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Property (Z_{E_a}) for direct sums

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Abstract. We show that generally the properties (Z_{E_a}) and (Z_{Π_a}) introduced by the author are not preserved under direct sum of operators. Moreover, If S and T are Banach spaces operators satisfying property (Z_{E_a}) or (Z_{Π_a}), we give conditions on S and T to ensure the preservation of these properties by the direct sum $S \oplus T$. Some crucial applications are also given.

1. Introduction

For T in the Banach algebra $L(X)$ of bounded linear operators acting on an infinite dimensional complex Banach space X, we will denote by $\sigma(T)$ the spectrum of T, by $\sigma_a(T)$ the approximate point spectrum of T, by $\mathcal{N}(T)$ the null space of T, by $\alpha(T)$ the nullity of T, by $\mathcal{R}(T)$ the range of T and by $\beta(T)$ its defect. If $\alpha(T) < \infty$ and $\beta(T) < \infty$, then T is called a *Fredholm* operator and its index is defined by $\text{ind}(T) = \alpha(T) - \beta(T)$. A *Weyl* operator is a Fredholm operator of index 0 and the Weyl spectrum is defined by $\sigma_w(T) = \{\lambda \in \mathbb{C} \,|\, T - \lambda I \text{ is not a Weyl operator}\}$. $T \in L(X)$ is called a *semi-Fredholm* if $\mathcal{R}(T)$ is closed and $\alpha(T) < \infty$ (resp., $\beta(T) < \infty$).

For a bounded linear operator T and $n \in \mathbb{N}$, let $T_{[n]} : \mathcal{R}(T^n) \to \mathcal{R}(T^n)$ be the restriction of T to $\mathcal{R}(T^n)$. $T \in L(X)$ is said to be *b-Weyl* if for some integer $n \geq 0$, the range $\mathcal{R}(T^n)$ is closed and $T_{[n]}$ is Weyl; its index is defined as the index of the Weyl operator $T_{[n]}$. The respective *b-Weyl spectrum* is defined by $\sigma_{bw}(T) = \{\lambda \in \mathbb{C} \,|\, T - \lambda I \text{ is not a b-Weyl operator}\}$. $T \in L(X)$ is called a *semi-b-Fredholm* if for some integer $n \geq 0$, the range $\mathcal{R}(T^n)$ is closed and $T_{[n]}$ is semi-Fredholm; and the *semi-b-Fredholm spectrum* is defined by $\sigma_{sbf}(T) = \{\lambda \in \mathbb{C} \,|\, T - \lambda I \text{ is not a semi-b-Fredholm operator}\}$, see [4].

The *ascent* of an operator T is defined by $a(T) = \inf\{n \in \mathbb{N} \,|\, \mathcal{N}(T^n) = \mathcal{N}(T^{n+1})\}$, and the *descent* of T is defined by $\delta(T) = \inf\{n \in \mathbb{N} \,|\, \mathcal{R}(T^n) = \mathcal{R}(T^{n+1})\}$, with $\inf \emptyset = \infty$. According to [10], a complex number $\lambda \in \sigma(T)$ is a *pole* of the resolvent of T if $T - \lambda I$ has finite ascent and finite descent, and in this case they are equal. We recall that $T \in L(X)$ is said to be *left Drazin invertible* if $a(T) < \infty$ and $\mathcal{R}(T^{a(T)+1})$ is closed; and the *left-Drazin spectrum* of T is defined by $\sigma_{ld}(T) = \{\lambda \in \mathbb{C} \,|\, T - \lambda I \text{ is not left Drazin invertible}\}$. A complex number $\lambda \in \sigma_a(T)$ is a *left pole* of T if $T - \lambda I$ is left Drazin invertible.

In the following, we recall the definition of a property which has a relevant role in local spectral theory.

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Definition 1.1. *[11] An operator $T \in L(X)$ is said to have the single valued extension property (SVEP) at $\lambda_0 \in \mathbb{C}$, if for every open neighborhood $\mathcal{U}$ of λ_0, the only analytic function $f : \mathcal{U} \longrightarrow X$ which satisfies the equation $(T - \lambda I)f(\lambda) = 0$ for all $\lambda \in \mathcal{U}$ is the function $f \equiv 0$. An operator $T \in L(X)$ is said to have the SVEP if T has the SVEP at every point $\lambda \in \mathbb{C}$.*

It follows easily that $T \in L(X)$ has the SVEP at every point of the boundary $\partial\sigma(T)$ of the spectrum $\sigma(T)$. In particular, T has the SVEP at every point of $\operatorname{iso}\sigma(T)$.

Evidently, $T \in L(X)$ has SVEP at every isolated point of the spectrum. We summarize in the following list the usual notations and symbols needed later.

Notations and symbols:

$\operatorname{iso} A$: isolated points of a subset $A \subset \mathbb{C}$,
$\operatorname{acc} A$: accumulations points of a subset $A \subset \mathbb{C}$,
$D(0, 1)$: the closed unit disc in $\mathbb{C}$,
$C(0, 1)$: the unit circle of $\mathbb{C}$,
$\Pi(T)$: poles of T,
$\Pi^0(T)$: poles of T of finite rank,
$\Pi_a(T)$: left poles of T,
$\Pi_a^0(T)$: left poles of T of finite rank,
$\sigma_p(T)$: eigenvalues of T,
$\sigma_p^0(T)$: eigenvalues of T of finite multiplicity,
$E^0(T) := \operatorname{iso}\sigma(T) \cap \sigma_p^0(T)$,
$E(T) := \operatorname{iso}\sigma(T) \cap \sigma_p(T)$,
$E_a^0(T) := \operatorname{iso}\sigma_a(T) \cap \sigma_p^0(T)$,
$E_a(T) := \operatorname{iso}\sigma_a(T) \cap \sigma_p(T)$,
$\sigma_b(T) = \sigma(T) \setminus \Pi^0(T)$: Browder spectrum of T,
$\sigma_{ub}(T) = \sigma_a(T) \setminus \Pi_a^0(T)$: upper-Browder spectrum of T,
$\sigma_w(T)$: Weyl spectrum of T,
$\sigma_{bw}(T)$: b-Weyl spectrum of T; the symbol $\sqcup$ stands for the disjoint union.

Definition 1.2. *[3], [5], [13], [14] Let $T \in L(X)$. We say that T satisfies:*
i) Property (ab) if $\sigma(T) \setminus \sigma_w(T) = \Pi_a^0(T)$.
ii) Property (gab) if $\sigma(T) \setminus \sigma_{bw}(T) = \Pi_a(T)$.
iii) Property (Bab) if $\sigma(T) \setminus \sigma_{bw}(T) = \Pi_a^0(T)$.
iv) Browder's theorem if $\sigma(T) \setminus \sigma_w(T) = \Pi^0(T)$.
v) Property (Z_{E_a}) if $\sigma(T) \setminus \sigma_w(T) = E_a(T)$.
vi) Property (Z_{Π_a}) if $\sigma(T) \setminus \sigma_w(T) = \Pi_a(T)$.

Definition 1.3. *Let $T \in L(X)$ and $S \in L(X)$. We will say that T and S have a shared stable sign index if for each $\lambda \notin \sigma_{sbf}(T)$ and $\mu \notin \sigma_{sbf}(S)$, $ind(T - \lambda I)$ and $ind(S - \mu I)$ have the same sign.*

For examples we have:

1. Here and elsewhere, $\mathcal{H}$ denotes a Hilbert space. Two hyponormal operators T and S acting on $\mathcal{H}$ have a shared stable sign index, since $\operatorname{ind}(S - \lambda I) \leq 0$ and $\operatorname{ind}(T - \mu I) \leq 0$ for every $\lambda \notin \sigma_{sbf}(S)$ and $\mu \notin \sigma_{sbf}(T)$. Recall that $T \in L(\mathcal{H})$, is said to be *hyponormal* if $T^*T - TT^* \geq 0$ (or equivalently $\|T^*x\| \leq \|Tx\|$ for all $x \in \mathcal{H}$). The class of hyponormal operators includes also *subnormal* operators and *quasinormal* operators, see [7].
2. It is easily verified that if $T \in L(X)$ has SVEP then $\operatorname{ind}(T - \mu I) \leq 0$ for every $\mu \notin \sigma_{sbf}(T)$. So if S and T have SVEP, then they have a shared stable sign index.

In this paper, we focus on the problem of giving conditions on the direct summands to ensure that the Fredholm-type spectral properties introduced very recently by the author in [13], hold for the direct sum. More recently, several authors have worked in this direction, see for examples [6], [8], [9], [12]. The paper is organized as follows: after giving an introduction and some definitions in the first section, we prove in the second section that property (Z_{Π_a}) is not preserved under direct sum of operators and we prove that if S and T satisfy property (Z_{Π_a}) with the supplementary condition $\Pi_a(S) \cap \rho_a(T) = \Pi_a(T) \cap \rho_a(S) = \emptyset$, then $S \oplus T$ satisfies property (Z_{Π_a}) if and only if $\sigma_w(S \oplus T) = \sigma_w(S) \cup \sigma_w(T)$. We obtain an analogous preservation result for property (Z_{E_a}). Some applications to quasisimilar hyponormal operators are given.

2. Properties (Z_{E_a}) and (Z_{Π_a}) for direct sum of operators

We start this section by citing the following two results (see also [13]) which will be used in the proof of the main results of this paper. And in order to give a global overview of the subject, we also include their proofs.

Lemma 2.1. *Let $T \in L(X)$. The following assertions hold:*
i) If T satisfies property (Z_{E_a}), then
$E_a(T) = E_a^0(T) = \Pi_a^0(T) = \Pi_a(T) = \Pi^0(T) = \Pi(T) = E^0(T) = E(T)$.
ii) If T satisfies property (Z_{Π_a}), then $\Pi_a^0(T) = \Pi_a(T) = \Pi^0(T) = \Pi(T)$.

Proof. i) Suppose that T satisfies property (Z_{E_a}), then $\sigma(T) = \sigma_w(T) \sqcup E_a(T)$. Thus $\mu \in E_a(T) \Longleftrightarrow \mu \in \operatorname{iso}\sigma_a(T) \cap \sigma_w(T)^C \Longrightarrow \mu \in \Pi_a^0(T)$, where $\sigma_w(T)^C$ is the complement of the Weyl spectrum of T. Hence $E_a(T) = E_a^0(T) = \Pi_a^0(T) = \Pi_a(T)$, $\Pi(T) = \Pi^0(T)$ and $E(T) = E^0(T)$. Consequently, $\sigma(T) = \sigma_w(T) \sqcup E_a^0(T)$. This implies that $E^0(T) = \Pi^0(T)$. Hence $E_a(T) = E_a^0(T) = \Pi_a^0(T) = \Pi_a(T)$ and $\Pi^0(T) = \Pi(T) = E^0(T) = E(T)$. Since the inclusion $\Pi(T) \subset \Pi_a(T)$ is always true, it suffices to show its opposite. If $\mu \in \Pi_a(T)$, then $a(T - \mu I)$ is finite and since T satisfies property (Z_{E_a}), it follows that $\mu \in \Pi(T)$ and hence the equality desired..
ii) Goes similarly with the proof of the first assertion. □

Theorem 2.2. *Let $T \in L(X)$. The following statements are equivalent:*
i) T satisfies property (Z_{Π_a});
ii) T satisfies property (gab) and $\sigma_{bw}(T) = \sigma_w(T)$;
iii) T satisfies property (ab) and $\Pi_a(T) = \Pi_a^0(T)$;
iv) T satisfies property (Bab) and $\Pi_a(T) = \Pi_a^0(T)$.
v) T satisfies Browder's theorem and $\Pi_a(T) = \Pi^0(T)$.

Proof. (i) $\Longleftrightarrow$ (ii) Suppose that T satisfies property (Z_{Π_a}), that's $\sigma(T) = \sigma_w(T) \sqcup \Pi_a(T)$. From Lemma 2.1, $\sigma(T) = \sigma_w(T) \sqcup \Pi_a^0(T)$. So T satisfies property (ab). As $\Pi(T) = \Pi_a(T)$, then from [5, Theorem 2.8], T satisfies property (gab). Moreover, $\sigma_{bw}(T) = \sigma(T) \setminus \Pi_a(T) = \sigma_w(T)$. The reverse implication is obvious.
(i) $\Longleftrightarrow$ (iii) Follows directly from Lemma 2.1.
(i) $\Longleftrightarrow$ (iv) If T satisfies property (Z_{Π_a}), then $\sigma(T) \setminus \sigma_{bw}(T) = \sigma(T) \setminus \sigma_w(T) = \Pi_a^0(T) = \Pi_a(T)$. So T satisfies property (Bab). Conversely, the property (Bab) for T implies from [14, Theorem 3.6] that $\sigma_{bw}(T) = \sigma_w(T)$. So $\sigma_w(T) = \sigma(T) \setminus \Pi_a^0(T) = \sigma(T) \setminus \Pi_a(T)$ and this means that T satisfies property (Z_{Π_a}). The equivalence between assertions (i) and (v) is clear. □

Now, we give the following proposition which will play an important role in this paper. Hereafter, Y denotes an infinite dimensional complex Banach space.

Proposition 2.3. (See also [12, Lemma 3]) Let $S \in L(X)$ and $T \in L(Y)$. Then

$$\sigma_w(S \oplus T) \subseteq \sigma_w(S) \cup \sigma_w(T).$$

Proof. If $\lambda \notin \sigma_w(S) \cup \sigma_w(T)$ be arbitrary, then $S - \lambda I$ and $T - \lambda I$ are Fredholm operators of index zero. Hence $(S \oplus T) - \lambda I$ is a Fredholm operator and $\operatorname{ind}((S \oplus T) - \lambda I) = \operatorname{ind}(S - \lambda I) + \operatorname{ind}(T - \lambda I) = 0$. So $\lambda \notin \sigma_w(S \oplus T)$ and then $\sigma_w(S \oplus T) \subseteq \sigma_w(S) \cup \sigma_w(T)$. □

Generally, the inclusion showed in Proposition 2.3 is proper.To see this, here and elsewhere the operators R and U are defined on the Hilbert space $\ell^2(\mathbb{N})$ by

$$R(x_1, x_2, \ldots) = (0, x_1, x_2, x_3, \ldots) \text{ and } U(x_1, x_2, x_3, \ldots) = (x_2, x_3, \ldots).$$

Then $\sigma_w(R) = \sigma_w(U) = D(0, 1)$. Since $\alpha(R \oplus U) = \beta(R \oplus U) = 1$, then $0 \notin \sigma_w(R \oplus U)$ and hence $\sigma_w(R \oplus U) \neq \sigma_w(R) \cup \sigma_w(U)$. Observe that this example shows also that $\sigma_{bw}(R \oplus U) \neq \sigma_{bw}(R) \cup \sigma_{bw}(U)$.

However, we have the following corollary:

Corollary 2.4. *Let $S \in L(X)$ and $T \in L(Y)$. The following assertions hold:*
i) If $\sigma_{bw}(S \oplus T) = \sigma_{bw}(S) \cup \sigma_{bw}(T)$, then $\sigma_w(S \oplus T) = \sigma_w(S) \cup \sigma_w(T)$.
ii) If S and T have a shared stable sign index, then $\sigma_w(S \oplus T) = \sigma_w(S) \cup \sigma_w(T)$.
iii) If $S \oplus T$ satisfies Browder's theorem, then $\sigma_w(S \oplus T) = \sigma_w(S) \cup \sigma_w(T)$.

Proof. i) Let $\lambda \notin \sigma_w(S \oplus T)$ be arbitrary and without loss of generality we can assume that $\lambda = 0$. Then $S \oplus T$ is a Weyl operator and so is B-Weyl operator. Thus S and T are B-Weyl operators. Since $\alpha(S) \leq \alpha(S \oplus T) < \infty$ and $\alpha(T) \leq \alpha(S \oplus T) < \infty$, then S and T are Weyl operators. Hence $\sigma_w(S \oplus T) \subset \sigma_w(S) \cup \sigma_w(T)$, and by Proposition 2.3, we conclude that $\sigma_w(S \oplus T) = \sigma_w(S) \cup \sigma_w(T)$.
ii) If S and T have a shared stable sign index, then from [6, Lemma 2.2] we have $\sigma_{bw}(S \oplus T) = \sigma_{bw}(S) \cup \sigma_{bw}(T)$. So $\sigma_w(S \oplus T) = \sigma_w(S) \cup \sigma_w(T)$.
iii) If $S \oplus T$ satisfies Browder's theorem, then $\sigma_w(S \oplus T) = \sigma_b(S \oplus T)$. As $\sigma_b(S \oplus T) = \sigma_b(S) \cup \sigma_b(T)$, then $\sigma_w(S \oplus T) = \sigma_b(S) \cup \sigma_b(T)$. Since the inclusion $\sigma_w(S) \cup \sigma_w(T) \subset \sigma_b(S) \cup \sigma_b(T)$ is always true, we then have $\sigma_w(S) \cup \sigma_w(T) \subset \sigma_w(S \oplus T)$. Hence $\sigma_w(S \oplus T) = \sigma_w(S) \cup \sigma_w(T)$. □

The following example shows that, in general the property (Z_{Π_a}) is not preserved under direct sum of operators.

Example 2.5. *Let $T \in L(\mathbb{C}^n)$ be a nilpotent operator and let $R \in L(\ell^2(\mathbb{N})$ be the operator defined above. Then $\sigma(T) = \{0\}$, $\sigma_w(T) = \emptyset$, $\Pi_a(T) = \{0\}$. Thus $\sigma(T) \setminus \sigma_w(T) = \Pi_a(T)$ and the property (Z_{π_a}) is satisfied by T. Moreover, $\sigma(R) = D(0, 1)$, $\sigma_w(R) = D(0, 1)$, $\Pi_a(R) = \emptyset$. So $\sigma(R) \setminus \sigma_w(R) = \Pi_a(R)$ and R satisfies property (Z_{Π_a}). But their direct sum $T \oplus R$ defined on the Banach space $\mathbb{C}^n \oplus \ell^2(\mathbb{N})$ does not satisfy property (Z_{Π_a}), because $\sigma(T \oplus R) = D(0, 1)$, $\sigma_w(T \oplus R) = D(0, 1)$ and $\Pi_a(T \oplus R) = \{0\}$. Here $\Pi_a(T) \cap \rho_a(R) = \{0\}$ and $\sigma_w(T \oplus R) = \sigma_w(T) \cup \sigma_w(R)$; where $\rho_a(.) = \mathbb{C} \setminus \sigma_a(.)$.*

Nonetheless, in the next theorem we explore certain sufficient conditions which ensure the preservation of property (Z_{Π_a}) under direct sum of operators.

Theorem 2.6. *Suppose that $S \in L(X)$ and $T \in L(Y)$ are such that $\Pi_a(S) \cap \rho_a(T) = \Pi_a(T) \cap \rho_a(S) = \emptyset$. If S and T satisfy property (Z_{Π_a}), then the following assertions are equivalent:*
(i) $S \oplus T$ satisfies property (Z_{Π_a});
(ii) $\sigma_w(S \oplus T) = \sigma_w(S) \cup \sigma_w(T)$.

Proof. (ii) $\Longrightarrow$ (i) Since S and T satisfy property (Z_{Π_a}), we then have

$$\begin{aligned}
[\sigma(S) \cup \sigma(T)] \setminus [\sigma_w(S) \cup \sigma_w(T)] &= [(\sigma(S) \setminus \sigma_w(S)) \cap \rho(T)] \cup [(\sigma(T) \setminus \sigma_w(T)) \cap \rho(S)] \\
&\quad \cup [(\sigma(S) \setminus \sigma_w(S)) \cap (\sigma(T) \setminus \sigma_w(T))] \\
&= [\Pi_a(S) \cap \rho(T)] \cup [\Pi_a(T) \cap \rho(S)] \cup [\Pi_a(S) \cap \Pi_a(T)]
\end{aligned}$$

The assumption $\Pi_a(S) \cap \rho_a(T) = \Pi_a(T) \cap \rho_a(S) = \emptyset$ implies that $\Pi_a(S) \cap \rho(T) = \Pi_a(T) \cap \rho(S) = \emptyset$; where $\rho(.) = \mathbb{C} \setminus \sigma(.)$. Thus

$$[\sigma(S) \cup \sigma(T)] \setminus [\sigma_w(S) \cup \sigma_w(T)] = \Pi_a(S) \cap \Pi_a(T).$$

On the other hand, as we know that $\sigma_{ld}(S \oplus T) = \sigma_{ld}(S) \cup \sigma_{ld}(T)$, we then have

$$\begin{aligned}
\Pi_a(S \oplus T) &= \sigma_a(S \oplus T) \setminus \sigma_{ld}(S \oplus T) \\
&= [\sigma_a(S) \cup \sigma_a(T)] \setminus [\sigma_{ld}(S) \cup \sigma_{ld}(T)] \\
&= [(\sigma_a(S) \setminus \sigma_{ld}(S)) \cap \rho_a(T)] \cup [(\sigma_a(T) \setminus \sigma_{ld}(T)) \cap \rho_a(S)] \\
&\quad \cup [(\sigma_a(S) \setminus \sigma_{ld}(S)) \cap (\sigma_a(T) \setminus \sigma_{ld}(T))] \\
&= [\Pi_a(S) \cap \rho_a(T)] \cup [\Pi_a(T) \cap \rho_a(S)] \cup [\Pi_a(S) \cap \Pi_a(T)] \\
&= \Pi_a(S) \cap \Pi_a(T).
\end{aligned}$$

Hence $\Pi_a(S \oplus T) = [\sigma(S) \cup \sigma(T)] \setminus [\sigma_w(S) \cup \sigma_w(T)]$. As by hypothesis $\sigma_w(S \oplus T) = \sigma_w(S) \cup \sigma_w(T)$, then $\Pi_a(S \oplus T) = \sigma(S \oplus T) \setminus \sigma_w(S \oplus T)$ and this shows that $S \oplus T$ satisfies property (Z_{Π_a}).
(i) $\Longrightarrow$ (ii) If $S \oplus T$ satisfies property (Z_{Π_a}) then from Theorem 2.2, $S \oplus T$ satisfies Browder's theorem. Thus by Corollary 2.4, $\sigma_w(S \oplus T) = \sigma_w(S) \cup \sigma_w(T)$. □

Remark 2.7. *Generally, we cannot ensure the transmission of the property* (Z_{Π_a}) *from two operators S and T to their direct sum even if* $\Pi_a(S) \cap \rho_a(T) = \Pi_a(T) \cap \rho_a(S) = \emptyset$. *For this, the operators R and U defined above satisfy property* (Z_{Π_a}), *because* $\sigma(U) = \sigma_w(U) = D(0,1)$ *and* $\Pi_a(U) = \emptyset$. *But this property is not satisfied by their direct sum, since* $\Pi_a(R \oplus U) = \emptyset$, $\sigma(R \oplus U) = D(0,1)$ *and* $\sigma_w(R \oplus L) \subsetneq D(0,1)$. *Remark that* $\Pi_a(R) \cap \rho_a(U) = \Pi_a(U) \cap \rho_a(R) = \emptyset$.

A bounded linear operator $A \in L(X, Y)$ is said to be *quasi-invertible* if it is injective and has dense range. Two bounded linear operators $T \in L(X)$ and $S \in L(Y)$ on complex Banach spaces X and Y are *quasisimilar* provided there exist quasi-invertible operators $A \in L(X, Y)$ and $B \in L(Y, X)$ such that $AT = SA$ and $BS = TB$.

Corollary 2.8. *If* $S \in L(\mathcal{H})$ *and* $T \in L(\mathcal{H})$ *are quasisimilar hyponormal operators and satisfy property* (Z_{Π_a}), *then* $S \oplus T$ *satisfies property* (Z_{Π_a}).

Proof. As S and T are quasisimilar hyponormal, then by [6, Lemma 2.8] we have $\Pi(T) = \Pi(S)$. The property (Z_{Π_a}) for S and for T entails from Lemma 2.1, that $\Pi(T) = \Pi_a(T)$ and $\Pi(S) = \Pi_a(S)$. So $\Pi_a(S) \cap \rho_a(T) = \Pi_a(T) \cap \rho_a(S) = \emptyset$. Moreover, since S and T are hyponormal operators, then they have a shared stable sign index. This implies from Corollary 2.4 that $\sigma_w(S \oplus T) = \sigma_w(S) \cup \sigma_w(T)$. But this is equivalent from Theorem 2.6, to say that $S \oplus T$ satisfies property (Z_{Π_a}). □

Similarly to theorem 2.6, we prove a preservation result for property (Z_{E_a}) under direct sum of operators. Firstly remark that in general, we cannot expect that property (Z_{E_a}) will be satisfied by the direct sum $S \oplus T$ if its components satisfy property (Z_{E_a}). For instance, we give the following example:

Example 2.9. *Let T and R be the operators defined in Example 2.5, then T and R satisfy property* (Z_{E_a}), *because* $\sigma(T) \setminus \sigma_w(T) = E_a(T) = \{0\}$, $\sigma(R) \setminus \sigma_w(R) = E_a(R) = \emptyset$. *But* $T \oplus R$ *does not satisfy property* (Z_{E_a}), *because* $\sigma(T \oplus R) \setminus \sigma_w(T \oplus R) = \emptyset \neq E_a(T \oplus R) = \{0\}$. *Here, observe that* $\sigma_p(R) = \emptyset$, $\sigma_p(T) = \{0\}$ *and* $\sigma_w(T \oplus R) = \sigma_w(T) \cup \sigma_w(R) = D(0,1)$.

However, we characterize in the next theorem the stability of property (Z_{E_a}) under direct sum via union of Weyl spectra of its summands, which in turn are supposed to have the same eigenvalues. Before this, we recall that $\sigma_p(S \oplus T) = \sigma_p(S) \cup \sigma_p(T)$. Moreover, if A and B are bounded subsets of complex plane $\mathbb{C}$ then $\mathrm{acc}(A \cup B) = \mathrm{acc}(A) \cup \mathrm{acc}(B)$.

Theorem 2.10. *Let* $S \in L(X)$ *and* $T \in L(Y)$ *be such that* $\sigma_p(S) = \sigma_p(T)$. *If S and T satisfy property* (Z_{E_a}), *then the following assertions are equivalent:*
(i) $S \oplus T$ *satisfies property* (Z_{E_a});
(ii) $\sigma_w(S \oplus T) = \sigma_w(S) \cup \sigma_w(T)$.

Proof. (ii) $\Longrightarrow$ (i) Suppose that $\sigma_w(S\oplus T)=\sigma_w(S)\cup\sigma_w(T)$. As S and T satisfy property (Z_{E_a}), i.e. $\sigma(S)\setminus\sigma_w(S)=E_a(S)$ and $\sigma(T)\setminus\sigma_w(T)=E_a(T)$, we then have

$$\begin{aligned}\sigma(S\oplus T)\setminus\sigma_w(S\oplus T) &= [(\sigma(S)\setminus\sigma_w(S))\cap\rho(T)]\cup[(\sigma(T)\setminus\sigma_w(T))\cap\rho(S)]\\ &\quad\cup[(\sigma(S)\setminus\sigma_w(S))\cap(\sigma(T)\setminus\sigma_w(T))]\\ &= [E_a(T)\cap\rho(S)]\cup[E_a(S)\cap\rho(T)]\cup[E_a(S)\cap E_a(T)].\end{aligned}$$

Since by hypothesis $\sigma_p(T)=\sigma_p(S)$, then $E_a(T)\cap\rho_a(S)=E_a(S)\cap\rho_a(T)=\emptyset$ which implies that $E_a(T)\cap\rho(S)=E_a(S)\cap\rho(T)=\emptyset$. Thus

$$\sigma(S\oplus T)\setminus\sigma_w(S\oplus T)=E_a(S)\cap E_a(T).$$

On the other hand, $\sigma_p(S\oplus T)=\sigma_p(S)=\sigma_p(T)$. This implies that

$$\begin{aligned}E_a(S\oplus T) &= \{\mathrm{iso}\,\sigma_a(S\oplus T)\}\cap\sigma_p(S\oplus T)\\ &= \{\mathrm{iso}[\sigma_a(S)\cup\sigma_a(T)]\}\cap\sigma_p(S)\\ &= \{[\sigma_a(S)\cup\sigma_a(T)]\setminus\mathrm{acc}[\sigma_a(S)\cup\sigma_a(T)]\}\cap\sigma_p(S)\\ &= \{[\sigma_a(S)\cup\sigma_a(T)]\setminus[\mathrm{acc}\,\sigma_a(S)\cup\mathrm{acc}\,\sigma_a(T)]\}\cap\sigma_p(S)\\ &= \{[\mathrm{iso}\,\sigma_a(S)\cap\rho_a(T)]\cup[\mathrm{iso}\,\sigma_a(T)\cap\rho_a(S)]\cup[\mathrm{iso}\,\sigma_a(S)\cap\mathrm{iso}\,\sigma_a(T)]\}\cap\sigma_p(S)\\ &= [E_a(S)\cap\rho_a(T)]\cup[E_a(T)\cap\rho_a(S)]\cup[E_a(S)\cap E_a(T)]\\ &= E_a(S)\cap E_a(T).\end{aligned}$$

Hence $\sigma(S\oplus T)\setminus\sigma_w(S\oplus T)=E_a(S\oplus T)$ and this shows that property (Z_{E_a}) is satisfied by $S\oplus T$.
(i) $\Longrightarrow$ (ii) If $S\oplus T$ satisfies property (Z_{E_a}), then by Lemma 2.1, $S\oplus T$ satisfies property (Z_{Π_a}). Therefore we have the equality $\sigma_w(S\oplus T)=\sigma_w(S)\cup\sigma_w(T)$, as seen in the proof of Theorem 2.6. □

Corollary 2.11. *Let $S\in L(X)$ and $T\in L(Y)$ be quasisimilar operators satisfying property (Z_{E_a}). If S or T has SVEP, then $S\oplus T$ satisfies property (Z_{E_a}).*

Proof. The quasisimilarity of S and T implies that $\sigma_p(S)=\sigma_p(T)$. It implies also from [1, Theorem 2.15] that S and T have SVEP. So they have a shared stable sign index and hence $\sigma_w(S\oplus T)=\sigma_w(S)\cup\sigma_w(T)$. But this is equivalent from Theorem 2.10, to say that $S\oplus T$ satisfies property (Z_{E_a}). □

Examples 2.12.

1. A bounded linear operator $T\in L(\mathcal{H})$ is said to be p-hyponormal, with $0<p\leq 1$, if $(T^*T)^p\geq(TT^*)^p$ and is said to be log-hyponormal if T is invertible and satisfies $\log(T^*T)\geq\log(TT^*)$. According to [2], if $T\in L(\mathcal{H})$ is invertible and p-hyponormal, there exists $S\in L(\mathcal{H})$ log-hyponormal quasisimilar to T. Then $\sigma_p(S)=\sigma_p(T)$. Since S has SVEP, then S and T have a shared stable sign index and so $\sigma_w(S\oplus T=\sigma_w(S)\cup\sigma_w(T)$. Moreover, if S and T satisfy property (Z_{E_a}), then $S\oplus T$ satisfies property (Z_{E_a}).
2. Let V denote the Volterra operator on the Banach space $C[0,1]$ defined by $V(f)(x)=\int_0^x f(t)dt$ for all $f\in C[0,1]$. V is injective and quasinilpotent. $\sigma(V)=\sigma_w(V)=\{0\}$ and $\Pi_a(V)=\emptyset$. So V satisfies property (Z_{Π_a}). It is already mentioned that R satisfies property (Z_{Π_a}). As R and V have SVEP, then they have a shared stable sign index. On the other hand, $\Pi_a(R)\cap\rho_a(V)=\Pi_a(V)\cap\rho_a(R)=\emptyset$. Hence $V\oplus R$ satisfies property (Z_{Π_a}).

We finish this paper by posing the following two questions arising from Corollary 2.4.
Let $S\in L(X)$ and $T\in L(Y)$. Is it true that?
1. If $\sigma_w(S\oplus T)=\sigma_w(S)\cup\sigma_w(T)$, then $\sigma_{bw}(S\oplus T)=\sigma_{bw}(S)\cup\sigma_{bw}(T)$.
2. If $\sigma_w(S\oplus T)=\sigma_w(S)\cup\sigma_w(T)$, then $S\oplus T$ satisfies Browder's theorem.

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