Functional Analysis,
Approximation and
Computation
10 (1) (2018), 21–27

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: `http://www.pmf.ni.ac.rs/faac`

Perturbed Browder, Weyl theorems and their variations: An addendum

B. P. Duggal[a], **I. H. Kim**[b]

[a] *8 Redwood Grove, London W5 4SZ, United Kingdom*
[b] *Department of Mathematics, Incheon National University, Incheon, 406-772, Korea*

Abstract. We generalize some results of Zariouh [13] on properties $(\mathbf{Z_{\Pi^a}})$ and $(\mathbf{Z_{E^a}})$ from the direct sum $A \oplus B$ (of Banach space operators A, B) to upper triangular matrix operators with main diagonal $\{A, B\}$ and answer two questions from [13], one of them affirmatively and the other in the negative.

1. Introduction

Let $B(\mathcal{X})$ (resp., $B(\mathcal{H})$) denote the algebra of operators, equivalently bounded linear transformations, on a complex infinite dimensional Banach space $\mathcal{X}$ (resp., Hilbert space $\mathcal{H}$) into itself. An operator $T \in B(\mathcal{X})$ satisfies Browder's theorem if $\sigma(T) \cap \Phi_w(T) = \Pi_0(T)$ (resp., Weyl's theorem if $\sigma(T) \cap \Phi_w(T) = E_0(T)$), where $\sigma(T)$, $\Phi_w(T)$, $\Pi_0(T)$ and $E_0(T)$ denote respectively the spectrum, the complement of the Weyl spectrum $\sigma_w(T)$ of T in the complex plane $\mathbb{C}$, the set of finite rank poles (of the resolvent) of T and the set of isolated points iso(T) of T which are finite multiplicity eigenvalues of T. Browder and Weyl theorems, and their variations, have drawn the attention of a large number of authors in the recent past, and there is a large body of information available on these topics in extant literature (see [1, 5–7, 12, 13] for further references). Let $\Pi^a(T)$ and $E^a(T)$ denote respectively the set of left poles and the set of isolated points of the approximate point spectrum $\sigma_a(T)$ which are eigenvalues of T. Amongst the many variations on Browder, Weyl theorems to have been introduced in the very recent past are the properties $(\mathbf{Z_{\Pi^a}})$ and $(\mathbf{Z_{E^a}})$, where $T \in B(\mathcal{X})$ satisfies property $(\mathbf{Z_{\Pi^a}})$, $T \in (\mathbf{Z_{\Pi^a}})$, if $\sigma(T) \cap \Phi_w(T) = \Pi^a(T)$ and $T \in B(\mathcal{X})$ satisfies property $(\mathbf{Z_{E^a}})$, $T \in (\mathbf{Z_{E^a}})$, if $\sigma(T) \cap \Phi_w(T) = E^a(T)$. Let $\sigma_p(T)$, $\rho_a(T)$ and $\Phi_{Bw}(T)$ denote, respectively, the point spectrum of T, the complement of $\sigma_a(T)$ in $\mathbb{C}$ and the complement of the B-Weyl spectrum $\sigma_{Bw}(T)$ of T in $\mathbb{C}$. In studying direct sums of operators satisfying properties $(\mathbf{Z_{\Pi^a}})$ and $(\mathbf{Z_{E^a}})$, Zariouh [13] proves that if $A, B \in B(\mathcal{X})$ are such that: (i) $A, B \in (\mathbf{Z_{\Pi^a}})$ and $\rho_a(A) \cap \Pi^a(B) = \rho_a(B) \cap \Pi^a(A) = \emptyset$, then $A \oplus B \in (\mathbf{Z_{\Pi^a}})$ if and only if $\sigma_w(A \oplus B) = \sigma_w(A) \cup \sigma_w(B)$; (ii) $A, B \in (\mathbf{Z_{E^a}})$ and $\sigma_p(A) = \sigma_p(B)$, then $A \oplus B \in (\mathbf{Z_{E^a}})$ if and only if $\sigma_w(A \oplus B) = \sigma_w(A) \cup \sigma_w(B)$. (Our notation differs slightly from that of Zariouh: We write $(\mathbf{Z}_{\Pi^a})$ instead of $(\mathbf{Z}_{\Pi_a})$ and $(\mathbf{Z}_{E^a})$ instead of $(\mathbf{Z}_{E_a})$.) Zariouh also raises the question of whether (a) $\sigma_w(A \oplus B) = \sigma_w(A) \cup \sigma_w(B)$ implies $\sigma_{Bw} A \oplus B) = \sigma_{Bw} A) \cup \sigma_{Bw}(B)$, and (b)

2010 *Mathematics Subject Classification*. 47A10, 47A55, 47A53, 47B40.
Keywords. Banach space operator; Fredholm operator; Variations on Browder and Weyl theorems; Properties $(\mathbf{Z_{\Pi^a}})$ and $(\mathbf{Z_{E^a}})$.
Received: 9 February 2018; Accepted: 26 March 2018
Communicated by Dragan S. Djordjević
The work of the second author was supported by a grant from the National Research Foundation of Korea (NRF), funded by the Korean government (No. NRF-2016R1D1A1B03930744).)
Email addresses: `bpduggal@yahoo.co.uk` (B. P. Duggal), `ihkim@inu.ac.kr` (I. H. Kim)

$\sigma_w(A \oplus B) = \sigma_w(A) \cup \sigma_w(B)$ implies $A \oplus B$ satisfies Browder's theorem. In this short addendum to our study of Browder, Weyl theorems and their variations, we generalize the results of Zariouh to upper triangular operator matrices, give an answer to [13, Question (1), Page 6] in the affirmative and give an example to show that [13, Question (2), Page 6] has a negative answer.

2. Notation and terminology

Almost all our terminology in the following is standard, and explained in [7] (see also [1, 9, 10]). In addition to the notation already introduced, we shall use the following further notation. Given a subset S of $\mathbb{C}$, we shall write S^C for the complement of S in $\mathbb{C}$, iso(S) for the isolated points of S and $\rho(A)$ for the resolvent of the operator $A \in B(\mathcal{X})$. The operator $A \in B(\mathcal{X})$ is: *upper semi–Fredholm* at $\lambda \in \mathbb{C}$, $\lambda \in \Phi_+(A)$), if $(A - \lambda I)(\mathcal{X})$ is closed and the deficiency index $\alpha(A-\lambda I) = \dim(\mathrm{A}-\lambda\mathrm{I})^{-1}(0) < \infty$; *lower semi–Fredholm at* $\lambda \in \mathbb{C}$, $\lambda \in \Phi_-(A)$, if $\beta(A-\lambda I) = \dim(\mathcal{X}/(\mathrm{A}-\lambda\mathrm{I})(\mathcal{X})) < \infty$; $A-\lambda I$ is semi–Fredholm, $A-\lambda I \in \Phi_\pm(\mathcal{X})$, if $A-\lambda I$ is either upper or lower semi–Fredholm, and A is Fredholm at $\lambda \in \mathbb{C}$, $\lambda \in \Phi(A)$, if $A - \lambda I$ is both upper and lower semi–Fredholm. The index of a semi–Fredholm operator $A \in B(\mathcal{X})$ is the integer $\mathrm{ind}(A) = \alpha(A) - \beta(A)$. Corresponding to these classes of one sided Fredholm operators, we have the following spectra: The *upper semi Fredholm spectrum* $\sigma_{uf}(A) = \{\lambda \in \sigma(A) : \lambda \notin \Phi_+(A)\}$, the *lower semi Fredholm spectrum* $\sigma_{lf}(A) = \{\lambda \in \sigma(A) : \lambda \notin \Phi_-(A)\}$ and the *Fredholm spectrum* $\sigma_e(A) = \sigma_{uf}(A) \cup \sigma_{lf}(A) = \{\lambda \in \sigma(A) : \lambda \notin \Phi(A)\}$. $A \in B(\mathcal{X})$ is upper Weyl (resp., lower Weyl, (simply) Weyl) at λ, $\lambda \in \Phi_+^-(A)$ (resp., $\lambda \in \Phi_-^+(A)$, $\lambda \in \Phi_w(A)$), if it is upper semi Fredholm with $\mathrm{ind}(A - \lambda I) \leq 0$ (resp., lower semi Fredholm with $\mathrm{ind}(A - \lambda I) \geq 0$, Fredholm with $\mathrm{ind}(A - \lambda I) = 0$). The upper Weyl spectrum, the lower Weyl spectrum and the Weyl spectrum of A are respectively the sets $\sigma_{aw}(A) = \{\lambda \in \sigma_a(A) : \lambda \notin \Phi_+(A) \text{ or } \mathrm{ind}(A-\lambda I) \not\leq 0\} = \{\lambda \in \sigma_a(A) : \lambda \notin \Phi_+^-(A)\}$, $\sigma_{sw}(A) = \{\lambda \in \sigma_s(A) : \lambda \notin \Phi_-(A)$ or $\mathrm{ind}(A - \lambda I) \not\geq 0\} = \{\lambda \in \sigma_s(A) : \lambda \notin \Phi_-^+(A)\}$ and $\sigma_w(A) = \sigma_{aw}(A) \cup \sigma_{sw}(A)$. (Here $\sigma_s(A)$ is the surjectivity spectrum of A.)

A generalization of Fredholm and Weyl spectrum is obtained as follows. An operator $A \in B(\mathcal{X})$ *is semi B-Fredholm* if there exists an integer $n \geq 1$ such that $A^n(\mathcal{X})$ is closed and the induced operator $A_{[n]} = A|_{A^n(\mathcal{X})}$, $A_{[0]} = A$, is semi Fredholm (in the usual sense). It is seen that if $A_{[n]} \in \Phi_\pm(\mathcal{X})$ for an integer $n \geq 1$, then $A_{[m]} \in \Phi_\pm(\mathcal{X})$ for all integers $m \geq n$, and one may unambiguously define the index of A by $\mathrm{ind}(A) = \alpha(A)-\beta(A)$ $(= \mathrm{ind}(A_{[n]}))$ (see any of [2–4, 6, 7, 12, 13] for further information). Upper semi B-Fredholm, lower semi B-Fredholm and B-Fredholm spectra of A are then the sets

$\sigma_{uBf}(A) = \{\lambda \in \sigma(A) : A - \lambda$ is not upper semi B-Fredholm$\}$,

$\sigma_{lBf}(A) = \{\lambda \in \sigma(A) : A - \lambda$ is not lower semi B-Fredholm$\}$, and

$\sigma_{Bf}(A) = \sigma_{uBf}(A) \cup \sigma_{lBf}(A)$.

Letting

$\Phi_{Bw}(A) = \{\lambda \in \sigma(A) : \lambda \notin \sigma_{Bf}(A) \text{ and } \mathrm{ind}(A - \lambda) = 0\}$,

$\Phi_{uBw}(A) = \{\lambda \in \sigma_a(A) : \lambda \notin \sigma_{uBf}(A) \text{ and } \mathrm{ind}(A - \lambda) \leq 0\}$,

$\Phi_{lBw}(A) = \{\lambda \in \sigma_s(A) : \lambda \notin \sigma_{lBf}(A) \text{ and } \mathrm{ind}(A - \lambda) \geq 0\}$

denote, respectively, the *the B-Weyl, the upper B-Weyl and the lower B-Weyl points* of A, we define the B-Weyl, the upper B-Weyl and the lower B-Weyl spectrum of A, respectively, by

$\sigma_{Bw}(A) = \{\lambda \in \sigma(A) : \lambda \notin \Phi_{Bw}(A)\}$,

$\sigma_{uBw}(A) = \{\lambda \in \sigma_a(A) : \lambda \notin \Phi_{uBw}(A)\}$,

$\sigma_{lBw}(A) = \{\lambda \in \sigma_s(A) : \lambda \notin \Phi_{lBw}(A)\}$.

Clearly, $\sigma_{Bw}(A) = \sigma_{uBw}(A) \cup \sigma_{lBw}(A)$ and $\sigma_{lBw}(A) = \sigma_{uBw}(A^*)$.

3. Results

Given operators A, B and C in $B(\mathcal{X})$, define the direct sum operator S, and the upper triangular operator T, $\in B(\mathcal{X} \oplus \mathcal{X})$ by

$$S = A \oplus B \text{ and } T = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}.$$

Then

$$\sigma_x(S) = \sigma_x(A) \cup \sigma_x(B), \quad \sigma_x = \sigma \text{ or } \sigma_a,$$

but a similar spectral equality fails for the operator T. Neither of the operators S and T satisfies a spectral equality of type $\sigma_x(Z) = \sigma_x(A) \cup \sigma_x(B)$, $Z = S$ or T, for $\sigma_x = \sigma_w$ or σ_{aw}. However, if either $\sigma_w(T) = \sigma_w(A) \cup \sigma_w(B)$ or $\sigma_{aw}(T) = \sigma_{aw}(A) \cup \sigma_{aw}(B)$, then $\sigma(T) = \sigma(A) \cup \sigma(B)$ [5, Theorem 2.3]. The following proposition says that one may replace σ_w and σ_{aw} by σ_{Bw} and σ_{uBw}, respectively.

Proposition 3.1. *If $\sigma_x(T) = \sigma_x(A) \cup \sigma_x(B)$, where $\sigma_x = \sigma_{Bw}$ or σ_{uBw}, then $\sigma(T) = \sigma(A) \cup \sigma(B)$.*

Proof. We start by considering the case $\sigma_x = \sigma_{uBw}$. Let $\lambda \notin \sigma(T)$. Then $\lambda \in \Phi_+^-(A) \cap \Phi_-^+(B)$, $A - \lambda I$ is injective and $B - \lambda I$ is surjective. Since $\lambda \notin \sigma(T)$ ensures $\lambda \notin \sigma_{uBw}(T)$, $\lambda \in \Phi_{uBw} \cap \Phi_{uBw}(B)$. Consequently, $\lambda \in \Phi_-^+(B) \cap \Phi_{uBw}(B)$. But then $\lambda \in \Phi_w(B)$ and $\mathrm{ind}(B - \lambda I) = 0$. Already we have $\beta(B - \lambda I) = 0$; hence $\alpha(B - \lambda I) = \beta(B - \lambda I) = 0$ and $B - \lambda I$ is invertible. This, in view of the fact that already $T - \lambda I$ is invertible, implies $A - \lambda I$ is also invertible. Hence $\sigma(T) \supseteq \sigma(A) \cup \sigma(B)$. Since $\sigma(T) \subseteq \sigma(A) \cup \sigma(B)$ always, $\sigma(T) = \sigma(A) \cup \sigma(B)$.

A bit more is true in the case in which $\sigma_{Bw}(T) = \sigma_{Bw}(A) \cup \sigma_{Bw}(B)$. We prove in this case that $\sigma_w(T) = \sigma_w(A) \cup \sigma_w(B)$ (and the proof then follows from [5, Theorem 2.3]). Let $\lambda \notin \sigma_w(T)$. Then $\lambda \in \Phi_w(T) \subseteq \Phi_{Bw}(T) = \Phi_{Bw}(A) \cap \Phi_{Bw}(B)$. Since $\lambda \notin \sigma_w(T)$ implies $\alpha(A - \lambda) < \infty$, $\lambda \in \Phi_{Bw}(A)$ implies $\lambda \in \Phi_w(A)$; hence, since already $\lambda \in \Phi_w(T)$, $\lambda \in \Phi_w(B)$. Conclusion: $\lambda \in \Phi_w(A) \cap \Phi_w(B)$ for all $\lambda \in \Phi_w(T)$, hence $\Phi_w(T) \subseteq \Phi_w(A) \cap \Phi_w(B)$. Trivially, consider the operator

$$T - \lambda I = \begin{pmatrix} I & 0 \\ 0 & B - \lambda I \end{pmatrix} \begin{pmatrix} I & C \\ 0 & I \end{pmatrix} \begin{pmatrix} A - \lambda I & 0 \\ 0 & I \end{pmatrix},$$

$\Phi_w(A) \cap \Phi_w(B) \subseteq \Phi_w(T)$. Hence $\Phi_w(A) \cap \Phi_w(B) = \Phi_w(T)$, equivalently, $\sigma_w(T) = \sigma_w(A) \cup \sigma_w(B)$. □

The reverse implication $\sigma_w(T) = \sigma_w(A) \cup \sigma_w(B)$ implies $\sigma_{Bw}(T) = \sigma_{Bw}(A) \cup \sigma_{Bw}(B)$ holds under additional hypotheses. For an operator $E \in B(\mathcal{X})$, let $\Phi^i_{Bw}(E) = \mathrm{iso}\sigma_{\mathrm{w}}(\mathrm{E}) \cap \Phi_{\mathrm{Bw}}(\mathrm{E})$.

Proposition 3.2. *If $\sigma_w(T) = \sigma_w(A) \cup \sigma_w(B)$, then a sufficient condition for $\sigma_{Bw}(T) = \sigma_{Bw}(A) \cup \sigma_{Bw}(B)$ is that $\sigma_{Bw}(T) \subseteq \sigma_{Bw}(A) \cup \sigma_{Bw}(B)$ and $\{\sigma_{Bw}(A) \cup \sigma_{Bw}(B)\} \cap \Phi^i_{Bw}(T) = \emptyset$. .*

Proof. Recall from [7], Pages 41–42, that

$$\sigma_w(E) = \sigma_{Bw}(E) \cup \Phi^i_{Bw}(E), \quad \sigma_{Bw}(E) \cap \Phi^i_{Bw}(E) = \emptyset, \quad \Phi^i_{Bw}(E) = \mathrm{iso}\sigma_{\mathrm{w}}(\mathrm{E}) \cap \Phi_{\mathrm{Bw}}(\mathrm{E})$$

for every $E \in B(\mathcal{X})$. Hence, given $\sigma_w(T) = \sigma_w(A) \cup \sigma_w(B)$,

$$\begin{aligned} \sigma_{Bw}(T) \cup \Phi^i_{Bw}(T) &= \{\sigma_{Bw}(A) \cup \Phi^i_{Bw}(A)\} \cup \{\sigma_{Bw}(B) \cup \Phi^i_{Bw}(B)\} \\ &= \{\sigma_{Bw}(A) \cup \sigma_{Bw}(B)\} \cup \{\Phi^i_{Bw}(A) \cup \Phi^i_{Bw}(B)\}. \end{aligned}$$

This , since $\sigma_{Bw}(T) \cap \Phi^i_{Bw}(T) = \emptyset$ and

$$\{\sigma_{Bw}(A) \cup \sigma_{Bw}(B)\} \cap \Phi^i_{Bw}(T) = \emptyset \Longrightarrow \{\sigma_{Bw}(A) \cup \sigma_{Bw}(B)\} \subseteq \Phi^i_{Bw}(T)^C,$$

implies

$$\begin{aligned} \sigma_{Bw}(T) &= \{(\sigma_{Bw}(A) \cup \sigma_{Bw}(B)) \cap \Phi^i_{Bw}(T)^C\} \cup \{(\Phi^i_{Bw}(A) \cup \Phi^i_{Bw}(B)) \cap \Phi^i_{Bw}(T)^C\} \\ &\supseteq (\sigma_{Bw}(A) \cup \sigma_{Bw}(B)) \cap \Phi^i_{Bw}(T)^C \\ &= \sigma_{Bw}(A) \cup \sigma_{Bw}(B). \end{aligned}$$

Already, by hypothesis, $\sigma_{Bw}(T) \subseteq \sigma_{Bw}(A) \cup \sigma_{Bw}(B)$; hence $\sigma_{Bw}(T) = \sigma_{Bw}(A) \cup \sigma_{Bw}(B)$. □

Proposition 3.2 answers Problem 1, Page 6 of [13] in the affirmative.

Theorem 3.3. $\sigma_w(S) = \sigma_w(A) \cup \sigma_w(B)$ *implies* $\sigma_{Bw}(S) = \sigma_{Bw}(A) \cup \sigma_{Bw}(B)$.

Proof. We start by proving $\sigma_{Bw}(S) \subseteq \sigma_{Bw}(A) \cup \sigma_{Bw}(B)$: this does not require the hypothesis $\sigma_w(S) = \sigma_w(A) \cup \sigma_w(B)$. Define the operators $V, W \in B(\mathcal{X} \oplus \mathcal{X})$ by $V = A \oplus I$ and $W = I \oplus B$. Then $S = VW$, where V and W commute. If we now assume that A, B are B-Fredholm operators, then V and W, hence also S [4], are B-Fredholm operators. Furthermore, [2, Theorem 1.1], $\text{ind}(S) = \text{ind}(V) + \text{ind}(W) = \text{ind}(A) + \text{ind}(B)$. Consequently, if A and B are B-Weyl operators, then S is a B-Weyl operator satisfying $\Phi_{Bw}(A) \cap \Phi_{Bw}(B) \subseteq \Phi_{Bw}(S)$. Equivalently, $\sigma_{Bw}(S) \subseteq \sigma_{Bw}(A) \cup \sigma_{Bw}(B)$. To complete the proof we have now to prove that $\{\sigma_{Bw}(A) \cup \sigma_{Bw}(B)\} \cap \Phi^i_{Bw}(S) = \emptyset$. For this we start by observing that $\text{iso}\sigma_{\text{w}}(\text{S}) = \{\text{iso}\sigma_{\text{w}}(\text{A}) \cap \Phi_{\text{w}}(\text{B})\} \cup \{\text{iso}\sigma_{\text{w}}(\text{A}) \cap \text{iso}\sigma_{\text{w}}(\text{B})\} \cup \{\Phi_{\text{w}}(\text{A}) \cap \text{iso}\sigma_{\text{w}}(\text{B})\}$. Recall, [3], that the operator $S = VW$, operators V, W as above, is B-Fredholm if and only if V, W (hence also, A, B) are B-Fredholm (and then $\text{ind}(S) = \text{ind}(V) + \text{ind}(W) = \text{ind}(A) + \text{ind}(B)$). Thus, if a $\lambda \in \Phi^i_{Bw}(S)$ satisfies the hypothesis $\lambda \in \text{iso}\sigma_{\text{w}}(\text{A}) \cap \Phi_{\text{Bw}}(\text{B})$ or $\lambda \in \Phi_{Bw}(A) \cap \text{iso}\sigma_{\text{w}}(\text{B})$, then $\lambda \in \Phi_{Bw}(A) \cap \Phi_{Bw}(B)$, i.e., $\lambda \notin \sigma_{Bw}(A) \cup \sigma_{Bw}(B)$. This leaves us with the case $\lambda \in \text{iso}\sigma_{\text{w}}(\text{A}) \cap \text{iso}\sigma_{\text{w}}(\text{B})$ such that $\lambda \in \Phi_{Bw}(T)$. (Then both $A - \lambda I$ and $B - \lambda I$ are B-Fredholm and there exists a neighbourhood of λ such that $\text{ind}(A - \mu I) = \text{ind}(B - \mu I) = 0$ in the deleted neighbourhood.) There exists an $\epsilon > 0$ and an ϵ–neighbourhood of λ such that $A - \lambda I - \mu I$ is Fredholm, with $\text{ind}(A - \lambda I - \mu I) = \text{ind}(A - \lambda I)$, for all $0 < |\mu| < \epsilon$ [8]. Hence $A - zI$ is Fredholm with $ind(A - zI) = \text{ind}(A - \lambda I) = 0$ for every z such that $0 < |z - \lambda| < \epsilon$. Since a similar argument works for B, we conclude that $\lambda \in \Phi_{Bw}(A) \cap \Phi_{Bw}(B)$. This implies $\lambda \notin \sigma_{Bw}(A) \cup \sigma_{Bw}(B)$ for every $\lambda \in \Phi^i_{Bw}(T)$. □

A result similar to that of Proposition 3.2 holds for operators T such that $\sigma_{uBw}(T) \subseteq \sigma_{uBw}(A) \cup \sigma_{uBw}(B)$. Given $E \in B(\mathcal{X})$, let $\Phi^i_{uBw}(E) = \text{iso}\sigma_{\text{aw}}(\text{E}) \cap \Phi_{\text{uBw}}(\text{E})$.

Proposition 3.4. *If $\sigma_{uBw}(T) \subseteq \sigma_{uBw}(A) \cup \sigma_{uBw}(B)$ and $(\sigma_{uBw}(A) \cup \sigma_{uBw}(B)) \cap \Phi^i_{uBw}(T) = \emptyset$, then $\sigma_{aw}(T) = \sigma_{aw}(A) \cup \sigma_{aw}(B)$ implies $\sigma_{uBw}(T) = \sigma_{uBw}(A) \cup \sigma_{uBw}(B)$.*

Proof. The proof being similar to that of Proposition 3.2, we shall be brief. Recall from [7] that

$$\sigma_{aw}(E) = \sigma_{uBw}(E) \cup \Phi^i_{uBw}(E), \quad \sigma_{uBw}(E) \cap \Phi^i_{uBw}(E) = \emptyset$$

for every $E \in B(\mathcal{X})$. If $\sigma_{aw}(T) = \sigma_{aw}(A) \cup \sigma_{aw}(B)$, then

$$\begin{aligned}
\sigma_{uBw}(T) &= \{(\sigma_{uBw}(A) \cup \sigma_{uBw}(B)) \cup (\Phi^i_{uBw}(A) \cup \Phi^i_{uBw}(B))\} \cap \Phi^i_{uBw}(T)^C \\
&= \{(\sigma_{uBw}(A) \cup \sigma_{uBw}(B)) \cap \Phi^i_{uBw}(T)^C\} \cup \{(\Phi^i_{uBw}(A) \cup \Phi^i_{uBw}(B)) \cap \Phi^i_{uBw}(T)^C\} \\
&\supseteq (\sigma_{uBw}(A) \cup \sigma_{uBw}(B)) \cap \Phi^i_{uBw}(T)^C \\
&= \sigma_{uBw}(A) \cup \sigma_{uBw}(B).
\end{aligned}$$

The proof now follows from the hypothesis $\sigma_{uBw}(T) \subseteq \sigma_{uBw}(A) \cup \sigma_{uBw}(B)$. □

Following (almost verbatim, having made sure to replace "B-Fredholm" by "semi B-Fredholm") the argument of the proof of [2, Theorem 1.1], it is seen that S is upper semi B-Fredholm if and only if (V and W, hence) A and B are upper semi B-Fredholm, and then $\text{ind}(S) = \text{ind}(V) + \text{ind}(W) = \text{ind}(A) + \text{ind}(B)$. This, if A and B are upper semi B-Weyl, implies S is upper semi B-Weyl, i.e., $\Phi_{uBw}(A) \cap \Phi_{uBw}(B) \subseteq \Phi_{uBw}(S)$. Equivalently, $\sigma_{uBw}(S) \subseteq \sigma_{uBw}(A) \cup \sigma_{uBw}(B)$. The following theorem is an analogue of Theorem 3.3 for upper semi B-Weyl spectrum.

Theorem 3.5. *$\sigma_{aw}(S) = \sigma_{aw}(A) \cup \sigma_{aw}(B)$ implies $\sigma_{uBw}(S) = \sigma_{uBw}(A) \cup \sigma_{uBw}(B)$.*

Proof. We have already seen that $\sigma_{uBw}(S) \subseteq \sigma_{uBw}(A) \cup \sigma_{uBw}(B)$. For the reverse inclusion, we start by observing that

$$\text{iso}\sigma_{\text{aw}}(\text{S}) = \{\text{iso}\sigma_{\text{aw}}(\text{A}) \cap \Phi_{\text{aw}}(\text{B})\} \cup \{\text{iso}\sigma_{\text{aw}}(\text{A}) \cap \text{iso}\sigma_{\text{aw}}(\text{B})\} \cup \{\Phi_{\text{aw}}(\text{A}) \cap \text{iso}\sigma_{\text{aw}}(\text{B})\}.$$

Here, argue as in the proof of Theorem 3.3, to conclude

$$\lambda \in \{\text{iso}\sigma_{\text{aw}}(\text{A}) \cap \Phi_{\text{aw}}(\text{B})\} \cup \{\Phi_{\text{aw}}(\text{A}) \cap \text{iso}\sigma_{\text{aw}}(\text{B})\} \Longrightarrow \lambda \in \Phi_{\text{uBw}}(\text{A}) \cap \Phi_{\text{uBw}}(\text{B}).$$

This leaves us the case $\lambda \in \text{iso}\sigma_{\text{aw}}(\text{A}) \cap \text{iso}\sigma_{\text{aw}}(\text{B})$ such that $\lambda \in \Phi_{uBw}(T)$. Then both $A - \lambda I$ and $B - \lambda I$ are upper semi B-Fredholm and there exists a neighbourhood of λ such that $\text{ind}(A - \mu I), \ \text{ind}(B - \mu I) \leq 0$ in the deleted neighbourhood. There exists an $\epsilon > 0$ and an ϵ–neighbourhood of λ such that $A - \lambda I - \mu I$ is upper semi Fredholm, with $\mathrm{ind}(A - \lambda I - \mu I) = \mathrm{ind}(A - \lambda I)$, for all $0 < |\mu| < \epsilon$ [3]. Hence $A - zI$ is upper semi Fredholm with $\mathrm{ind}(A - zI) = \mathrm{ind}(A - \lambda I) \leq 0$ for every z such that $|z - \lambda| < \epsilon$. Since a similar argument works for B, we conclude that $\lambda \in \Phi_{uBw}(A) \cap \Phi_{uBw}(B)$. $\square$

Remark 3.6. The following argument shows that $\sigma_{uBw}(S) = \sigma_{uBw}(A) \cup \sigma_{uBw}(B)$ implies $\sigma_{aw}(S) = \sigma_{aw}(A) \cup \sigma_{aw}(B)$. Consider a $\lambda \notin \sigma_{aw}(S)$. Then $\lambda \in \Phi_{aw}(S) \subseteq \Phi_{uBw}(S) = \Phi_{uBw}(A) \cap \Phi_{uBw}(B)$. Since $\lambda \notin \sigma_{aw}(S)$ implies $\alpha(A - \lambda I)$ and $\alpha(B - \lambda I)$ are finite, $\lambda \in \Phi_{uBw}(A) \cap \Phi_{uBw}(B)$ implies $\lambda \in \Phi_+^-(A) \cap \Phi_+^-(B)$, equivalently, $\lambda \notin \sigma_{aw}(A) \cup \sigma_{aw}(B)$. Thus $\sigma_{aw}(A) \cup \sigma_{aw}(B) \subseteq \sigma_{aw}(S)$. Since the reverse inclusion is always true, we have equality. (A similar argument proves that $\sigma_{lBw}(S) = \sigma_{lBw}(A) \cup \sigma_{lBw}(B)$ implies $\sigma_{sw}(S) = \sigma_{sw}(A) \cup \sigma_{sw}(B)$.) A natural question here is: Does $\sigma_{uBw}(T) = \sigma_{uBw}(A) \cup \sigma_{uBw}(B)$ imply $\sigma_{aw}(T) = \sigma_{aw}(A) \cup \sigma_{aw}(B)$?

We consider next operators T satisfying $T \in (\mathbf{Z}_{\Pi^a})$ and $T \in (\mathbf{Z}_{E^a})$. It is clear that if $T \in (\mathbf{Z}_{\Pi^a})$, then $\Pi^a(T) = \Pi(T) = \Pi_0(T) = \Pi_0^a(T)$ and if $T \in (\mathbf{Z}_{E^a})$, then $E^a(T) = E(T) = E_0(T) = E_0^a(T)$. Let $\mathrm{asc}(T - \lambda I)$ and $\mathrm{dsc}(T - \lambda I)$ denote, respectively the ascent and the descent of T at λ. Then, see [11, Exercise 7, P. 293],

$$\mathrm{asc}(A - \lambda I) \leq \mathrm{asc}(T - \lambda I) \leq \mathrm{asc}(A - \lambda I) + \mathrm{asc}(B - \lambda I) \text{ and}$$
$$\mathrm{dsc}(B - \lambda I) \leq \mathrm{dsc}(T - \lambda I) \leq \mathrm{dsc}(A - \lambda I) + \mathrm{dsc}(B - \lambda I)$$

for all complex λ. Assume that the operators $A, B \in (\mathbf{Z}_{\Pi^a})$; then $\Pi^a(A) = \Pi_0(A)$ and $\Pi^a(B) = \Pi_0(B)$. Assume now that $T \in (\mathbf{Z}_{\Pi^a})$, and consider a $\lambda \in \Pi^a(T) = \Pi_0(T)$. Since $\lambda \in \Phi(T)$ implies $\lambda \in \Phi_+(A) \cap \Phi_-(B)$ and $\lambda \in \Pi^a(T)$ implies $\mathrm{asc}(A - \lambda I) < \infty$, $\lambda \in \Pi^a(A) = \Pi_0(A)$. In particular, $\lambda \in \Phi(A)$, and this (since already $\lambda \in \Phi(T)$) implies ($\lambda \in \Phi(B)$, and hence since $\mathrm{dsc}(B - \lambda I) < \infty$) $\lambda \in \Pi_0(B) = \Pi^a(B)$. Since $\lambda \in \Pi^a(A) \cap \Pi^a(B)$ trivially implies $\lambda \in \Pi^a(T)$, we have:

If $A, B \in (\mathbf{Z}_{\Pi^a})$, then $T \in (\mathbf{Z}_{\Pi^a})$ implies $\Pi^a(T) = \Pi^a(A) \cap \Pi^a(B)$.

Given operators $A, B \in (\mathbf{Z}_{\Pi^a})$ such that $\Pi^a(A) \cap \rho(B) = \emptyset = \Pi^a(B) \cap \rho(A)$, the following theorem gives a sufficient, and an almost necessary, condition for $T \in (\mathbf{Z}_{\Pi^a})$.

Theorem 3.7. *Let the operators $A, B \in (\mathbf{Z}_{\Pi^a})$ be such that $\Pi^a(A) \cap \rho(B) = \emptyset = \Pi^a(B) \cap \rho(A)$.*

(i) If $\sigma_w(T) = \sigma_w(A) \cup \sigma_w(B)$, then $\sigma(T) \cap \Phi_w(T) = \Pi^a(A) \cap \Pi^a(B)$.

(ii) If $\sigma(T) = \sigma(A) \cup \sigma(B)$ and $T \in (\mathbf{Z}_{\Pi^a})$, then $\sigma_w(T) = \sigma_w(A) \cup \sigma_w(B)$.

Proof. The hypothesis $A, B \in (\mathbf{Z}_{\Pi^a})$ implies

$$\Phi_w(A) = \rho(A) \cup \Pi^a(A) \text{ and } \Phi_w(B) = \rho(B) \cup \Pi^a(B),$$

where $\Pi^a(A) = \Pi_0(A)$ and $\Pi^a(B) = \Pi_0(B)$.

(i) Recall from Proposition 3.1 that the hypothesis $\sigma_w(T) = \sigma_w(A) \cup \sigma_w(B)$ implies $\sigma(T) = \sigma(A) \cup \sigma(B)$, and hence

$$\begin{aligned}
\sigma(T) \cap \Phi_w(T) &= \{\sigma(A) \cup \sigma(B)\} \cap \{\Phi_w(A) \cap \Phi_w(B)\} \\
&= \{\Phi_w(A) \cap \Phi_w(B) \cap \sigma(A)\} \cup \{\Phi_w(A) \cap \Phi_w(B) \cap \sigma(B)\} \\
&= \{\Pi^a(A) \cap \Phi_w(B)\} \cup \{\Pi^a(B) \cap \Phi_w(A)\} \\
&= \{(\Pi^a(A) \cap \Phi_w(B)) \cup \Pi^a(B)\} \cap \{(\Pi^a(A) \cap \Phi_w(B)) \cup \Phi_w(A)\} \\
&= (I) \cap (II) \quad \text{(say)}.
\end{aligned}$$

Recalling $\Pi^a(A) \cap \rho(B) = \emptyset = \Pi^a(B) \cap \rho(A)$, we simplify to obtain:

$$\begin{aligned}
(I) &= \{\Pi^a(A) \cap (\Pi^a(B) \cup \rho(B))\} \cup \Pi^a(B) \\
&= \{(\Pi^a(A) \cap \Pi^a(B)) \cup (\Pi^a(A) \cap \rho(B))\} \cup \Pi^a(B) \\
&= (\Pi^a)(A) \cap \Pi^a(B))) \cup \Pi^a(B) = \Pi^a(B),
\end{aligned}$$

$$\begin{aligned}(II) &= \{\Pi^a(A) \cap (\Pi^a(B) \cup \rho(B))\} \cup \Phi_w(A)\\ &= \{(\Pi^a(A) \cap \Pi^a(B)) \cup (\Pi^a(A) \cap \rho(B))\} \cup \Phi_w(A)\\ &= (\Pi^a(A) \cap \Pi^a(B)) \cup \Phi_w(A),\end{aligned}$$

and

$$\begin{aligned}(I) \cap (II) &= \{(\Pi^a(A) \cap \Pi^a(B)) \cap \Pi^a(B)\} \cup \{\Phi_w(A) \cap \Pi^a(B)\}\\ &= (\Pi^a(A) \cap \Pi^a(B)) \cup \{(\Pi^a(A) \cup \rho(A)) \cap \Pi^a(B)\}\\ &= (\Pi^a(A) \cap \Pi^a(B)) \cup \{(\Pi^a(A) \cap \Pi^a(B)) \cup (\rho(A) \cap \Pi^a(B))\}\\ &= \Pi^a(A) \cap \Pi^a(B).\end{aligned}$$

(ii) If $T \in (\mathbf{Z}_{\Pi^a})$, then (as seen above) $\Pi^a(T) = \Pi^a(A) \cap \Pi^a(B)$. Working backwards with the argument of the proof of (i) above, it is seen that if $A, B, T \in (\mathbf{Z}_{\Pi^a})$ and $\sigma(T) = \sigma(A) \cup \sigma(B)$, then

$$\sigma(T) \cap \Phi_w(T) = \Pi^a(T) = \Pi^a(A) \cap \Pi^a(B) = \sigma(T) \cap \{\Phi_w(A) \cap \Phi_w(B)\}.$$

Observe that $\sigma(T) = \sigma(A) \cup \sigma(B)$ implies $\rho(T) \subseteq \Phi_w(A) \cap \Phi_w(B)$. Hence. since

$$\rho(T) \cup \{\sigma(T) \cap \Phi_w(T)\} = \mathbb{C} \cap \{\rho(T) \cup \Phi_w(T)\} = \mathbb{C} \cap \Phi_w(T) = \Phi_w(T)$$

and

$$\begin{aligned}&\rho(T) \cup \{\sigma(T) \cap (\Phi_w(A) \cap \Phi_w(B))\} = \mathbb{C} \cap \{\rho(T) \cup (\Phi_w(A) \cap \Phi_w(B))\}\\ =\ &\Phi_w(A) \cap \Phi_w(B) \subseteq \Phi_w(T),\end{aligned}$$

we have

$$\Phi_w(T) = \Phi_w(A) \cap \Phi_w(B), \quad \text{equivalently,} \quad \sigma_w(T) = \sigma_w(A) \cup \sigma_w(B).$$

This completes the proof. □

Theorem 3.7 subsumes [13, Theorem 2.6]. Recall that $\rho_a(R)$ denotes the complement of $\sigma_a(R)$ in $\mathbb{C}$.

Corollary 3.8. *If $A, B \in (\mathbf{Z}_{\Pi^a})$ and $\rho_a(A) \cap \Pi^a(B) = \rho_a(B) \cap \Pi^a(A) = \emptyset$, then $S \in (\mathbf{Z}_{\Pi^a})$ if and only if $\sigma_w(S) = \sigma_w(A) \cup \sigma_w(B)$.*

Proof. The proof follows from Theorem 3.7 since the hypotheses imply $\sigma(S) = \sigma(A) \cup \sigma(B)$, $\rho(A) \cap \Pi^a(B) \subseteq \rho_a(A) \cap \Pi^a(B) = \emptyset$, $\rho(B) \cap \Pi^a(A) \subseteq \rho_a(B) \cap \Pi^a(A) = \emptyset$ and $\Pi^a(A) \cap \Pi^a(B) \subseteq \Pi^a(S) \subseteq (\rho(A) \cap \Pi^a(B)) \cup (\Pi^a(A) \cap \Pi^a(B)) \cup (\rho(B) \cap \Pi^a(A)) = \Pi^a(A) \cap \Pi^a(B)$. □

Theorem 3.7 has a property $(\mathbf{Z}_{\mathbf{E}^a})$ analogue.

Theorem 3.9. *If $A, B \in (\mathbf{Z}_{\mathbf{E}^a})$ are such that $\rho(A) \cap E^a(B) = \rho(B) \cap E^a(A) = \emptyset$, and:*
(i) if $\sigma_w(T) = \sigma_w(A) \cup \sigma_w(B)$, then $\sigma(T) \cap \Phi_w(T) = E^a(A) \cap E^a(B)$;
(ii) if $\sigma(T) = \sigma(A) \cup \sigma(B)$ and $T \in (\mathbf{Z}_{\mathbf{E}^a})$, then $\sigma_w(T) = \sigma_w(A) \cup \sigma_w(B)$.

Proof. The proof follows from Theorem 3.7 since the hypothesis $A, B \in (\mathbf{Z}_{\mathbf{E}^a})$ implies $E^a(A) = \Pi^a(A)$ and $E^a(B) = \Pi^a(B)$. □

Evidently, if $\sigma_p(A) = \sigma_p(B)$, then $\rho(A) \cap E^a(B) = \rho(B) \cap E^a(A) = \emptyset$ and Theorem 3.9 holds with the hypothesis $\rho(A) \cap E^a(B) = \rho(B) \cap E^a(A) = \emptyset$ replaced by the hypothesis $\sigma_p(A) = \sigma_p(B)$. In particular:

Corollary 3.10. *[13, Theorem2.10] If $A, B \in (\mathbf{Z}_{\mathbf{E}^a})$ are such that $\sigma_p(A) = \sigma_p(B)$, then $S \in (\mathbf{Z}_{\mathbf{E}^a})$ if and only if $\sigma_w(S) = \sigma_w(A) \cup \sigma_w(B)$.*

We conclude with a couple of remarks, the first giving an example showing that the answer to Problem 2, [13, Page 6] is in the negative and the second announcing an erratum to [6].

Remark 3.11. (i) Let $\overline{\mathcal{D}}$ denote the closure and $\partial\mathcal{D}$ the boundary of the open unit disc $\mathcal{D}$ in $\mathbb{C}$. Let $A = U \oplus U^* \in B(\mathcal{H} \oplus \mathcal{H})$ and $B = Q \in B(\mathcal{H})$, where U denotes the forward unilateral shift and Q is a quasinilpotent operator. Then $\sigma_w(S) = \sigma_w(A) \cup \sigma_w(B) = \partial\mathcal{D} \cup \{0\}$, $\sigma(S) = \overline{\mathcal{D}}$ and $\Pi^a(S) = \Pi_0(S) = \emptyset$. Since $\sigma(S) \cap \sigma_w(S) = \mathcal{D} \setminus \{0\} \neq \Pi_0(S)$, S does not satisfy Browder's theorem.

(ii) The first author regrets that the statement on Page 20, Lines 17- to 15- (starting with "For operators $A \in B(\mathcal{X})$ such that $\sigma(A) = \sigma_w(A)$ ...") of [6] is false, as can be seen from a consideration of the operators $A = \begin{pmatrix} U & 1 - UU^* \\ 0 & U^* \end{pmatrix}$ and $K = \begin{pmatrix} 0 & -1 + UU^* \\ 0 & 0 \end{pmatrix}$. Here, as in the above, $U \in B(\mathcal{H})$ is the forward unilateral shift, and the operator K is compact. (The author's have since "binned" reference 22 of [6].)

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