \in (Bt), E(A) \cap \sigma_{Bw}(A) = \emptyset \\ &\Longrightarrow A \in (Bt), E_0(A) \cap \sigma_{Bw}(A) = \emptyset \\ &\iff A \in (Bt), E_0(A) \cap \sigma_w(A) = \emptyset \iff A \in (Wt), \end{aligned}$$

since

$$\begin{aligned} E_0(A) \cap \sigma_w(A) &= \{E_0(A) \cap \sigma_{Bw}(A)\} \cup \{E_0(A) \cap (\sigma_w(A) \setminus \sigma_{Bw}(A))\} \\ &= E_0(A) \cap (\sigma_w(A) \setminus \sigma_{Bw}(A)) \\ &\quad (\text{since } E_0(A) \cap \sigma_{Bw}(A) = \emptyset) \\ &= (E_0(A) \cap \sigma_{Bw}(A)^{\mathcal{C}}) \cap \sigma_w(A) \\ &\subseteq \Pi_0(A) \cap \sigma_w(A) = \emptyset. \end{aligned}$$

Conversely,

$$\begin{aligned} A \in (Wt) &\iff A \in (Bt), E_0(A) \cap \sigma_w(A) = \emptyset \\ &\iff A \in (gBt), E_0(A) \cap \sigma_w(A) = \emptyset. \end{aligned}$$

Thus

$$A \in (Wt) \Longrightarrow A \in (gBt), E(A) \cap \sigma_{Bw}(A) = \emptyset \iff A \in (gWt)$$

if and only if

$$\begin{aligned} E(A) \cap \sigma_{Bw}(A) &= \{E_0(A) \cap \sigma_{Bw}(A)\} \cup \{E_\infty(A) \cap \sigma_{Bw}(A)\} \\ &= E_\infty(A) \cap \sigma_{Bw}(A) \quad (\text{since } E_0(A) \cap \sigma_{Bw}(A) \subseteq E_0(A) \cap \sigma_w(A) = \emptyset) \\ &= \emptyset. \end{aligned}$$

(B). The forward implication (once again, is well known and) follows from

$$\begin{aligned} A \in (a-Wt) &\iff A \in (a-Bt), E^a_0(A) \cap \sigma_{aw}(A) = \emptyset \\ &\Longrightarrow A \in (Bt), E^a_0(A) \cap \sigma_{aw}(A) = \emptyset \\ &\Longrightarrow A \in (Bt), E_0(A) \cap \sigma_{aw}(A) = \emptyset \\ &\iff A \in (Bt), E_0(A) \cap \sigma_w(A) = \emptyset, \end{aligned}$$

since

$$\begin{aligned} E_0(A) \cap \sigma_w(A) &= \{E_0(A) \cap \sigma_{aw}(A)\} \cup \{E_0(A) \cap (\sigma_w(A) \setminus \sigma_{aw}(A))\} \\ &= E_0(A) \cap (\sigma_w(A) \setminus \sigma_{aw}(A)) \\ &\quad (\text{since } E_0(A) \cap \sigma_{aw}(A) = \emptyset) \\ &= \{E_0(A) \cap \sigma_{aw}(A)^{\mathcal{C}}\} \cap \sigma_w(A) \\ &\subseteq \Pi_0(A) \cap \sigma_w(A) = \emptyset. \end{aligned}$$

Conversely, if $A \in (Wt)$ (equivalently, $\sigma(A) \cap \sigma_w(A)^{\mathcal{C}} = \Pi_0(A) = E_0(A)$), then

$$E^a_0(A) \setminus E_0(A) = \{E^a_0(A) \cap \sigma(A)^{\mathcal{C}}\} \cup \{E^a_0(A) \cap \sigma_w(A)\} = E^a_0(A) \cap \sigma_w(A)$$

and $E_0^a(A) = E_0(A)$ if and only if $E_0^a(A) \cap \sigma_w(A) = \emptyset$. (Similarly, $\Pi_0^a(A) = \Pi_0(A)$ if and only if $\Pi_0^a(A) \cap \sigma_w(A) = \emptyset$.) Hence

$$\begin{aligned} A \in (Wt) &\iff \sigma(A) \cap \sigma_w(A)^{\mathcal{C}} = \Pi_0(A) = E_0(A) \\ &\Longrightarrow \sigma(A) \cap \sigma_w(A)^{\mathcal{C}} = \Pi_0(A) = E_0^a(A) \end{aligned}$$

if and only if $E_0^a(A) \cap \sigma_w(A) = \emptyset$. It is clear from $E_0(A) = E_0^a(A)$ that A^* has SVEP on $E_0^a(A)$. Since the equivalence $\lambda \in \sigma_w(A)^{\mathcal{C}} \iff \lambda \in \sigma_{aw}(A)^{\mathcal{C}}$ holds if and only if A^* has SVEP at λ, we have:

$$A \in (Wt) \Longrightarrow \sigma_a(A) \cap \sigma_{aw}(A)^{\mathcal{C}} = E_0^a(A) \iff A \in (a - Wt)$$

if and only if $E_0^a(A) \cap \sigma_w(A) = \emptyset$.

(C). The implication $A \in (a - gWt) \Longrightarrow A \in (gWt)$ (again, well known) follows from

$$\begin{aligned} A \in (a - gWt) &\iff A \in (a - gBt), E^a(A) \cap \sigma_{uBw}(A) = \emptyset \\ &\Longrightarrow A \in (gBt), E^a(A) \cap \sigma_{uBw}(A) = \emptyset \\ &\Longrightarrow A \in (gBt), E(A) \cap \sigma_{uBw}(A) = \emptyset \\ &\qquad (\text{since } E(A) \cap \sigma_{uBw}(A) \subseteq E^a(A) \cap \sigma_{uBw}(A)) \\ &\Longrightarrow A \in (gBt), E(A) \cap \sigma_{Bw}(A) = \emptyset \ (\iff A \in (gWt)), \end{aligned}$$

since

$$\begin{aligned} E(A) \cap \sigma_{Bw}(A) &= \{E(A) \cap \sigma_{uBw}(A)\} \cup \{E(A) \cap (\sigma_{Bw}(A) \setminus \sigma_{uBw}(A))\} \\ &= E(A) \cap (\sigma_{Bw}(A) \setminus \sigma_{uBw}(A)) \\ &= \{E(A) \cap \sigma_{uBW}(A)^{\mathcal{C}}\} \cap \sigma_{Bw}(A) \\ &= \Pi(A) \cap \sigma_{Bw}(A) = \emptyset. \end{aligned}$$

For the reverse implication, we start by observing that if $A \in (gWt)$ (equivalently, if $\sigma(A) \cap \sigma_{Bw}(A)^{\mathcal{C}} = \Pi(A) = E(A)$), then

$$E^a(A) \setminus E(A) = \{E^a(A) \cap \sigma(A)^{\mathcal{C}}\} \cup \{E^a(A) \cap \sigma_{Bw}(A)\} = E^a(A) \cap \sigma_{Bw}(A)$$

implies

$$E^a(A) = E(A) \iff E^a(A) \cap \sigma_{Bw}(A) = \emptyset.$$

(Similarly, $\Pi^a(A) = \Pi(A)$ if and only if $\Pi^a(A) \cap \sigma_{Bw}(A) = \emptyset$.) Hence

$$A \in (gWt) \iff \sigma(A) \cap \sigma_{Bw}(A) = E(A) = E^a(A)$$

if and only if $E^a(A) \cap \sigma_{Bw}(A) = \emptyset$. Since $E(A) = E^a(A)$ implies A^* has SVEP on $E^a(A)$, and the equivalence $\lambda \in \sigma_{Bw}(A)^{\mathcal{C}} \iff \lambda \in \sigma_{uBw}(A)^{\mathcal{C}}$ holds if and only if A^* has SVEP at λ,

$$\begin{aligned} A \in (gWt) &\iff \sigma_a(A) \cap \sigma_{Bw}(A)^{\mathcal{C}} = E^a(A), \quad E(A) = E^a(A) = \Pi(A) = \Pi^a(A) \\ &\Longrightarrow \sigma_a(A) \cap \sigma_{uBw}(A)^{\mathcal{C}} = E^a(A) (\iff A \in (a - gWt)) \end{aligned}$$

if and only if $E^a(A) \cap \sigma_{Bw}(A) = \emptyset$.

(D).The forward implication $A \in (a - gWt) \Longrightarrow A \in (a - Wt)$ (again, well known) follows from

$$\begin{aligned} A \in (a - gWt) &\iff A \in (a - gBt), E^a(A) \cap \sigma_{uBw}(A) = \emptyset \\ &\Longrightarrow A \in (a - Bt), E^a(A) \cap \sigma_{uBw}(A) = \emptyset \\ &\Longrightarrow A \in (a - Bt), E_0^a(A) \cap \sigma_{aw}(A) = \emptyset \ (\iff A \in (a - Wt)), \end{aligned}$$

since

$$\begin{aligned}E_0^a(A)\cap\sigma_{aw}(A) &= \{E_0^a(A)\cap\sigma_{uBw}(A)\}\cup\{E_0^a(A)\cap(\sigma_{aw}(A)\setminus\sigma_{uBw}(A))\}\\ &= E_0^a(A)\cap(\sigma_{aw}(A)\setminus\sigma_{uBw}(A))\\ &= \{E(A)\cap\sigma_{uBW}(A)^{\mathcal{C}}\}\cap\sigma_{aw}(A)\\ &= \Pi_0^a(A)\cap\sigma_{aw}(A)=\emptyset.\end{aligned}$$

For the reverse implication, we argue

$$\begin{aligned}A\in(a-Wt) &\iff A\in(a-Bt), E_0^a(A)\cap\sigma_{aw}(A)=\emptyset\\ &\iff A\in(a-gBt), E_0^a(A)\cap\sigma_{aw}(A)=\emptyset\\ &\Longrightarrow A\in(a-gBt), E^a(A)\cap\sigma_{uBw}(A)=\emptyset(\iff A\in(a-gWt))\end{aligned}$$

if and only if

$$\begin{aligned}E^a(A)\cap\sigma_{uBw}(A) &= \{E_0^a(A)\cap\sigma_{uBw}(A)\}\cup\{E_\infty^a(A)\cap\sigma_{uBw}(A)\}\\ &= E_\infty^a(A)\cap\sigma_{uBw}(A)\\ &\quad(\text{since}E_0^a(A)\cap\sigma_{uBw}(A)\subseteq E_0^a(A)\cap\sigma_{aw}(A)=\emptyset)\\ &= \emptyset.\end{aligned}$$

This completes the proof. □

The following well known corollaries [1, 16, 17, 27**?**] demonstrate instances of operators satisfying the hypotheses of Theorem 5.1.

Corollary 5.2. *A sufficient condition for $A\in B(\mathcal{X})$ to satisfy:*
(A). $A\in(Wt)\Longrightarrow A\in(gWt)$ is that A is polaroid.
(B). $A\in(Wt)\Longrightarrow A\in(a-Wt)$ is that A is a-polaroid.
(C). $A\in(gWt)\Longrightarrow A\in(a-gWt)$ is that A is a-polaroid.
(D). $A\in(a-Wt)\Longrightarrow A\in(a-gWt)$ is that A is left- polaroid.

Proof. (A). The hypothesis $A\in(Wt)$ implies

$$A\in(Bt)\iff A\in(gBt)\iff\sigma(A)\cap\sigma_{Bw}(A)^{\mathcal{C}}=\Pi(A)(\subseteq E(A)).$$

Since A is polaroid, $E(A)\subseteq\Pi(A)$ (equivalently, $E(A)=\Pi(A)$). Hence $A\in(gWt)$.
(B). If A is a-polaroid, then

$$\begin{aligned}&E_0^a(A)=\Pi_0(A)\subseteq E_0(A)\subseteq E_0^a(A)\Longrightarrow E_0^a(A)=\Pi_0(A)=E_0(A)\\ \Longrightarrow\ &\{E_0^a(A)\cap\sigma_w(A)=\emptyset\iff E_0(A)\cap\sigma_w(A)=\emptyset\}.\end{aligned}$$

Hence, $A\in(Wt)$ and A is a-polaroid imply

$$A\in(Wt), E_0^a(A)\cap\sigma_w(A)=\emptyset\Longrightarrow A\in(a-Wt).$$

(C). As in the proof of (B) above, if A is a -polaroid, then

$$\begin{aligned}&E^a(A)=\Pi(A)\subseteq E(A)\subseteq E^a(A)\Longrightarrow E^a(A)=\Pi(A)=E(A)\\ \Longrightarrow\ &\{E^a(A)\cap\sigma_w(A)=\emptyset\Longrightarrow E^a(A)\cap\sigma_{Bw}(A)=\emptyset\}.\end{aligned}$$

Hence, $A\in(gWt)$ and A is a-polaroid imply

$$A\in(gWt), E^a(A)\cap\sigma_{uBw}(A)=\emptyset\Longrightarrow A\in(a-gWt).$$

(D). If A is left polaroid, then $E^a(A)=\Pi^a(A)$. Hence if $A\in(a-Wt)$ and is left polaroid, then $A\in(a-Wt)$ and $E^a(A)\cap\sigma_{uBw}(A)=\emptyset$, equivalently $A\in(a-gWt)$. □

Remark 5.3. No advantage is to be gained by assuming A is left polaroid (or right polaroid, or even a-polaroid) in (A) of the Corollary for the reason that A is polaroid at a point in $E(A)$ if and only if it is left polaroid (resp., right polaroid, a-polaroid) at the point. (An operator $A \in B(\mathcal{X})$ is right polaroid if A^* is left polaroid.) One may, however, replace the requirement that A is a-polaroid in (B) by A is polaroid at points in $\text{iso}\sigma_a(A)$ which are finite multiplicity eigenvalues of A.

Perturbation by commuting Riesz operators Given an $A \in B(\mathcal{X})$ and a Riesz operator $R \in B(\mathcal{X})$ such that $[A, R] = AR - RA = 0$, a sufficient condition for $A \in (a - Wt) \Longrightarrow A + R \in (a - Wt)$ and $A \in (a - gWt) \Longrightarrow A + R \in (a - gWt)$ is that A is finitely a-isoloid [16, Theorem 4.10]. Since finitely polaroid operators are finitely isoloid, and finitely a-polaroid operators are finitely a-isoloid, Corollary 5.2 implies:

Corollary 5.4. *Given operators $A, R \in B(\mathcal{X})$ with R a Riesz operator which commutes with A, a sufficient condition for:*

(i) $A + R \in (Wt) \Longrightarrow A + R \in (gWt)$ is that A is finitely polaroid.

(ii) $A + R \in (Wt) \Longrightarrow A + R \in (a - Wt)$ and $A + R \in (gWt) \Longrightarrow A + R \in (a - gWt)$ is that A is finitely a-polaroid.

(iii) $A + R \in (Wt) \Longrightarrow A + R \in (a - gWt)$ is that A is finitely left polaroid.

Proof. The proof in all cases is similar: We prove (ii). If A is finitely a-polaroid, then $E_0^a(A) \cap \text{iso}\sigma_{aw}(A) \subseteq \Pi_0(A) \cap \text{iso}\sigma_{aw}(A) = \emptyset$. Hence, $A \in (Wt) \Longrightarrow A \in (a - Wt)$. Again, since A is finitely a-polaroid implies A is (both) finitely isoloid and finitely a-isoloid,

$$A + R \in (Wt) \Longleftrightarrow A \in (Wt) \text{ and } A + R \in (a - Wt) \Longleftrightarrow A \in (a - Wt).$$

Hence, if A is finitely a-polaroid, then

$$A + R \in (Wt) \Longleftrightarrow A \in (Wt) \Longrightarrow A \in (a - Wt) \Longleftrightarrow A + R \in (a - Wt).$$

□

6. Properties (w), (gw) and Weyl type theorems: Equivalences

It is immediate from

$$A \in (w) \Longleftrightarrow A \in (a - Bt), \Pi_0^a(A) = \Pi_0(A) = E_0(A), E_0(A) \cap \sigma_{aw}(A) = \emptyset$$

and

$$A \in (gw) \Longleftrightarrow A \in (a - gBt), \Pi^a(A) = \Pi(A) = E(A), E(A) \cap \sigma_{uBw}(A) = \emptyset$$

that

$$(w) \Longrightarrow (Wt) \text{ and } (gw) \Longrightarrow (gWt).$$

(Recall that $(a - Bt) \Longrightarrow (Bt)$ and $(a - gBt) \Longrightarrow (gBT)$.) Reverse implications do not hold; see example below. Property (w) neither implies nor is implied by $(a - Wt)$. For example, if $U \in B(\ell^2)$ is the forward unilateral shift, Q_1 and Q_2 are the operators $Q_1(x_1, x_2, x_3, ...) = (\frac{x_2}{2}, \frac{x_3}{3}, ...)$ and $Q_2(x_1, x_2, x_3, ...) = (0, x_2, x_3, ...)$, $A_1 = U \oplus Q_1$ and $A_2 = U \oplus Q_2$, then

$$\sigma_a(A_1) = \sigma_{aw}(A_1) = \partial\mathcal{D} \cup \{0\}, E_0(A_1) = \Pi_0^a(A_1) = \Pi_0(A_1) = \emptyset, E_0^a(A_1) = \{0\},$$
$$\sigma_a(A_2) = \partial\mathcal{D} \cup \{0\}, \sigma_{aw}(A_2) = \partial\mathcal{D}, E_0(A_2) = \emptyset, E_0^a(A_2) = \{0\}.$$

Clealrly,

$$A_1 \in (w), A_1 \notin (a - Wt), A_2 \in (a - Wt) \text{ (hence also) } A_2 \in (Wt)), \text{and } A_2 \notin (w).$$

Similarly (gw) neither implies nor is implied by $(a-gWt)$. The forward implication $(gw) \Longrightarrow (w)$ holds, as the following argument shows. Since $(a-gBt) \Longrightarrow (a-Bt)$, if $A \in (gw)$, then $\Pi^a(A) = \Pi(A)$ and $\Pi_0^a(A) = \Pi_0(A)$ $(\subseteq E_0(A))$. Let $\lambda \in E_0(A)$. Then $\lambda \in E(A)$, with $0 < \alpha(A-\lambda) < \infty$, and $A \in (gw)$ imply $\lambda \in \sigma_a(A) \cap \sigma_{aw}(A)^{\mathcal{C}}$. Consequently, $E_0(A) \subseteq \sigma_a(A) \cap \sigma_{aw}(A)^{\mathcal{C}} = \Pi_0^a(A)$ $(= \Pi_0(A) \subseteq E_0(A))$, and $A \in (w)$. The reverse implication fails, as follows from a consideration of the operator A_1 above (when it is seen that $\sigma_a(A) \cap \sigma_{uBw}(A)^{\mathcal{C}} = \{0\} \neq E(A_1)$). The following theorem considers the implications $(w) \Longleftrightarrow (a-Wt)$, $(gw) \Longleftrightarrow (a-gWt)$ and $(w) \Longrightarrow (gw)$.

Theorem 6.1. *Given $A \in B(\mathcal{X})$:*
(i). $A \in (w) \Longleftrightarrow A \in (a-Wt)$ if and only if $E_0^a(A) \cap \sigma_w(A) = \emptyset$.
(ii). $A \in (gw) \Longleftrightarrow A \in (a-gWt)$ if and only if $E^a(A) \cap \sigma_{Bw}(A) = \emptyset$.
(iii). $A \in (w) \Longrightarrow A \in (gw)$ if and only if $E_\infty(A) \cap \sigma_{uBw}(A) = \emptyset$.

Proof. (i). If $A \in (w)$, then $(\sigma_a(A) \cap \sigma_{aw}(A)^{\mathcal{C}} = E_0(A), E_0(A) = \Pi_0(A) = \Pi_0^a(A)$, and)

$$A \in (a-Bt) \wedge \{\Pi_0(A) = E_0(A)\} \Longrightarrow A \in (Bt) \wedge \{\Pi_0(A) = E_0(A)\} \Longleftrightarrow A \in (Wt).$$

Hence

$$\begin{aligned} A \in (a-Wt) &\Longleftrightarrow E_0(A) = E_0^a(A) \Longleftrightarrow E_0^a(A) \cap E_0(A)^{\mathcal{C}} = \emptyset \\ &\Longleftrightarrow \{E_0^a(A) \cap \sigma(A)^{\mathcal{C}}\} \cup \{E_0^a(A) \cap \sigma_w(A)\} = \emptyset \\ &\Longleftrightarrow E_0^a(A) \cap \sigma_w(A) = \emptyset. \end{aligned}$$

Conversely, $A \in (a-Wt)$ (implies $A \in (Wt)$) and $E_0^a(A) \cap \sigma_w(A) = \emptyset$ imply

$$\begin{aligned} & E_0^a(A) \cap E_0(A)^{\mathcal{C}} = E_0^a(A) \cap \{\sigma(A)^{\mathcal{C}} \cup \sigma_w(A)\} \\ = \; & E_0^a(A) \cap \sigma_w(A) = \emptyset. \end{aligned}$$

Thus, if $A \in (a-Wt)$ and $E_0^a(A) \cap \sigma_{aw}(A) = \emptyset$, then

$$\sigma_a(A) \cap \sigma_{aw}(A)^{\mathcal{C}} = E_0^a(A) = E_0(A) \Longrightarrow A \in (w).$$

(ii). If $A \in (gw)$, then $(\sigma_a(A) \cap \sigma_{uBw}(A)^{\mathcal{C}} = E(A), E(A) = \Pi(A) = \Pi^a(A)$, and)

$$A \in (a-gBt) \wedge \{\Pi(A) = E(A)\} \Longrightarrow A \in (gBt) \wedge \{\Pi(A) = E(A)\} \Longleftrightarrow A \in (gWt).$$

Hence

$$\begin{aligned} A \in (a-gWt) &\Longleftrightarrow E^a(A) = E(A) \\ &\Longleftrightarrow \{E^a(A) \cap \sigma(A)^{\mathcal{C}}\} \cup \{E^a(A) \cap \sigma_{Bw}(A)\} = \emptyset \\ &\Longleftrightarrow E^a(A) \cap \sigma_{Bw}(A) = \emptyset. \end{aligned}$$

Conversely, $A \in (a-gWt)$ (implies $A \in (gWt)$) and $E^a(A) \cap \sigma_{Bw}(A) = \emptyset$ imply

$$\begin{aligned} & E^a(A) \cap E(A)^{\mathcal{C}} = E^a(A) \cap \{\sigma(A)^{\mathcal{C}} \cup \sigma_{Bw}(A)\} \\ = \; & E^a(A) \cap \sigma_{Bw}(A) = \emptyset. \end{aligned}$$

Thus, if $A \in (a-gWt)$ and $E^a(A) \cap \sigma_{Bw}(A) = \emptyset$, then

$$\sigma_a(A) \cap \sigma_{uBw}(A)^{\mathcal{C}} = E^a(A) = E(A) \Longleftrightarrow A \in (gw).$$

(iii). By definition,

$$\begin{aligned} A \in (w) &\Longleftrightarrow A \in (a-Bt), \;\; E_0(A) = \Pi_0(A) = \Pi_0^a(A) \\ &\Longleftrightarrow A \in (a-gBt), \;\; E_0(A) = \Pi_0(A) = \Pi_0^a(A), \end{aligned}$$

and

$$A \in (gw) \Longleftrightarrow A \in (a - gBt), \;\; E(A) = \Pi(A) = \Pi^a(A).$$

Hence a necessary and sufficient condition for $A \in (w)$ to imply $A \in (gw)$ is that $E(A) = \Pi^a(A)$. Since

$$\begin{aligned} E(A) \setminus \Pi^a(A) &= \{E(A) \cap \sigma_a(A)^{\mathcal{C}}\} \cup \{E(A) \cap \sigma_{uBw}(A)\} \\ &\quad (A \in (w) \Longrightarrow A \in (a - gBt)) \\ &= E(A) \cap \sigma_{uBw}(A) = \{E_0(A) \cap \sigma_{uBw}(A)\} \cup \{E_\infty(A) \cap \sigma_{uBw}(A)\} \\ &= E_\infty(A) \cap \sigma_{uBw}(A) \quad (A \in (w) \Longrightarrow E_0(A) \cap \sigma_{uBw}(A) = \emptyset), \end{aligned}$$

it follows that our necessary and sufficient condition reduces to $E_\infty(A) \cap \sigma_{uBw}(A) = \emptyset$. $\square$

Theorem 6.1 has a number of consequences, amongst them the following corollary from [**?** , Corollary 3.8].

Corollary 6.2. *A sufficient condition for the equivalences*

$$A \in (w) \Longleftrightarrow A \in (gw) \Longleftrightarrow A \in (a - gWt) \Longleftrightarrow A \in (a - Wt)$$

is that A is a-polaroid.

Proof. If A is a-polaroid, then

$$\lambda \in E^a(A) \Longrightarrow \lambda \in \Pi(A), \;\; \lambda \in E_0^a(A) \Longrightarrow \lambda \in \Pi_0(A).$$

Hence

$$\begin{aligned} E(A) \cap \sigma_{Bw}(A) = \emptyset = E^a(A) \cap \sigma_{Bw}(A), \;\; \text{and} \\ E_0(A) \cap \sigma_w(A) = \emptyset = E_0^a(A) \cap \sigma_w(A). \end{aligned}$$

$\square$

Remark 6.3. The example of the operator A_1 above shows that the condition $E_0^a(A) \cap \sigma_w(A) = \emptyset$ can not be replaced in Theorem 6.1(i) by $E_0(A) \cap \sigma_w(A) = \emptyset$. The same examples shows also that $E^a(A) \cap \sigma_{Bw}(A) = \emptyset$ can not be replaced in (ii) of the theorem by $E(A) \cap \sigma_{Bw}(A) = \emptyset$.

7. Properties (R), (aR) etc. and equivalences

Given an operator $A \in B(\mathcal{X})$, A satisfies property:

(R) if $E_0(A) = \Pi_0^a(A)$ (equivalently, $E_0(A) = \sigma_a(A) \cap \sigma_{ab}(A)^{\mathcal{C}}$);

(aR) if $E_0^a(A) = \Pi_0(A)$ (equivalently, $E_0^a(A) = \sigma(A) \cap \sigma_b(A)^{\mathcal{C}}$);

(gR) if $E(A) = \Pi^a(A)$ (equivalently, $E(A) = \sigma_a(A) \cap \sigma_{uBb}(A)^{\mathcal{C}}$);

(agR) if $E^a(A) = \Pi(A)$ (equivalently, $E^a(A) = \sigma(A) \cap \sigma_{Bb}(A)^{\mathcal{C}}$);

(aw) if $\sigma(A) \cap \sigma_w(A)^{\mathcal{C}} = E_0^a(A)$;

(gaw) if $\sigma(A) \cap \sigma_{Bw}(A)^{\mathcal{C}} = E^a(A)$;

(Bgw) if $\sigma_a(A) \cap \sigma_{uBw}(A)^{\mathcal{C}} = E_0(A)$.

These properties (alongwith a few others which we have chosen not to consider here), and their relationship with various versions of Browder and Weyl type theorems, have been studied in a number of papers in the recent past; see [4, 18, 29, 33] for further references. We study these properties in the following, concentrating upon the role they play in defining equivalences between various versions of the Browder and Weyl theorems. In the process we obtain a number of previously known results. We start with some observations (the proofs of which being straight forward are left to the reader). An operator $A \in B(\mathcal{X})$ satisfies:

Property (R) if and only if $\Pi_0^a(A) = E_0(A) = \Pi_0(A)$, equivalently, if and only if $E_0(A) \cap \sigma_{aw}(A) = \emptyset$. Left polaroid operators A satisfy (R).

Property (aR) if and only if $\Pi_0(A) = \Pi_0^a(A) = E_0^a(A)$, equivalently, if and only if $E_0^a(A) \cap \sigma_w(A) = \emptyset$. A a-polaroid implies A satisfies (aR).

Property (gR) if and only if $E(A) = \Pi(A) = \Pi^a(A)$, equivalently if and only if $E(A) \cap \sigma_{uBw}(A) = \emptyset$ ($\Longleftrightarrow E(A) \cap \sigma_{uBb}(A) = \emptyset$). Left polaroid operators A satisfy (gR).

Property (agR) if and only if $E^a(A) = \Pi(A) = \Pi^a(A)$, equivalently if and only if $E^a(A) \cap \sigma_{Bw}(A) = \emptyset$. A a-polaroid implies A satisfies (agR).

Furthermore:

$$(aR) \Longrightarrow (R), (agR) \Longrightarrow (gR), (agR) \wedge (gWt) \Longrightarrow (a - gWt), (aR) \wedge (Wt) \Longrightarrow (a - Wt).$$

The reverse implications in the above, in general, fail: We give here a simple example to show $(a - gWt) \not\Longrightarrow (agR) \wedge (gWt)$ and $(a - gWt) \not\Longrightarrow (agR) \wedge (a - Wt)$ (leaving it for the reader to construct examples to show that the remaining reverse implication also fail). Let $U \in B(\mathcal{H})$ be the forward unilateral shift and let $A = U \oplus 0 \in B(\mathcal{H} \oplus \mathcal{H})$. Then

$$\begin{aligned}&\sigma(A) = \sigma_w(A) = \sigma_{Bw}(A) = \overline{\mathcal{D}}, \sigma_a(A) = \sigma_{aw}(A) = \sigma_{uBw}(A) = \partial\mathcal{D} \cup \{0\},\\ &\Pi(A) = E(A) = \emptyset, \Pi^a(A) = E^a(A) = \{0\}, \text{ and } \Pi_0^a(A) = E_0^a(A) = \emptyset.\end{aligned}$$

Clearly,

$$A \in (a - Wt), (gWt) \text{ and } (a - gWt) \text{ but } A \notin (agR).$$

Theorem 7.1. *Operators in $B(\mathcal{X})$ satisfy the following equivalences:*
(I). $(aR) \wedge (Wt) \Longleftrightarrow (a - Wt) \wedge (R)$, $(agR) \wedge (gWt) \Longleftrightarrow (a - gWt) \wedge (gR) \Longleftrightarrow (agR) \wedge (Wt)$.
(II). $(gw) \Longleftrightarrow (w) \wedge (gR) \Longleftrightarrow (gWt) \wedge (gR) \Longleftrightarrow (Wt) \wedge (gR) \Longleftrightarrow (Bt) \wedge (gR) \Longleftrightarrow (gw) \wedge (R)$.
(III). $(ab) \wedge (R) \Longleftrightarrow (Wt) \wedge (R) \Longleftrightarrow (b) \wedge (R)$.

Proof. (I). Given $A \in B(\mathcal{X})$,

$$A \in (a - Wt) \Longrightarrow A \in (Wt) \Longrightarrow E_0(A) \cap \sigma_w(A) = \emptyset;$$

hence if also $A \in (R)$ (so that $E_0^a(A) = E_0(A)$), then $E_0^a(A) \cap \sigma_w(A) = \emptyset$. Applying Theorem 5.1(B) we have:

$$\begin{aligned}A \in (a - Wt) \wedge (R) &\Longleftrightarrow A \in (Wt) \wedge (R), \Pi_0^a(A) = E_0^a(A)\\ &\Longleftrightarrow A \in (Wt), \Pi_0^a(A) = E_0^a(A) = E_0(A) = \Pi_0(A)\\ &\Longleftrightarrow A \in (Wt) \wedge (aR).\end{aligned}$$

Again, if $A \in (a - gWt) \wedge (gR)$, then $E^a(A) \cap \sigma_{Bw}(A) = \emptyset$. Hence, applying Theorem 5.1(C) and (B), we have:

$$\begin{aligned}A \in (a - gWt) \wedge (gR) &\Longleftrightarrow A \in (gWt) \wedge (gR), \Pi^a(A) = E^a(A)\\ &\Longleftrightarrow A \in (gWt), \Pi^a(A) = E^a(A) = E(A) = \Pi(A)\\ &\Longleftrightarrow A \in (gWt) \wedge (agR)\end{aligned}$$

and

$$\begin{aligned}A \in (a - gWt) \wedge (gR) &\Longleftrightarrow A \in (Wt) \wedge (gR), \Pi^a(A) = E^a(A)\\ &\qquad (\text{since } A \in (Wt) \Longrightarrow A \in (Bt) \Longleftrightarrow A \in (gBt))\\ &\Longleftrightarrow A \in (Wt), \Pi^a(A) = E^a(A) = E(A) = \Pi(A)\\ &\Longleftrightarrow A \in (Wt) \wedge (agR).\end{aligned}$$

(II). The equivalence $(Bt) \wedge (gR) \iff (gBt) \wedge (gR)$ is immediate from the equivalence $(Bt) \iff (gBt)$. Given an $A \in B(\mathcal{X})$, we have:

$$\begin{aligned} A \in (gw) &\iff A \in (a - gBt), \Pi^a(A) = \Pi(A) = E(A) \\ &\iff A \in (a - Bt), \Pi_0^a(A) = \Pi_0(A) = E_0(A), \Pi^a(A) = E(A) \\ &\iff A \in (w) \wedge (gR) \\ &\iff A \in (Bt), \Pi_0^a(A) = \Pi_0(A) = E_0(A), \Pi^a(A) = E(A)(\iff A \in (Bt) \wedge (gR)) \\ &\iff A \in (gBt), \Pi^a(A) = \Pi(A) = E(A) \\ &\iff A \in (gBt) \wedge (gR), A \in (gWt) \wedge (gR). \end{aligned}$$

Here the implication $A \in (Bt) \wedge (gR) \implies A \in (w) \wedge (gR)$ follows from $\Pi^a(A) = E(A)$ (implies $\Pi_0^a(A) = E_0(A)$) and $A \in (a - Bt)$ since A has SVEP on

$$\begin{aligned} \sigma_{aw}(A)^{\mathcal{C}} &= \sigma_w(A)^{\mathcal{C}} \cup \{\sigma_{aw}(A)^{\mathcal{C}} \setminus \sigma_w(A)^{\mathcal{C}}\} \\ &= \sigma_w(A)^{\mathcal{C}} \cup \{\sigma_{aw}(A)^{\mathcal{C}} \cap \{\sigma(A) \cup \Pi_0(A)\}\} \\ &= \sigma_w(A)^{\mathcal{C}} \cup \{\{\sigma_{aw}(A)^{\mathcal{C}} \cap \sigma(A)^{\mathcal{C}}\} \cup \{\sigma_{aw}(A)^{\mathcal{C}} \cap \Pi_0^a(A)\}\} \\ &= \sigma_w(A)^{\mathcal{C}} \cup \{\sigma_{aw}(A)^{\mathcal{C}} \cap \sigma(A)^{\mathcal{C}}\}. \end{aligned}$$

Again, since $A \in (gw)$ implies $A \in (Bt)$, and since $A \in (Bt)$ (equivalently, $A \in (gBt)$) and $\Pi^a(A) = E(A)$ together imply A has SVEP on $\sigma_{aw}(A)^{\mathcal{C}}$ (see the argument above), $A \in (a - Bt) \wedge \{\Pi^a(A) = E(A)\}$ (equivalently, $A \in (a-gBt) \wedge \{\Pi^a(A) = E(A)\}$. We have $A \in (Bt) \wedge \{\Pi^a(A) = E(A)\}$ implies $A \in (gw) \wedge (R)$. Hence

$$\begin{aligned} A \in (gw) \wedge (R) &\iff A \in (Bt), \Pi^a(A) = E(A) = \Pi(A) \\ &\iff A \in (Bt) \wedge (gR). \end{aligned}$$

(III). Since

$$\begin{aligned} &\sigma(A) \cap \sigma_w(A)^{\mathcal{C}} = \Pi_0(A), \Pi_0(A) = \Pi_0^a(A) \\ \iff &\sigma_a(A) \cap \sigma_{aw}(A)^{\mathcal{C}} = \Pi_0(A), \Pi_0(A) = \Pi_0^a(A) \end{aligned}$$

(see the argument above, part (II), proving that $A \in (Bt)$ and $\Pi_0(A) = \Pi_0^a(A)$ implies $A \in (a - Bt)$ and $\Pi_0(A) = \Pi_0^a(A)$), we have:

$$\begin{aligned} A \in (Wt) \wedge (R) &\iff \sigma(A) \cap \sigma_w(A)^{\mathcal{C}} = E_0(A), E_0(A) = \Pi_0^a(A) \\ &\iff \sigma(A) \cap \sigma_w(A)^{\mathcal{C}} = \Pi_0^a(A), E_0(A) = \Pi_0^a(A) \\ &\iff A \in (ab) \wedge (R) \\ &\iff \sigma(A) \cap \sigma_w(A)^{\mathcal{C}} = \Pi_0(A), \Pi_0(A) = E_0(A) = \Pi_0^a(A) \\ &\iff \sigma_a(A) \cap \sigma_{aw}(A)^{\mathcal{C}} = \Pi_0(A), E_0(A) = \Pi_0^a(A) \\ &\iff (b) \wedge (R). \end{aligned}$$

This completes the proof. $\square$

Neither of the properties (R), (aR), (gR) and (agR) implies either of the properties (aw), (gaw) and (Bgw). It is straightforward to see that (aw) implies (Bt), (gaw) implies (gBt) and (Bgw) implies (gBt) (so that, in particular, all three properties imply (Bt)). Properties (R), (aR), (gR) and (agR) do not, however, imply (Bt). To see this consider the operator $A = U \oplus U^*$, where $U \in B(\mathcal{H})$ is the forward unilateral shift, when it is seen that

$$\begin{aligned} &\sigma(A) = \sigma_a(A) = \overline{\mathcal{D}}, \sigma_w(A) = \sigma_{aw}(A) = \partial\mathcal{D}, \Pi_0(A) = \Pi_0^a(A) = \Pi(A) = \Pi^a(A) = E_0(A) \\ = &E_0^a(A) = E(A) = E^a(A) = \emptyset, A \in (R) \wedge (aR) \wedge (gR) \wedge (agR), A \notin (aw) \vee (gaw) \vee (Bgw). \end{aligned}$$

The a-polaroid property, $\lambda \in \text{iso}\sigma_a(A) \Longleftrightarrow \lambda \in \Pi(A)$, acts as "an equalizer" for certain pairs of these properties in the sense that if A is a-polaroid, then:

$$A \in (R) \Longleftrightarrow E_0(A) = \Pi_0(A) = \Pi_0^a(A) = E_0^a(A) \Longleftrightarrow A \in (aR),$$
$$A \in (gR) \Longleftrightarrow E(A) = \Pi(A) = \Pi^a(A) = E^a(A) \Longleftrightarrow A \in (agR),$$

and if A is finitely a-polaroid, then

$$A \in (R) \Longleftrightarrow A \in (aR) \Longleftrightarrow A \in (gR) \Longleftrightarrow A \in (agR), A \in (aw) \Longleftrightarrow A \in (gaw),$$
$$A \in (R) \wedge (Bt) \Longleftrightarrow A \in (aR) \wedge (Bt) \Longleftrightarrow A \in (gR) \wedge (Bt) \Longleftrightarrow A \in (agR) \wedge (Bt)$$
$$\Longleftrightarrow A \in (aw) \Longleftrightarrow A \in (gaw).$$

The a-polaroid hypothesis can not be replaced by the polaroid hypothesis.Thus

$$(R) \wedge (\text{polaroid}) \not\Longrightarrow (\text{aR}) \wedge (\text{polaroid}), (\text{gR}) \wedge (\text{polaroid}) \not\Longrightarrow (\text{agR}) \wedge (\text{polaroid}),$$

as the following example shows. Let $A = U \oplus Q$, where $U \in B(\ell^2)$ is the forward unilateral shift and $Q \in B(\ell^2)$ is the quasinilpotent operator $Q(x_1, x_2, x_3, \cdots) = (\frac{x_2}{2}, \frac{x_3}{3}, \cdots)$. Then

$$\text{iso}\sigma(A) = \emptyset (\Longrightarrow A \text{ is polaroid}), E_0(A) = \Pi_0^a(A) = \emptyset = E(A) = \Pi(A),$$
$$\emptyset = \Pi_0(A) \neq E_0^a(A) = \{0\}, \emptyset = \Pi(A) \neq E^a(A) = \{0\}.$$

Evidently,

$$A \in (R) \wedge (gR) \text{ and } A \notin (aR) \vee (agR).$$

Again, since

$$\sigma(A) \cap \sigma_w(A)^{\mathcal{C}} = \emptyset \neq \{0\} \text{ and}$$
$$\sigma(A) \cap \sigma_{Bw}(A)^{\mathcal{C}} = \emptyset \neq \{0\} = E^a(A),$$

it follows that

$$A \notin (aw) \vee (gaw).$$

Finally, if we define $B \in B(\ell^2 \oplus \ell^2)$ by $B = U \oplus 0$, then B is polaroiod, $\sigma(B) \cap \sigma_w(B)^{\mathcal{C}} = \emptyset = E_0^a(A)$ (i.e., $B \in (aw)$) and $\sigma(A) \cap \sigma_{Bw}(A)^{\mathcal{C}} = \emptyset \neq E^a(A) = \{0\}$ (so that $B \notin (gaw)$). B, however, satisfies properties (aR) and (Bgw).

8. Perturbation by commuting Riesz operators: Preservation

If $A, R \in B(\mathcal{X})$, R is a Riesz operator and $[A, R] = AR - RA = 0$, then A (similarly, A^*) has SVEP at a point if and only if $A + R$ (resp., $A^* + R^*$) has SVEP at the point [6], and $A + R$ has the same Weyl and a-Weyl spectrum as A [28]. The following lemma is a more general version of a known result [27].

Lemma 8.1. *Given operators $A, R \in B(\mathcal{X})$, with R a Riesz operator which commutes with A,*

$$E_x(A + R) \cap \sigma_x(A) \subseteq iso\sigma_x(A),$$

where Z_x stands for Z or Z_a (exclusive 'or').

Proof. The proof in both the cases being similar, we consider $E_a(A + R) \cap \sigma_a(A) \subseteq \text{iso}\sigma_a(A)$. Take a $\lambda \in E_a(A + R) \cap \sigma_a(A)$. Since λ is isolated in $\sigma_a(A + R)$, there exists a deleted neighbourhood $\mathcal{N}_\epsilon(\lambda)$ of λ such that $\mu \notin \sigma_a(A + R)$ for all $\mu \in \mathcal{N}_\epsilon(\lambda)$. The hypothesis $\lambda \in \sigma_a(A)$ implies that either $\lambda \in \text{acc}\sigma_\text{a}(\text{A})$ or $\lambda \in \text{iso}\sigma_a(A)$: We prove that $\lambda \notin \text{acc}\sigma_\text{a}(\text{A})$. Suppose to the contrary that $\lambda \in \text{acc}\sigma_\text{a}(\text{A})$. Then there exists a sequence $\{\mu_n\} \subset \mathcal{N}_\epsilon(\lambda) \cap \sigma_a(A)$ converging to λ. Since $\mu_n \notin \sigma_a(A + R)$, $\mu_n \in \sigma_{aw}(A + R)^{\mathcal{C}}$, and hence $\mu_n \in \sigma_{aw}(A)^{\mathcal{C}}$ for all natural numbers n. Furthermore, since $A + R$ has SVEP at μ_n implies A has SVEP at μ_n, $\mu_n \in \Pi_0^a(A)$ for all n. But then this, by the punctured neighbourhood theorem [1], implies $\lambda \in \sigma_{aw}(A)^{\mathcal{C}}$, and hence (since A has SVEP at λ) $\lambda \in \Pi_0^a(A)$ - a contradiction. $\square$

In the following, we start by considering the preservation of properties (R) and (aR) under perturbation by commuting Riesz operators. Throughout the following $R \in B(\mathcal{X})$ shall denote a Riesz operator which commutes with $A \in B(\mathcal{X})$. (Reader be warned that R, on its own- no parentheses, denotes a Riesz operator and (R) denotes property (R).)

Theorem 8.2. *If $A \in B(\mathcal{X})$ is such that* $\mathrm{iso}\sigma_\mathrm{a}(\mathrm{A}+\mathrm{R}) = \mathrm{iso}\sigma_\mathrm{a}(\mathrm{A})$*, then:*
(I). $\{A \in (R) \Longrightarrow A+R \in (R)\} \Longleftrightarrow \{E_0(A+R) \subseteq \sigma_a(A), \mathrm{iso}\sigma_\mathrm{a}(\mathrm{A}) \cap \mathrm{E}_0(\mathrm{A}+\mathrm{R}) \subseteq \Pi_0^\mathrm{a}(\mathrm{A})\}$.
(II). $\{A \in (aR) \Longrightarrow A+R \in (aR)\} \Longleftrightarrow \{E_0^a(A+R) \subseteq \sigma_a(A), \mathrm{iso}\sigma_\mathrm{a}(\mathrm{A}) \cap \mathrm{E}_0^\mathrm{a}(\mathrm{A}+\mathrm{R}) \subseteq \Pi_0^\mathrm{a}(\mathrm{A})\}$.

Proof. (I). Since $A \in (R)$ if and only if $E_0(A) = \Pi_0^a(A)$, the Browder and a-Browder spectra are stable under perturbation by commuting Riesz operators, and SVEP at points is preserved by commuting Riesz operators, the hypothesis $\mathrm{iso}\sigma_\mathrm{a}(\mathrm{A}) = \mathrm{iso}\sigma_\mathrm{a}(\mathrm{A}+\mathrm{R})$ implies (see Proposition 3.1) that

$$A \in (R) \Longleftrightarrow E_0(A) = \Pi_0^a(A) = \Pi_0^a(A+R) = \Pi_0(A+R) = \Pi_0(A).$$

Evidently, $\Pi_0(A+R) \subseteq E_0(A+R)$. Hence $A \in (R)$ implies $A+R \in (R)$ if and only if

$$\begin{aligned} & E_0(A+R) \subseteq \Pi_0(A+R) = \Pi_0^a(A) = E_0(A) \\ \Longleftrightarrow \quad & E_0(A+R) \subseteq \sigma(A), \mathrm{iso}\sigma_a(A) \cap E_0(A+R) \subseteq \Pi_0^a(A). \end{aligned}$$

(II). The proof here is similar, so we shall be brief. If $\mathrm{iso}\sigma_\mathrm{a}(\mathrm{A}) = \mathrm{iso}\sigma_\mathrm{a}(\mathrm{A}+\mathrm{R})$, then

$$A \in (aR) \Longleftrightarrow E_0^a(A) = \Pi_0(A) = \Pi_0(A+R) = \Pi_0^a(A+R) = \Pi_0^a(A),$$

and hence

$$\begin{aligned} \{A \in (R) \Longrightarrow A+R \in (aR)\} \quad \Longleftrightarrow \quad & E_0^a(A+R) \subseteq \Pi_0^a(A+R) = \Pi_0^a(A) = E_0^a(A) \\ \Longleftrightarrow \quad & E_0^a(A+R) \subseteq \sigma_a(A), \mathrm{iso}\sigma_a(A) \cap E_0^a(A+R) \subseteq \Pi_0^a(A). \end{aligned}$$

This completes the proof. $\square$

Perturbation by a commuting nilpotent $N \in B(\mathcal{X})$ preserves the spectrum, the approximate point spectrum and the eigenvalues of A. Hence: $A \in (R) \Longrightarrow A+N \in (R)$ *and* $A \in (aR) \Longrightarrow A+R \in (aR)$. Recall that $\sigma(A)$ and $\sigma_a(A)$ are also preserved under perturbation by commuting quasinilpotent operators $Q \in B(\mathcal{X})$. Ensuring $\mathrm{iso}\sigma_a(A) \cap E_0(A+Q) \subseteq \Pi_0(A)$ (similarly, ensuring $\mathrm{iso}\sigma_a(A) \cap E_0^a(A+Q) \subseteq \Pi_0^a(A)$) however requires an additional hypothesis. One such hypothesis is that A is polaroid (resp., left polaroid). More generally, if $\mathrm{iso}\sigma_w(A) = \emptyset$ (resp., $\mathrm{iso}\sigma_{aw}(A) = \emptyset$), then

$$\begin{aligned} & \lambda \in \mathrm{iso}\sigma_a(A) \cap E_0(A+Q) \Longrightarrow \lambda \in \sigma_w(A)^{\mathcal{C}} \cap E_0(A+Q) \subseteq \Pi_0(A) \\ & (\text{resp., } \lambda \in \mathrm{iso}\sigma_a(A) \cap E_0^a(A+Q) \Longrightarrow \lambda \in \sigma_{aw}(A)^{\mathcal{C}} \cap E_0^a(A+Q) \subseteq \Pi_0^a(A)). \end{aligned}$$

For finite rank operators $F \in B(\mathcal{X})$ (or, operators $F \in B(\mathcal{X})$ such that F^n is finite rank for some natural number n) such that $[A,F] = 0$ and $\mathrm{iso}\sigma_a(A+F) = \mathrm{iso}\sigma_a(A)$, $\sigma_x(A+F) = \sigma_x(A)$, $\sigma_x = \sigma$ or σ_a. Once again, if A is polaroid then $A \in (R) \Longrightarrow A+R \in (R)$, and if A is left polaroid then $A \in (aR) \Longrightarrow A+R \in (aR)$. Putting it altogether, we have:

Corollary 8.3. *Given an operator $A \in B(\mathcal{X})$, if:*
(a) $N \in B(\mathcal{X})$ is a nilpotent operator such that $[A,N] = 0$, then

$$A \in (R) \Longleftrightarrow A+N \in (R), A \in (aR) \Longleftrightarrow A+N \in (aR).$$

(b) $Q \in B(\mathcal{X})$ is a quasinilpotent operator such that $[A,Q] = 0$, then a sufficient condition for (i) $A \in (R) \Longleftrightarrow A+Q \in (R)$ is that $iso\sigma_w(A) = \emptyset$ and a sufficient condition for $A \in (aR) \Longleftrightarrow A+Q \in (aR)$ is that $iso\sigma_{aw}(A) = \emptyset$.

(c) $F \in B(\mathcal{X})$ is a finite rank operator such that $[A,F] = 0$ and $iso\sigma_a(A) = iso\sigma_a(A+F)$, then a sufficient condition for (i) $A \in (R) \Longleftrightarrow A+F \in (R)$ is that $iso\sigma_w(A) = \emptyset$ and for (ii) $A \in (aR) \Longleftrightarrow A+F \in (aR)$ is that $iso\sigma_{aw}(A) = \emptyset$.

The hypothesis $\mathrm{iso}\sigma_{\mathrm{aw}}(\mathrm{A}) = \emptyset$ for an operator $A \in B(\mathcal{X})$ implies $\lambda \in \sigma_{aw}(A)^{\mathcal{C}}$ for $\lambda \in \mathrm{iso}\sigma_{\mathrm{a}}(\mathrm{A})$; hence, since A has SVEP at all such λ, $\lambda \in \Pi_0^a(A)$. Thus, if $\lambda \in E_0(A)$ for an operator $A \in B(\mathcal{X})$ such that $\mathrm{iso}\sigma_{\mathrm{aw}}(\mathrm{A}) = \emptyset$, then ($A^*$ has SVEP at λ, therefore) $\lambda \in \Pi_0(A)$, and hence for A to satisfy $A \in (R)$ we must have $\Pi_0(A) = \Pi_0^a(A)$. (Observe that whereas $\Pi_0(A) \subseteq E_0(A)$, we do not in general have $\Pi_0^a(A) \subseteq E_0(A)$.) The following theorem shows that the hypothesis $\mathrm{iso}\sigma_{aw}(A) = \emptyset$ is sufficient for the implication $A \in (R)$ implies $A + R \in (R)$ for commuting Riesz operators R ssuch that $\mathrm{iso}\sigma_{\mathrm{a}}(\mathrm{A}) = \mathrm{iso}\sigma_{\mathrm{a}}(\mathrm{A+R})$.

Theorem 8.4. *Given operators $A, R \in B(\mathcal{X})$ such that R is a Riesz operator, $[A, R] = 0$ and $\mathrm{iso}\sigma_{\mathrm{a}}(\mathrm{A}) = \mathrm{iso}\sigma_{\mathrm{a}}(\mathrm{A+R})$, if:*

(i). $\mathrm{iso}\sigma_w(A) = \emptyset$, then $A \in (R) \iff A + R \in (R)$.

(ii). $\mathrm{iso}\sigma_{aw}(A) = \emptyset$, then $A \in (aR) \iff A + R \in (aR)$.

Proof. (i). Given a $\lambda \in E_0(A+R)$, either $\lambda \in \sigma(A)$ or $\lambda \notin \sigma(A)$. If $\lambda \in \sigma(A)$, then $\lambda \in E_0(A+R) \cap \sigma(A) \subseteq \mathrm{iso}\sigma(A)$, and this (since $\mathrm{iso}\sigma_w(A) = \emptyset$) implies $\lambda \in \sigma_w(A)^{\mathcal{C}} = \sigma_w(A+R)^{\mathcal{C}}$. Since $\lambda \notin \sigma(A)$ automatically implies $\lambda \in \sigma_w(A+R)^{\mathcal{C}}$, it follows (from $\Pi_0(A+R) \subseteq E_0(A+R)$) that

$$\lambda \in E_0(A+R) \iff \lambda \in \Pi_0(A+R), \text{ equivalently } E_0(A+R) = \Pi_0(A+R).$$

Consequently, $A + R \in (R)$ if and only if $\Pi_0(A+R) = \Pi_0^a(A+R)$, equivalently, if and only if $(A+R)^*$ has SVEP on $\Pi_0^a(A+R)$. Recall from Proposition 3.1 that the hypothesis $\mathrm{iso}\sigma_{\mathrm{a}}(\mathrm{A}) = \mathrm{iso}\sigma_{\mathrm{a}}(\mathrm{A+R})$ implies $\Pi_0(A) = \Pi_0(A+R)$ and $\Pi_0^a(A) = \Pi_0^a(A+R)$. Hence if $A \in (R)$, then $E_0(A) = \Pi_0^a(A)$ and A^* has SVEP at points in $\Pi_0^a(R)$. Since $(A+R)^*$ has SVEP at a point if and only if A^* has SVEP at the point, $(A+R)^*$ has SVEP on $\Pi_0^a(A+R)$. Consequently, $A \in (R)$ implies $A + R \in (R)$. The reverse implication follows by symmetry.

(ii). The proof here being similar to that of part (i), we shall be brief. For every $\lambda \in E_0^a(A+R)$, $A+R$ has SVEP at λ and $\lambda \in \sigma_{aw}(A+R)^{\mathcal{C}}$. Hence $E_0^a(A+R) = \Pi_0^a(A+R)$, and $A+R \in (aR)$ if and only if $\Pi_0^a(A+R) = \Pi_0(A+R)$. Since the hypothesis $\mathrm{iso}\sigma_{\mathrm{a}}(\mathrm{A}) = \mathrm{iso}\sigma_{\mathrm{a}}(\mathrm{A+R})$ implies $\Pi_0^a(A) = \Pi_0^a(A+R)$, and since $A \in (aR)$ if and only if $E_0^a(A) = \Pi_0(A)$, ($\Pi_0(A) \subseteq \Pi_0^a(A) \subseteq E_0^a(A)$ ensures $\Pi_0^a(A) = \Pi_0(A)$, and hence that) $(A+R)^*$ has SVEP on $\Pi_0^a(A+R)$. Thus $A \in (aR)$ implies $A + R \in (aR)$. The reverse implication being evident by symmetry, the proof is complete. □

It is obvious from $\mathrm{iso}\sigma_w(A) \subseteq \mathrm{iso}\sigma_{aw}(A)$ that Theorem 8.4(i) holds with the hypothesis $\mathrm{iso}\sigma_w(A) = \emptyset$ replaced by the hypothesis $\mathrm{iso}\sigma_{aw}(A) = \emptyset$. The example of the operator $B = A \oplus R \in B(\ell^2 \oplus \ell^2)$, where $A \in B(\ell^2)$ is the forward unilateral shift and $R \in B(\ell^2)$ is the quasinilpotent $R(x_1, x_2, x_3, \cdots) = (\frac{x_2}{2}, \frac{x_3}{3}, \cdots)$, $\mathrm{iso}\sigma_w(B) = \emptyset$, $\mathrm{iso}\sigma_{aw}(B) = \{0\}$, $E_0(A) = \Pi_0^a(A) = \Pi_0(A) = \emptyset$, $\Pi_0(B) = \emptyset$ and $E_0^a(B) = \{0\}$, proves that the condition $\mathrm{iso}\sigma_{\mathrm{w}}(\mathrm{A}) = \emptyset$ is not sufficient for the transfer of property (aR) from A to its perturbation $A + R$. Neither of the conditions $\mathrm{iso}\sigma_w(A) = \emptyset$ and $\mathrm{iso}\sigma_{aw}(A) = \emptyset$ is necessary in Theorem 8.4: Taking $A \in B(\mathcal{H})$ to be the zero operator, and $R \in B(\mathcal{H})$ to be a (non-trivial) nilpotent operator, it is seen that $\sigma(A) = \sigma_a(A) = \sigma(A+R) = \sigma_a(A+R) = \{0\}$, $E_0(A) = \Pi_0^a(A) = E_0(A+R) = \Pi_0^a(A+R) = \emptyset$, $\sigma_w(A) = \sigma_{aw}(A) = \{0\}$, and both A and $A + R \in (R)$. Observe that the hypotheses of Theorem 8.2 are satisfied. The example of the operators A and R of Example 4.3 shows that Theorem 8.4, as also Theorem 8.2, is liable to fail in the absence of the hypothesis $\mathrm{iso}\sigma_{\mathrm{a}}(\mathrm{A}) = \mathrm{iso}\sigma_{\mathrm{a}}(\mathrm{A+R})$.

The argument of the proof of Theorem 8.2 does not extend to prove the preservation of properties (gR) and (agR) under commuting Riesz perturbations. The problem (just as for the case of operators in (gb) and (gab)) lies with the failure of the stability of B-Weyl and upper B-Weyl spectra under commuting Riesz perturbations. The removal of the points $\mathrm{iso}\sigma_{\mathrm{aw}}(\mathrm{A})$ for an operator $A \in B(\mathcal{X})$ ensures ($\sigma_{Bw}(A) = \sigma_w(A)$ and) $\sigma_{uBw}(A) = \sigma_{aw}(A)$, and then, given $\mathrm{iso}\sigma_a(A) = \mathrm{iso}\sigma_{\mathrm{a}}(\mathrm{A+R})$, $A \in (gR)$ if and only if $A + R \in (gR)$ and $A \in (agR)$ if and only if $A + R \in (agR)$.

Theorem 8.5. *Given an operator $A \in B(\mathcal{X})$, if $\mathrm{iso}\sigma_{aw}(A) = \emptyset$ and $\mathrm{iso}\sigma_{\mathrm{a}}(\mathrm{A}) = \mathrm{iso}\sigma_{\mathrm{a}}(\mathrm{A+R})$, then $A \in (gR) \iff A + R \in (gR)$ and $A \in (agR) \iff A + R \in (agR)$.*

Proof. Since $\mathrm{iso}\sigma_{\mathrm{aw}}(\mathrm{A}) = \mathrm{iso}\sigma_{\mathrm{aw}}(\mathrm{A}+\mathrm{R})$, the hypothesis $\mathrm{iso}\sigma_{aw}(A) = \emptyset$ implies $\sigma_{uBw}(A) = \sigma_{aw}(A) = \sigma_{aw}(A+R) = \sigma_{uBw}(A+R)$. By definition $A \in (gR)$ if and only if $E(A) = \Pi^a(A)$ (equivalently, $E(A) = \Pi^a(A)$, $\Pi^a(A) = \Pi(A)$) and $A \in (agR)$ if and only if $E^a(A) = \Pi(A)$ (equivalently,$E^a(A) = \Pi^a(A) = \Pi(A)$). Hence, see Proposition 3.2,

$$A \in (gR) \Longrightarrow E(A) = \Pi^a(A) = \Pi^a(A+R) = \Pi(A+R) \subseteq E(A+R) \text{ and}$$

$$A \in (agR) \Longrightarrow E^a(A) = \Pi(A) = \Pi^a(A) = \Pi^a(A+R) \subseteq E^a(A+R), \ \ \Pi^a(A+R) = \Pi(A+R)$$

(since A^* has SVEP at points in $E(A) = \Pi^a(A+R)$, respectively at points in $\Pi(A) = \Pi^a(A+R)$, implies $(A+R)^*$ has SVEP on $\Pi^a(A+R)$). Consider now a $\lambda \in E(A)$ or $E^a(A)$. Then $\lambda \in E^a(A)$ and, as seen in the proof of Theorem 8.4, $\lambda \in \sigma_{uBw}(A+R)^{\mathcal{C}}$ $(= \sigma_{aw}(A+R)^{\mathcal{C}})$ and $A+R$ has SVEP at λ. Consequently, $\lambda \in \Pi^a(A+R)$, proving thereby that $E(A+R) = \Pi^a(A+R)$ if $A \in (gR)$ and $E^a(A+R) = \Pi(A+R)$ if $A \in (agR)$. Since the reverse implication follows by symmetry, the proof is complete. $\square$

The hypothesis $\mathrm{iso}\sigma_{\mathrm{aw}}(\mathrm{A}) = \emptyset$ (on its own) is not sufficient for the validity of Theorem 8.5, and (just as Theorem 8.4) the theorem may fail in the absence of the hypothesis $\mathrm{iso}\sigma_{\mathrm{a}}(\mathrm{A}) = \mathrm{iso}\sigma_{\mathrm{a}}(\mathrm{A}+\mathrm{R})$: Consider, once again, the operators A and R of Example 4.3, when it is seen that $E(A) = \Pi^a(A) = \emptyset$ (so that $A \in (gR)$) and $E(A+R) = \emptyset \neq \{\frac{1}{2}\} = \Pi^a(A+R)$ (and $A+R \notin (gR)$).

Perturbation by commuting nilpotents preserves properties (gR) and (agR); for commuting finite rank operators which preserve isolated points of the point spectrum, a sufficient condition for the preservation of the properties (gR) and $agR)$ is that the operator is left-polaroid (- a condition guaranteed by SVEP and the hypothesis $\mathrm{iso}\sigma_{\mathrm{aw}}(\mathrm{A}) = \emptyset$).

Corollary 8.6. *(a). $A \in (gR)$ ($A \in (agR)$) if and only if $A+N \in (gR)$ (resp., $A+N \in (agR)$) for nilpotent operators $N \in B(\mathcal{X})$ satisfying $[A,N] = 0$.*

(b). Given a finite rank operators $F \in B(\mathcal{X})$ such that $[A,F] = 0$ and $iso\sigma_a(A+F) = iso\sigma_a(A)$, if: (i) A is polaroid, then $A \in (gR)$ if and only if $A+F \in (gR)$; (ii) A is left-polaroid, then $A \in (agR)$ if and only if $A+F \in (agR)$.

Proof. (a). For commuting nilpotent N, $\sigma_x(A+N) = \sigma_x(A)$, $\sigma_x = \sigma$ or σ_a, and $E^x(A+N) = E^x(A)$, $E^x = E$ or E^a. Recall from [19, Theorem 2.6] that $\Pi(A+N) = \Pi(A)$. Since either of the hypotheses $A \in (gR)$ and $A \in (agR)$ implies A^* has SVEP on $\Pi^a(A)$, $\Pi^a(A) = \Pi(A) = \Pi(A+N) = \Pi^a(A+N)$ (see the statement following the proof of Proposition 3.3). Hence the proof.

(b). The hypotheses imply $\sigma_x(A+F) = \sigma_x(A)$, $\sigma_x = \sigma$ or σ_a, and (see Proposition 3.3) $\lambda \in \Pi(A)$ if and only if $\lambda \in \Pi(A+F)$ (resp., $\lambda \in \Pi^a(A)$ if and only if $\lambda \in \Pi^a(A+F)$). We start by considering (i). If $A \in (gR)$, then $E^a(A) = \Pi(A) = \Pi^a(A)$ and

$$\lambda \in E(A+F) \Longrightarrow \lambda \in \mathrm{iso}\sigma(A+F) \Longleftrightarrow \lambda \in \mathrm{iso}\sigma(A) \Longleftrightarrow \lambda \in \Pi(A) = \Pi^a(A) \Longleftrightarrow \lambda \in \Pi^a(A+F).$$

Hence $E(A+F) \subseteq \Pi^a(A+F)$. Since the semi B-Fredholm spectrum of an operator is stable under finite rank perturbations [9],

$$\begin{aligned}\{\Pi(A) = \Pi^a(A)\} &\Longleftrightarrow \{\mathrm{iso}\sigma(A) \cap \sigma_{Be}(A)^{\mathcal{C}} = \mathrm{iso}\sigma_a(A) \cap \sigma_{uBe}(A)^{\mathcal{C}}\} \\ &\Longleftrightarrow \{\mathrm{iso}\sigma(A+F) \cap \sigma_{Be}(A+F)^{\mathcal{C}} = \mathrm{iso}\sigma_a(A+F) \cap \sigma_{uBe}(A+F)^{\mathcal{C}}\} \\ &\Longleftrightarrow \{\Pi(A+F) = \Pi^a(A+F)\}.\end{aligned}$$

But then

$$E(A+F) \subseteq \Pi^a(A+F) = \Pi(A+F) \subseteq E(A+F) \Longleftrightarrow A \in (gR) \Longrightarrow A+F \in (gR).$$

The reverse implication follows from a consideration of the operator $A = (A+F) - F$.

To complete the proof, we consider now case (ii). Then A is left ploaroid, and assuming $A \in (agR)$ (so that $E^a(A) = \Pi(A) = \Pi^a(A)$ $(= E(A))$ and

$$\begin{aligned} & \lambda \in E^a(A+F) \Longrightarrow \lambda \in \text{iso}\sigma_a(A+F) \Longleftrightarrow \lambda \in \text{iso}\sigma_a(A) \\ \Longleftrightarrow \quad & \lambda \in \Pi^a(A) \Longleftrightarrow \lambda \in \Pi^a(A+F) \Longrightarrow \lambda \in E^a(A+F). \end{aligned}$$

Hence $E^a(A+F) = \Pi^a(A+F)$. Since, see above, $\Pi(A) = \Pi^a(A)$ implies $\Pi(A+F) = \Pi^a(A+F)$, $E^a(A+F) = \Pi(A+F)$; equivalently, $A \in (agR)$ implies $A+F \in (agR)$. The reverse implication follows from a consideration of the operator $A = (A+F) - F$. $\square$

The Corollary fails for commuting quasinilpotents Q, i.e., A polaroid or left polaroid and $A \in (gR)$ does not imply $A+Q \in (gR)$. This follows from a consideration of the operator $A+Q = 0+Q \in B(\mathcal{H})$, Q an injective quasinilpotent, when it is seen that $A \in (gR) \wedge (agR)$ but $A+Q \notin (gR) \vee (agR)$.

It is easily seen from the definition that

$$A \in (aw) \Longleftrightarrow \sigma(A) \cap \sigma_w(A)^{\mathcal{C}} = E_0^a(A) = E_0(A) = \Pi_0(A) = \Pi_0^a(A).$$

In particular, $A \in (aw)$ implies $A \in (Bt)$, hence also that $A+R \in (Bt)$, i.e.,

$$A \in (aw) \Longrightarrow \sigma(A+R) \cap \sigma_w(A+R)^{\mathcal{C}} = \Pi_0(A+R)$$

(for Riesz operator R commuting with A). Assume now that $\text{iso}\sigma_{\text{a}}(\text{A}) = \text{iso}\sigma_{\text{a}}(\text{A}+\text{R})$. Then Proposition 3.1 implies that $\Pi_0^a(A+R) = \Pi_0^a(A)$, and hence since $\Pi_0^a(A) = \Pi_0(A)$ that $(A+R)^*$ has SVEP on $\Pi_0^a(A+R)$. Consequently, $\Pi_0^a(A+R) = \Pi_0(A+R)$ and $\sigma(A+R) \cap \sigma_w(A+R)^{\mathcal{C}} = \Pi_0^a(A+R)$.

Theorem 8.8(a) *If* $\text{iso}\sigma_{\text{a}}(\text{A}) = \text{iso}\sigma_{\text{a}}(\text{A}+\text{R})$, *then* $A \in (aw)$ *implies* $A+R \in (aw)$ *if and only if* $E_0^a(A+R) \cap \sigma_{aw}(A) = \emptyset$.

Proof. We have already seen that $\sigma(A+R) \cap \sigma_w(A+R)^{\mathcal{C}} = \Pi_0^a(A+R)$. Assume now that $E_0^a(A+R) \cap \sigma_{aw}(A) = \emptyset$. Then $\lambda \in E_0^a(A+R)$ implies $\lambda \in \sigma_{aw}(A+R)^{\mathcal{C}} = \sigma_{aw}(A)^{\mathcal{C}}$. Since $A+R$ has SVEP at λ, $\lambda \in \Pi_0^a(A+R)$. The necessity of the condition $E_0^a(A+R) \cap \sigma_{aw}(A) = \emptyset$ being evident from $\Pi_0^a(A+R) = E_0^a(A+R)$, the proof is complete. $\square$

Considering operators satisfying property (gaw) it is seen that

$$\begin{aligned} & A \in (gaw) \Longleftrightarrow \sigma(A) \cap \sigma_{Bw}(A)^{\mathcal{C}} = E^a(A) = E(A) = \Pi(A) = \Pi^a(A); \\ & A \in (gaw) \Longrightarrow A \in (gBt) \Longleftrightarrow A+R \in (gBt). \end{aligned}$$

If we assume now that $\text{iso}\sigma_{\text{a}}(\text{A}) = \text{iso}\sigma_{\text{a}}(\text{A}+\text{R})$ and $\Phi_{uBw}^{\text{iso}}(A+R) = \Phi_{uBw}^{\text{iso}}(A)$ (Recall: $\Phi_{uBw}^{\text{iso}}(A) = \text{iso}\sigma_{\text{aw}}(\text{A}) \cap \sigma_{\text{uBw}}(\text{A})^{\mathcal{C}}$), then Proposition 3.1 implies that $\Pi^a(A+R) = \Pi^a(A)$. Hence, since A^* has SVEP on $\Pi^a(A)$, $\sigma(A+R) \cap \sigma_{Bw}(A+R) = \Pi^a(A+R)$.

Theorem 8.8(b) *If* $\text{iso}\sigma_{\text{a}}(\text{A}) = \text{iso}\sigma_{\text{a}}(\text{A}+\text{R})$ *and* $\Phi_{uBw}^{\text{iso}}(A+R) = \Phi_{uBw}^{\text{iso}}(A)$, $A \in (gaw)$ *implies* $A+R \in (gaw)$ *if and only if* $E^a(A+R) \cap \sigma_{uBw}(A) = \emptyset$.

Proof. The necessity being evident, the sufficiency of the condition would follow once we have proved $E^a(A+R) = \Pi^a(A+R)$, and for this it would suffice to prove $E^a(A+R) \subseteq \Pi^a(A+R)$. The hypotheses $\Phi_{uBw}^{\text{iso}}(A+R) = \Phi_{uBw}^{\text{iso}}(A)$ and $E^a(A+R) \cap \sigma_{uBw}(A) = \emptyset$ imply

$$\begin{aligned} E^a(A+R) \cap \sigma_{uBw}^{\text{iso}}(A) &= \{E^a(A+R) \cap \sigma_{uBw}^{\text{iso}}(A)\} \cup \{E^a(A+R) \cap \sigma_{uBw}(A)\} \\ &= E^a(A+R) \cap \{\sigma_{uBw}(A) \cup \sigma_{uBw}^{\text{iso}}(A)\} \\ &= E^a(A+R) \cap \sigma_{aw}(A) = E^a(A+R) \cap \sigma_{aw}(A+R) \\ &= \{E^a(A+R) \cap \sigma_{uBw}(A+R)\} \cup \{E^a(A+R) \cap \sigma_{uBw}^{\text{iso}}(A+R)\} \\ &\Longrightarrow E^a(A+R) \cap \sigma_{uBw}^{\text{iso}}(A+R) = \emptyset. \end{aligned}$$

Hence $\lambda \in E^a(A+R)$ implies $\lambda \in \sigma_{uBw}(A+R)^{\mathcal{C}}$; since $A+R$ has SVEP at λ, $\lambda \in \Pi^a(A+R)$. $\square$

An argument similar to that above works for operators $A \in (Bgw)$. Since A^* has SVEP at points in $E_0(A)$, and since $\alpha(A-\lambda) < \infty$ at points $\lambda \in E_0(A)$,

$$\begin{aligned} A \in (Bgw) \quad &\Longleftrightarrow \quad \sigma_a(A) \cap \sigma_{uBw}(A)^{\mathcal{C}} = E_0(A) = \Pi_0(A) = \Pi_0^a(A) \\ &\Leftrightarrow \quad \sigma_a(A) \cap \sigma_{aw}(A)^{\mathcal{C}} = \Pi_0^a(A) = \Pi_0(A) = E_0(A). \end{aligned}$$

Thus $A \in (a - Bt)$ (equivqlently, $A \in (a - gBt)$) and A^* has SVEP on $\Pi_0^a(A)$ for operators $A \in (Bgw)$.

Theorem 8.8(c) *If* $\mathrm{iso}\sigma_\mathrm{a}(\mathrm{A}) = \mathrm{iso}\sigma_\mathrm{a}(\mathrm{A}+\mathrm{R})$*, then* $A \in (Bgw)$ *implies* $A + R \in (Bgw)$ *if and only if* $E_0(A+R) \cap \sigma_{aw}(A) = \emptyset$.

Proof. We start by observing that

$$\begin{aligned} & E_0(A+R) \cap \sigma_{aw}(A+R) = E_0(A+R) \cap \{\sigma_{uBw}(A+R) \cup \Phi_{uBw}^{\mathrm{iso}}(A+R)\} \\ = \quad & \{E_0(A+R) \cap \sigma_{uBw}(A+R)\} \cup \{E_0(A+R) \cap \Phi_{uBw}^{\mathrm{iso}}(A+R)\}. \end{aligned}$$

We claim that $E_0(A+R) \cap \Phi_{uBw}^{\mathrm{iso}}(A+R) = \emptyset$. For if not, then there exists a $\lambda \in E_0(A+R) \cap \sigma_{aw}(A+R)$ such that $0 < \alpha(A+R-\lambda) < \infty$ and $(A+R-\lambda)(\mathcal{X})$ is not closed. Since $\lambda \notin \sigma_{uBw}(A+R)$, there exists an integer $d > 1$ such that $(A+R-\lambda)^d(\mathcal{X})$ is closed. But then, since $\alpha(A+R-\lambda) < \infty$, $d = 1$ – a contradiction. Hence

$$E_0(A+R) \cap \sigma_{aw}(A) = \emptyset \Longleftrightarrow E_0(A+R) \cap \sigma_{uBw}(A+R) = \emptyset.$$

Consequently,

$$\begin{aligned} E_0(A+R) \cap \Pi^a(A+R) &= E_0(A+R) \cap \{\sigma_a(A+R)^{\mathcal{C}} \cup \sigma_{uBw}(A+R)\} \\ &= \{E_0(A+R) \cap \sigma_a(A+R)^{\mathcal{C}}\} \cup \{E_0(A+R) \cap \sigma_{uBw}(A+R)\} \\ &= \emptyset, \end{aligned}$$

and therefore that $\sigma_a(A+R) \cap \sigma_{uBw}(A+R)^{\mathcal{C}} = E_0(A+R)$, i.e., $A+R \in (Bgw)$. □

If $N \in B(\mathcal{X})$ is a nilpotent operator which commutes with A, then $A+N$ and A have the same eigenvalues (the same eigenvalues of finite multiplicity and the same eigenvalues of infinite multiplicity). Hence:

$$\begin{aligned} & E_0^a(A+N) \cap \sigma_w(A+N) = E_0^a(A) \cap \sigma_w(A), \quad \text{and} \\ & E_0(A+N)\sigma_{aw}(A+N) = E_0(A) \cap \sigma_{aw}(A); \end{aligned}$$

also, since $\sigma_{uBw}(.)$ is stable under perturbation by commuting nilpotents (see the proof of Proposition 4.4)

$$E^a(A+N) \cap \sigma_{uBw}(A+N) = E^a(A) \cap \sigma_{uBw}(A).$$

Hence

$$\begin{aligned} & A \in (aw) \;\; (\text{resp. } A \in (gaw), A \in (Bgw)) \Longrightarrow A+N \in (aw) \\ & (\text{resp. } A+N \in (gaw), A+N \in (Bgw)). \end{aligned}$$

The preceding argument works equally well for commuting finite rank operators F such that $\mathrm{iso}_a\sigma_a(A) = \mathrm{iso}_a\sigma(A+F)$. Perturbation by commuting quasinilpotent operators does not result in an as satisfactory a result. Additional hypotheses are required. We summarize this, and the conclusion for nilpotent and finite rank operators in the following.

Corollary 8.9 Let $A \in B(\mathcal{X})$ be such that A commutes with operators N, F and $Q \in B(\mathcal{X})$, where N is a nilpotent, F^n is finite rank for some positive integer n with $\mathrm{iso}\sigma_a(A) = \mathrm{iso}\sigma_a(A+F)$, and Q is a quasinilpotent. Then

$$A \in (aw) \Longleftrightarrow A+X \in (aw), A \in (gaw) \Longleftrightarrow A+X \in (gaw), A \in (Bgw) \Longleftrightarrow A+X \in (Bgw),$$

where $X = N$, or F. Furthermore:

If A is isoloid, then $A \in (Bgw) \Longrightarrow A+Q \in (Bgw)$;

if A is a-isoloid, then $A \in (gaw) \Longrightarrow A+Q \in (gaw)$;

if A is finitely a-isoloid, then $A \in (aw) \Longrightarrow A+Q \in (aw)$.

Proof. Observe that: if A is isoloid, then $\lambda \in E_0(A+Q) \Longrightarrow \lambda \in \text{iso}\sigma(A) \Longrightarrow \lambda \in E_0(A)$; if A is a-isoloid, then $\lambda \in E^a(A+Q) \Longrightarrow \lambda \in \text{iso}\sigma_a(A) \Longrightarrow \lambda \in E^a(A)$; if A is finitely a-isoloid, then $\lambda \in E_0^a(A+Q) \Longrightarrow \lambda \in \text{iso}\sigma_a(A) \Longrightarrow \lambda \in E_0^a(A)$. Hence

$$E_0(A+Q) \cap \sigma_{aw}(A) \subseteq E_0(A) \cap \sigma_{aw}(A) = \emptyset \text{ if } A \in (Bgw),$$

$$E^a(A+Q) \cap \{\text{iso}\sigma_{aw}(A)^{\mathcal{C}} \cap \sigma_{aw}(A)\} \subseteq E^a(A) \cap \sigma_{uBw}(A) = \emptyset \text{ if } A \in (gaw),$$

$$E_0^a(A+Q) \cap \sigma_w(A) \subseteq E_0^a(A) \cap \sigma_w(A) = \emptyset \text{ if } A \in (aw)$$

and the proof is complete. □

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