Functional Analysis,
Approximation and
Computation
10 (1) (2018), 29–34

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/faac

Matrix partial orders in indefinite inner product space

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Abstract. For a given rangesymmetric matrix B, conditions are obtained for all matrices A that lie below (above) B, to be range symmetric under a given partial ordering on complex matrices with respect to the indefinite matrix product in an indefinite inner product space.

1. Introduction

An indefinite inner product in C^n is a conjugate symmetric Sesquilinear from $[x, y]$ together with the regularity condition that $[x, y] = 0$ for all $y \in C^n$ only when $x = 0$. Any indefinite inner product is associated with a unique invertible complex matrix J (called a weight) such that $[x, y] =< x, Jy >$ where $<,>$ denotes the Euclidean inner product on C^n, we also make an additional assumption on J, that is $J^2 = I$, to present the results with much algebraic ease. Further, there are two different values for dot product of vectors in indefinite inner product spaces. To overcome these difficulties, a new matrix product, called indefinite matrix multiplication is introduced and some of its properties are investigated in [9]. A matrix $A \in C^{n\times n}$ is said to be positive semi definite (p.s.d) if $Re(x * Ax) \geq 0$ for all $x \in C^{n\times 1}$. If A is also Hermitian, then A is to be a Hermitian positive definite matrix (h.p.s.d) and denoted as $A \geq 0$. For $A, B \in C^{n\times n}$, $A \geq B \Leftrightarrow A - B \geq 0$. It is well known that, for nonsingular matrices A, B if $A \geq B \geq 0$, then $B^{-1} \geq A^{-1} \geq 0$. This was extended to generalized inverse of certain types of pairs of singular matrices $A \geq B \geq 0$ by Hans J.Warner [4] and independently by Hartvig [5].

The aim of this manuscript is to extend the *Löwner* order on Hermitian matrices, one of the standard partial orders on complex matrices with respect to the indefinite matrix product and discuss the reverse order laws for various types of generalized inverses as an extension of the results available in the literature [1, 2, 7, 8], for a wide class of range symmetric matrices in an indefinite inner product space. In section 2, we recall the definitions and preliminary results on complex matrices over an indefinite inner product space. Characterizations of a range symmetric matrix in an indefinite inner product space in the setting of an indefinite matrix product established in our work [8] are stated for reference. In section 3, for a given range symmetric matrix B, conditions are obtained for all matrices A that lie below (above) B, to be range

2010 *Mathematics Subject Classification*. 46C20,15A09.
Keywords. Indefinite matrix product; Indefinite inner product space; Matrix partial orders; Range symmetric matrix; EP matrix.
Received: 5 June 2017; Accepted: 27 March 2018
Communicated by Dragan S. Djordjević
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symmetric under a given partial ordering on complex matrices with respect to the indefinite matrix product in an indefinite inner product space. Wherever possible, we provide examples to illustrate our results.

2. Preliminaries

We first recall the notion of an indefinite multiplication of matrices in an indefinite inner product space $\wp$.

Definition 2.1. *Let $A \in C^{m\times n}$, $B \in C^{n\times p}$. Let J_n be an arbitrary but fixed $n \times n$ complex matrix such that $J_n = J_n{}^* = J_n{}^{-1}$. The indefinite matrix product of A and B (relative to J) is defined as $A \circ B = AJ_nB$.*

Definition 2.2. *For $A \in C^{m\times n}$, $A^{[*]} = J_nA^*J_m$ is the adjoint of A relative to J_n and J_m, the weights in the appropriate spaces. Also $[Ax, y] = [x, A^{[*]}y]$ hold.*

Remark 2.3. *When J_n is the identity matrix the product reduces to the usual product of matrices and it can be easily verified that with respect to the indefinite matrix product, $rank(A \circ A^{[*]}) = rank(A^{[*]} \circ A) = rank(A)$, where as this rank property fails under the usual matrix multiplication. Thus, the Moore-Penrose inverse of a complex matrix over an indefinite inner product space always exists, with respect to the indefinite matrix product exists and this is one of its main advantages.*

Definition 2.4. *For $A \in C^{n\times n}$ is said to be J-invertible if there exists $X \in C^{n\times n}$ such that $A \circ X = X \circ A = J_n$. Such an X is denoted as $A^{[-1]} = JA^{-1}J$.*

Definition 2.5. *For $A \in C^{m\times n}$, a matrix $X \in C^{n\times m}$ is called Moore-Penrose if it satisfies the following equations: $A \circ X \circ A = A$, $X \circ A \circ X = X$, $(A \circ X)^{[*]} = A \circ X$ and $(X \circ A)^{[*]} = X \circ A$. Such an X is denoted by $A^{[\dagger]}$ and represented as $A^{[\dagger]} = J_nA^\dagger J_m$.*

Definition 2.6. *The range space of $A \in C^{m\times n}$ is defined by*
$Ru(A) = \{y = A \circ x \in C^m / x \in C^n\}$.The null space of A is defined by $Nu(A) = \{x \in C^n / A \circ x = 0\}$.

Remark 2.7. *It is clear that $Ru(A) = R(A)$ and $Nu(A^{[*]}) = N(A^*)$.*

Property 2.8.
(i) $(A^{[*]})^{[*]} = A$.
(ii) $(A^{[\dagger]})^{\dagger]} = A$.
(iii) $(AB)^{[*]} = B^{[*]}A^{[*]}$.
(iv) $R(A^{[*]}) = R(A)^{[\dagger]}$.
(v) $R(A \circ A^{[*]}) = R(A)$,$R(A^{[*]} \circ A) = R(A^{[*]})$.
(vi) $N(A \circ A^{[*]}) = N(A^{[*]})$,$N(A^{[*]} \circ A) = N(A)$.

Definition 2.9. *$A \in C^{n\times n}$ is range symmetric in $\mathscr{P}$ if and only if $R(A) = R(A^{[*]})$.*

Remark 2.10. *In particular for $J = I_n$, this reduces to the definition of range symmetric matrix in unitary space (or) equivalently to an EP matrix* [1].

We make use of the following equivalent characterizations of a range symmetric matrix in $\wp$ established in our earlier work [7].

Theorem 2.11. *For $A \in C^{n\times n}$, the following are equivalent:*
(i)A is range symmetric in $\wp$.
(ii)AJ is EP.
(iii)JA is EP.
(iv)$N(A) = N(A^{[*]})$.
(v)$N(A^*) = N(AJ)$.
(vi)$A^{[*]} = AK = HA$, for some invertible matrices K and H.

(vii)$R(A^*) = R(JA)$.
(viii)$(A \circ A^{[\dagger]}) = (A^{[\dagger]} \circ A)$.
(ix)A *is* $J - EP$.
(x)$A^{[\dagger]}$ *is a polynomial in* A.
(xi)$A^{[\dagger]}$ *is range symmetric in* $\wp$.

3. Partial ordering on Range symmetric matrices in $\wp$.

In Sequal of the *Löwner* order on Hermitian matrices, several partial orders such as star partial order, Minus partial order, Sharp partial order on the subclasses of matrices have been studied by many researchers. Recently, these matrix orders have been extended for matrices in an indefinite inner product space with respect to the indefinite matrix product. In [3] it has been proven that star partial orders $A \leq^{[*]} B$ and $A \leq B$ are equivalent. The same holds for sharp partial orders $A \leq^{[\#]} B$ and $AJ \leq^{\#} BJ$. It can be verified for the minus partial order that, $A \leq^{[-]} B$ if and only if $A \leq^{-} B$. In this section, we extend *Löwner* order on Hermitian matrices for J-symmtric matrices in an indefinite inner product space $\wp$ with weight J. A matrix A is J-symmetric if $A = A^{[*]}$.

Definition 3.1. *A matrix* $A \in C^{n\times n}$ *is said to be J-symmetric positive semi definite in* $\wp$ *denoted as* $A \geq^{J} 0$ *if and only if A is J-symmetric and* $[Ax, x] \geq 0$, *for all* $x \in C^{n\times 1}$

Lemma 3.2. *For* $A \in C^{n\times n}$, $A \geq^{J} 0 \Leftrightarrow A^{[*]} \geq^{J} 0$.

Proof. This directly follows from the Definition 3.1 □

Theorem 3.3. *For* $A \in C^{n\times n}$, $A \geq^{J} 0 \Leftrightarrow AJ \geq 0 \Leftrightarrow JA \geq 0$.

Proof. Since A is J-symmetric, $A = A^{[*]} = JA^*J$ and $[Ax, x] = \langle Ax, Jx\rangle \geq 0$ for all $x \in C^{n\times 1}$.
Hence, $(AJ) = (AJ)^*$ and $\langle AJx, x\rangle \geq 0$ for all $x \in C^{n\times 1} \Leftrightarrow AJ \geq 0$.
$A \geq^{J} 0 \Leftrightarrow A$ is J-symmetric and $[Ax, x] \geq 0$, for all $x \in C^{n\times 1}$.
$\Leftrightarrow A = A^{[*]}$ and $[Ax, x] \geq 0$, for all $x \in C^{n\times 1}$.
$\Leftrightarrow JA = (JA)^*$ and $\langle JAx, x\rangle = \langle Ax, Jx\rangle = [Ax, x] \geq 0$,
for all $x \in C^{n\times 1}$.
$\Leftrightarrow JA \geq 0$.
Hence, the Theorem. □

Corollary 3.4. *For* $A \in C^{n\times n}$, $A \geq^{J} 0 \Leftrightarrow A = R^{[*]} \circ R$ *for some* $R \in C^{n\times n}$.

Proof. $A \geq^{J} 0 \Leftrightarrow JA \geq 0$(By Theorem 3.3)
$\Leftrightarrow JA = R * R$
$\Leftrightarrow A = JR * JJR$
$\Leftrightarrow A = R^{[*]} \circ R$(By Definition 2.1).
$\Leftrightarrow JA \geq 0$.
Hence, the Theorem. □

Theorem 3.5. *For* $A \in C^{n\times n}$, $A \geq^{J} 0$, *then for any* $P \in C^{n\times n}$,$P^{[*]}AP \geq^{J} 0$.

Proof. Let $B = P^{[*]}AP$, then by using $A = A^{[*]}$, property 2.8 (ii) and (iii), $B^{[*]} = P^{[*]}AP = B$.
Hence, B is J-symmetric. For all $x \in C^{n\times 1}$,$[Bx, x] = [P^{[*]}APx, Px] = [APx, Px] = [Ay, y] \geq 0$, where $y = Px$.
Thus $B = P^{[*]}AP \geq^{J} 0$. Hence the Theorem. □

Theorem 3.6. *For* $A \in C^{n\times n}$,$A \geq^{J} 0 \Leftrightarrow A^{[\dagger]} \geq^{J} 0$.

Proof. $A \geq^J 0 \Leftrightarr JA \geq 0$ (By Theorem 3.3)
$\Leftrightarrow (JA)^\dagger \geq 0$
$\Leftrightarrow A^\dagger J \geq 0$
$\Leftrightarrow A^\dagger J \geq^J 0$ (By Theorem 3.3)
$\Leftrightarrow JA^\dagger J \geq^J 0$ (By Theorem 3.5)
$\Leftrightarrow A^{[\dagger]} \geq^J 0$.
Hence, the Theorem. □

Definition 3.7. *For $A \in C^{n\times n}$, the J-symmetric part of A is defined as $(Sym\ A)_J = \frac{1}{2}(A + A^{[*]})$.*

In particular for $J = I_n$, it reduces to the symmetric part of A.
$Sym\ A = \frac{1}{2}(A + A^*)$

Property 3.8. *For $A \in C^{n\times n}$, we have the following:*
(i) $(Sym\ A)_J = (Sym\ A)_J^{[*]} = (Sym\ A^{[*]})_J$.
(ii) $J(Sym\ A)_J = J(\frac{1}{2}(A + A^{[*]})) = \frac{1}{2}(JA + (JA)^*) = Sym(JA)$.
(iii) $(Sym\ A)_J J = \frac{1}{2}(A + A^{[*]})J = \frac{1}{2}(AJ + (AJ)^*) = Sym(AJ)$.

Remark 3.9. *From property* 3.8 *(i) it is clear that $(SymA)_J$ is always $J - EP$ being J-Symmetric.*

Lemma 3.10. *For $A \in C^{n\times n}$, we have the following:*
(i) $N(SymA)_J = N(J(Sym\ A)_J) = N(Sym(JA))$.
(ii) $R(SymA)_J = R((Sym\ A)_J J) = R(Sym(AJ))$.

Proof. (i) $x \in N(SymA)_J \Leftrightarrow (Sym\ A)_J x = 0$
$\Leftrightarrow J(SymA)_J x = 0$
$\Leftrightarrow x \in N(J(Sym\ A)_J)$.
Hence, $N(SymA)_J = N(J(Sym\ A)_J)$. The last equality in (i) follows from property 3.8(ii)
(ii) $x \in R(SymA)_J \Leftrightarrow x = (Sym\ A)_J y$, for some $y \in C^{n\times 1}$
$\Leftrightarrow x = ((Sym\ A)_J J)u$, for some u=$Jy \in C^{n\times 1}$
$\Leftrightarrow x \in R((Sym\ A)_J J)$.
Thus, $R(SymA)_J = R((Sym\ A)_J J)$. The last equality in (ii) follows from property 3.8(iii).
Hence, the Lemmma.
□

Definition 3.11. *For some $A, B \in C^{n\times n}$, $A \geq^J B \Leftrightarrow A - B \geq^J 0 \Leftrightarrow A - B$ is J-Symmetric and $[Ax, x] \geq [Bx, x]$ for all $x \in C^{n\times 1}$.*

Since, A is $J - EP$ and A is range symmetric in $\wp$ are equivalent by Theorem 2.11 henceforth, we use A is $J - EP$. First, we shall prove lemmas on $J - EP$ matrices which will simply the proof of the main result.

Lemma 3.12. *If $A \in C^{n\times n}$ is $J - EP$, then $N(A) \subseteq N(Sym\ A)_J$*
$= N(Sym(JA))$.

Proof. Since A is $J - EP$, by Theorem 2.11 $N(A) = N(A^{[*]})$. For $x \in N(A)$, $Ax = A^{[*]}x = 0$.
Hence, $(Sym\ A)_J x = 0$ and $x \in N(Sym\ A)_J$. Therefore, by property 3.8(ii), it follows that, $N(A) \subseteq N(Sym\ A)_J = N(Sym(JA))$. □

Lemma 3.13. *Let $A \in C^{n\times n}$. Then A is $J - EP$ and $rk(A) = rk(Sym\ A)_J \Leftrightarrow N(A) = N(Sym\ A)_J$.*

Proof. Since A is $J - EP$, by Lemma 3.12 $N(A) \subseteq N(Sym\ A)_J$ and together with $rk(A) = rk(Sym\ A)_J$, it follows that $N(A) = N(Sym\ A)_J$.
Conversely, if $N(A) = N(Sym\ A)_J$, then $rk(Sym\ A)_J$ automatically holds. To prove A is $J - EP$. If possible, let us assume the contrary, that is, for $0 \neq x \in C^{n\times 1}$, $Ax = 0$ and $A^{[*]}x \neq 0$. Then, $(Sym\ A)_J x \neq 0$. Hence, $x \notin N((Sym\ A)_J)$. This contradicts that, $N(A) = n(Sym\ A)_J$.
Hence, A is $J - EP$. □

Remark 3.14. *We observe, that in Lemma 3.13, both the conditions on A are essential. This is illustrated in the following example.*

Example 3.15. *Let us consider*

$$A = \begin{bmatrix} 0 & -1 \\ -1 & 2 \end{bmatrix} J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ then } JA = \begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix}.$$

JA is EP being nonsingular. Hence, A is $-EP$.

$(Sym\ A)_J = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$*.$rk(Sym\ A)_J = 1 \neq rk$ and $N((Sym\ A)_J) \not\subset N(A)$.*

Theorem 3.16. *Let $A, B \in C^{n\times n}$ such that $A \geq^J B$. Then the following hold:*
(i) If B is $J - EP$ and $N(A) \subseteq N(B)$, then A is $J - EP$.
(ii) If A is $J - EP$ and $N(B) \subseteq N(A)$, then B is $J - EP$.

Proof. Since, $A \geq^J B$, by Definition 3.1, $A^{[*]} - B^{[*]} = (A - B)^{[*]} = A - B$ and $A - B \geq^J 0$. For any $x \in N(A)$, since $N(A) \subseteq N(B)$ and B is $J - EP$, $Ax = 0 \Rightarrow Bx = B^{[*]}x = 0 \Rightarrow A^{[*]}x = Ax - Bx + B^{[*]}x = 0$. Therefore,$A$ is $E - JP$. Thus, (i) holds. (ii) can be proved in a similar manner. Hence, the Theorem. □

Corollary 3.17. *Let $A, B \in C^{n\times n}$ such that $A \geq^J B$. If B is $J - EP$ then $N(A) \subseteq N(B) \Leftrightarrow R(B) \subseteq R(A)$.*

Proof. If B is $J - EP$ and $N(A) \subseteq N(B)$, then by Theorem 3.16(i) A is $J - EP$. Hence, $N(A) \subseteq N(B) \Leftrightarrow N(A^{[*]} \subseteq N(B^{[*]}) \Leftrightarrow R(B) \subseteq R(A)$. Hence, the Corollary. □

Remark 3.18. *In particular, for $J = I_n$, Corollary 3.4 reduces to the following:*
For $A, B \in C^{n\times n}$ such that $A \geq B$. If B is EP, then $N(A) \subseteq N(B) \Leftrightarrow R(B) \subseteq R(A)$. Further, if B is a.p.d, then the condition $N(A) \subseteq N(B)$ automatically holds and it reduces to Lemma (2) [6].

Theorem 3.19. *Let $A, B \in C^{n\times n}$ such that $A \geq^J B$ is $J - EP$ and $N(A) \subseteq N(B)$ then the following are equivalent:*
(i) $N(A) = N(Sym(JB))$.
(ii) $R(A) = R(Sym(BJ))$.
(iii) $rk(A) = rk(Sym(JB))$.

Proof. (i)⇔ (ii): Since B is $J - EP$ and $N(A) \subseteq N(B)$, by Theorem 3.16 (i) A is $J - EP$. Further, by Lemma 3.12,$N(B) \subseteq N(Sym(B))$. Hence, $N(A) \subseteq N(B) \subseteq N(Sym(JB))$. Then, by (iii) $N(A) = N(Sym(JB))$.
(i)⇒ (iii) is trivial. Thus (i) ⇔ (iii) hold.
(i)⇔ (ii): This equivalence follows from the fact that A and $(Sym\ B)_J$ are $J-EP$ and from Lemma 3.10. Hence, the Theorem.
□

Theorem 3.20. *Let $A, B \in C^{n\times n}$ such that $A \geq^J B$. If B is$J-EP$ and $(Sym\ B)_J \geq^J 0$, then the following are equivalent:*
(i) $R(Sym\ A)_J = R(Sym\ B)_J$.
(ii) $(Sym\ B)_J)^{[\dagger]} \geq^J ((Sym\ A)_J)^{[\dagger]}$.

Proof. (i)⇔ (ii): $A \geq^J B \Rightarrow A^{[*]} \geq^J A^{[*]} \Rightarrow (Sym\ A)_J \geq^J (Sym\ B)_J$. Then, under the given condition on B, we have $(Sym\ A)_J \geq^J (Sym\ B)_J \geq^J 0$. From Theorem 3.3, it follows that $((Sym\ A)_J)J \geq^J ((Sym\ B)_J)J \geq^J 0$. By using the Property 3.8 (iii), we get $Sym(AJ) \geq Sym(BJ) \geq 0$.

$R((Sym\ A)J) = R((Sym\ B)J) \Leftrightarrow R(Sym(AJ)) = R(Sym(BJ))$ (By Lemma 3.10)
$\Leftrightarrow (Sym(BJ))^\dagger \geq (Sym(AJ))^\dagger$ (By Theorem 1 of [5])
$\Leftrightarrow ((Sym\ B)_J J)^\dagger \geq ((Sym\ A)_J J)^\dagger$ (By Property 3.8(iii))
$\Leftrightarrow J(Sym\ B)_J)^\dagger \geq J(Sym\ A)_J)^\dagger$
$\Leftrightarrow ((Sym\ B)_J)^\dagger \geq^J ((Sym\ A)_J)^\dagger$ (By Theorem 3.3)
$\Leftrightarrow J(Sym\ B)_J{}^\dagger J \geq^J J(Sym\ A)_J)^\dagger J$ (By Theorem 3.5)
$\Leftrightarrow ((Sym\ B)_J)^\dagger \geq^J ((Sym\ A)_J)^\dagger$ (By Theorem 2.5)

Hence, the Theorem. □

Remark 3.21. *In particular, if* $A \geq^J B \geq^j 0$*, then* $(Sym\ A)_J = A$ *and* $(Sym\ B)_J = B$*. The condition* B *is* $J-EP$ *automatically hold. Theorem 3.19 and Theorem 3.20 reduce to the following:*

Corollary 3.22. *Let* $A, B \in C^{n\times n}$ *such that* $A \geq^J B \geq^J 0$*. Then the following are equivalent:*
(i) $N(A) = N(B)$.
(ii) $R(A) = R(B)$.
(iii) $B^{[\dagger]} \geq^J A^{[\dagger]}$.

Remark 3.23. *In particular, if* $A \geq B \geq 0$ *and* $J = I_n$*, then* $(Sym(A)_J) = A$ *and* $(Sym(B)_J) = B$ *and Theorem 3.19 and Theorem 3.20 reduce to the following results:*

Corollary 3.24. *(Theorem 1 [5]) Let* $A, B \in C^{n\times n}$*, such that* $A \geq B \geq 0$*, then* $B^\dagger \geq A^\dagger \Leftrightarrow R(A) = R(B)$.

Corollary 3.25. *(Theorem 1 [4]) For* $A, B \in C^{n\times n}$*, any two of the following conditions imply the other one.*
(i) $A \geq B \geq 0$. *(ii)* $rk(A) = rk(B)$
(iii) $B^\dagger \geq A^\dagger \geq 0$.

Remark 3.26. *We observe that, in Theorem 3.19, the condition* $N(A) \subseteq N(B)$ *is essential. This is illustrated in the following:*

Example 3.27. *Let* $A = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$ *and* $B = \begin{bmatrix} 0 & -1 \\ -1 & 2 \end{bmatrix}$*. For* $J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$,

$JA = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ *and* $JB = \begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix}$*. Here,* $A - B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = J$
is $J-$*symmetric and* $JA - JB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \geq 0$.

Hence, by Definition 3.1 $A \geq^J B$. B is $J-EP$, being non-singular. $N(A) = \{y = (x, 0)^t \setminus Ay = 0\}$. $N(A) \not\subset N(B)$.

$Sym(JB) = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$. $N(Sym(JB)) = \{y = [x\ x]^t \setminus (Sym(JB))y = 0\}$.
Here, $N(A) \neq N(Sym(JB))$ but $rk(A) = rk(sym(JB)) = 1$. In Theorem 3.19, the statement (i) fails and statement (iii) holds. Therefore, the condition $N(A) \subseteq N(B)$ is essential in Theorem 3.19.

4. Conclusion

We have extended matrix inequalities on a pair of h.p.s.d matrices in the references [1, 2, 4, 5] and on a.p.d matrices in [6] for a wider class of range symmetric matrices in an indefinite inner product space.

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