Functional Analysis,
Approximation and
Computation
10 (1) (2018), 35–39

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/faac

A Cline's formula for the generalized Drazin-Riesz inverses

Mohammed Karmouni[a], Abdelaziz Tajmouati[b]

[a] *Cadi Ayyad University, Multidisciplinary Faculty, Safi, Morocco.*
[b] *Sidi Mohamed Ben Abdellah University, Faculty of Sciences Dhar Al Mahraz, Laboratory of Mathematical Analysis and Applications, Fez, Morocco.*

Abstract. Let X be Banach space, A, B, C be bounded linear operators on X satisfying operator equation $ABA = ACA$. In this note, we show that AC is generalized Drazin-Riesz invertible if and only if BA is generalized Drazin-Riesz invertible. So, we generalize Cline's formula to the case of the generalized Drazin-Riesz invertibility.

1. Introduction and Preliminaries

Throughout, X denotes a complex Banach space and $\mathcal{B}(X)$ denotes the Banach algebra of all bounded linear operators on X. An operator $T \in \mathcal{B}(X)$ is Riesz, if $T - \lambda I$ is Fredholm in the usual sense for every $\lambda \in \mathbb{C}\backslash\{0\}$ [1]. Recall that a bounded operator $T \in \mathcal{B}(X)$ is said to be a Drazin invertible if there exists a positive integer k and an operator $S \in \mathcal{B}(X)$ such that

$$ST = TS, \quad S^2T = S \quad \textit{and } T^{k+1}S = T^k.$$

The concept of Drazin invertible operators has been generalized by Koliha [6] by replacing the third condition in this definition with the condition that $TST - T$ is quasi-nilpotent. Recently, Živković-Zlatanović SČ and M D. Cvetković [10] introduced and studied a new concept of pseudo-inverse to extend the Koliha concept to "generalized Drazin-Riesz invertible". In fact, an operator $T \in \mathcal{B}(X)$ is said to be generalized Drazin-Riesz invertible, if there exists $S \in \mathcal{B}(X)$ such that

$$TS = ST, \quad STS = S \quad \text{and } TST - T \text{ is Riesz}$$

In this case S is called a generalized Drazin-Riesz inverse of T. Until now we will be considered that the generalized Drazin-Riesz inverse is not unique. Živković-Zlatanović SČ and M D. Cvetković also showed that T is generalized Drazin-Riesz invertible iff it has a direct sum decomposition $T = T_1 \oplus T_0$ with T_1 is invertible and T_0 is Riesz. The generalized Drazin-Riesz spectrum of $T \in \mathcal{B}(X)$ is defined by

$$\sigma_{gDR}(T) = \{\lambda \in \mathbb{C}, \quad T - \lambda I \text{ is not generalized Drazin-Riesz invertible}\}$$

2010 *Mathematics Subject Classification*. Primary 15A09, 47A53, 47A10.
Keywords. Cline's formula; generalized Drazin-Riesz inverse.
Received: 7 April 2017; Accepted: 4 March 2018
Communicated by Dijana Mosić
Email addresses: med89karmouni@gmail.com (Mohammed Karmouni), abdelaziz.tajmouati@usmba.ac.ma (Abdelaziz Tajmouati)

Jacobson's Lemma [2] asserts that if $A, B \in \mathcal{B}(X)$, then

$$AB - I \text{ is invertible } \Longleftrightarrow \quad BA - I \text{ is invertible.} \tag{1}$$

As extensions of Jacobson's lemma, Corach et al. [4] investigated (1) under the assumption $ABA = ACA$. They studied common properties of AC and BA in algebraic viewpoint and also obtained some nice topological analogues. For an associative ring R with unit, R.E Cline [3] showed that if $a, b \in R$ such that ab is Drazin invertible then so is ba and in this case the Drazin inverse of ba is $(ba)^D = b((ab)^D)^2 a$. This formula is so-called Cline's formula. Recently, Cline's formula for Drazin and generalized Drazin in a ring under the condition $aba = aca$ was extended respectively by Zeng and Zhong [9] and Lian and Zeng [7]. In this note, we establish Cline's formula for the generalized Drazin-Riesz inverse for bounded linear operators under the condition $ABA = ACA$.

2. Main Results

The following lemma will be needed in the sequel.

Lemma 2.1. *Suppose that $A, B, C \in \mathcal{B}(X)$ satisfy $ABA = ACA$. Then*

$$AC \text{ is Riesz } \Longleftrightarrow BA \text{ is Riesz.}$$

Proof.

$$\begin{aligned} AC \text{ is Riesz } &\Longleftrightarrow \lambda I - AC \text{ is Fredholm for all } \lambda \in \mathbb{C} \setminus \{0\} \\ &\Longleftrightarrow \lambda I - BA \text{ is Fredholm for all } \lambda \in \mathbb{C} \setminus \{0\} \\ &\Longleftrightarrow BA \text{ is Riesz} \end{aligned}$$

see [8, Theorem 2.8]. □

Theorem 2.2. *If $A, B, C \in \mathcal{B}(X)$ satisfy $ABA = ACA$. Then*

$$AC \text{ is generalized Drazin-Riesz invertible} \Longleftrightarrow BA \text{ is generalized Drazin-Riesz invertible.}$$

In this case if S is a generalized Drazin-Riesz inverse of AC then $T = BS^2A$ is a generalized Drazin-Riesz inverse of BA.

Proof. Suppose that AC is generalized Drazin-Riesz invertible, then there exists $S \in \mathcal{B}(X)$ such that

$$S(AC) = (AC)S, \quad S(AC)S = S \quad \text{and} \quad (AC)S(AC) - (AC) \text{ is Riesz}$$

Let $T = BS^2A$. We have

$$T(BA) = BS^2ABA = BS^2ACA = BSA$$

and

$$\begin{aligned} (BA)T &= (BA)BS^2A \\ &= BABACS^2SA \\ &= BACACS^3A \\ &= BACS^2A = BSA. \end{aligned}$$

Then $T(BA) = (BA)T$.

$$\begin{aligned}
T(BA)T &= BS^2A(BA)BS^2A \\
&= BS^2ABABACS^3A \\
&= BS^2ACACACS^3A \\
&= BS^2ACSA \\
&= BSSA = BS^2A = T.
\end{aligned}$$

Hence $T(BA)T = T$.

Now, let $Q = I - ACS$.

$$QAC = (I - ACS)AC = AC - ACSAC \text{ is Riesz.}$$

We have

$$\begin{aligned}
BA - (BA)^2T &= BA - BABABS^2A \\
&= BA - BABABACS^2SA \\
&= BA - BACACACS^2SA \\
&= BA - BACSA \\
&= B(I - ACS)A \\
&= BQA
\end{aligned}$$

and

$$\begin{aligned}
ABQA &= AB(I - ACS)A \\
&= ABA - ABACSA \\
&= ACA - ACACSA \\
&= AC(I - ACS)A = ACQA
\end{aligned}$$

Then $(QA)B(QA) = QACQA = (QA)C(QA)$, and since QAC is Riesz by lemma 2.1 $BA - (BA)^2T = BQA$ is Riesz. Consequently, BA is generalized Drazin-Riesz invertible with $T = BS^2A$ is a generalized Drazin-Riesz inverse of BA.

Conversely, if BA is generalized Drazin-Riesz invertible with a generalized Drazin-Riesz inverse T, AC is generalized Drazin-Riesz invertible with AT^2C is a generalized Drazin-Riesz inverse of AC. Indeed:

$$(AC)AT^2C = ACAT^2C = ABAT^2C = ATC.$$

$$\begin{aligned}
(AT^2C)(AC) = AT^2CAC &= A(BAT^2)TCAC \\
&= AT^3BACAC \\
&= AT^3BABAC \\
&= ATC.
\end{aligned}$$

Hence $(AC)(AT^2C) = (AT^2C)(AC)$.

$$\begin{aligned}
(AT^2C)(AC)(AT^2C) &= AT^2CACAT^2C \\
&= AT^3BACACAT^2C \\
&= AT^3BABACAT^2C \\
&= AT^3BABABAT^2C \\
&= AT^2BABAT^2C \\
&= AT^2C.
\end{aligned}$$

Let $Q = I - BAT$

$BAQ = (I - BAT)BA = BA - BATBA$ is a Riesz operator.

And

$$\begin{aligned} AC - (AC)^2(AT^2C) &= AC - ACACAT^2C \\ &= AC - ACACA(BAT^2)TC \\ &= AC - ACACABAT^3C \\ &= AC - ABACABAT^3C \\ &= AC - ABABABAT^3C \\ &= AC - ABABAT^2C = AC - ABATC = A(I - BAT)C = AQC. \end{aligned}$$

$$\begin{aligned} AQCA &= A(I - BAT)CA \\ &= ACA - ABATCA \\ &= ABA - ATBACA \\ &= ABA - ATBABA \\ &= ABA - ABATBA \\ &= A(I - BAT)BA = AQBA. \end{aligned}$$

Now, we have $(AQ)B(AQ) = (AQ)C(AQ)$. Since BAQ is a Riesz operator, by lemma 2.1 $AC-(AC)^2(AT^2C) = AQC$ is Riesz.

□

In the case $B = C$, we have the following theorem.

Theorem 2.3. *If $A, B \in \mathcal{B}(X)$. Then*

AB is generalized Drazin-Riesz invertible $\Longleftrightarrow$ BA is generalized Drazin-Riesz invertible

Then from Theorem 2.2 we have

Theorem 2.4. *If $A, B, C \in \mathcal{B}(X)$ satisfy $ABA = ACA$. Then*

$$\sigma_{gDR}(AC) = \sigma_{gDR}(BA)$$

Corollary 2.5. *If $A, B \in \mathcal{B}(X)$. Then*

$$\sigma_{gDR}(AB) = \sigma_{gDR}(BA)$$

Let H be complex Hilbert space. For $T \in \mathcal{B}(H)$, let $T = U|T|$ be the polar decomposition of T, where $|T| = (T^*T)^{\frac{1}{2}}$. The Aluthge transform of T is given by $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$. Set $B = |T|^{\frac{1}{2}}$ and $A = U|T|^{\frac{1}{2}}$. Then $AB = T$ and $BA = \tilde{T}$. From corollary 2.5, we have the following corollary.

Corollary 2.6. *Let $T \in \mathcal{B}(H)$, then*

$$\sigma_{gDR}(T) = \sigma_{gDR}(\tilde{T})$$

Remark 2.7. *1) Generalized inverses are not unique in general. For example, consider a regular operator A and suppose that B is a generalized inverse of A. One can then easily verify that the operator BAB is also a generalized inverse of A. It is well known that if a generalized Drazin inverse (Drazin inverse) exists then it is unique. A logical question to ask is whether generalized Drazin-Riesz inverses are unique provided they exist.*

2) Živković-Zlatanović SČ and M D. Cvetković [10] showed that T is generalized Drazin-Riesz invertible iff there exists a bounded projection P on X which commutes with T such that $T + P$ is Browder in the usual sense [1] and TP is Riesz. Does it exist a unique projection satisfy previous conditions?

Now, we present an additive result concerning generalized Drazin-Riesz invertible operators.

Proposition 2.8. *Let $A, B \in \mathcal{B}(X)$ be generalized Drazin-Riesz invertible operators such that $AB = BA = 0$. Then $A + B$ is generalized Drazin-Riesz invertible.*

Proof. Suppose that A and B are generalized Drazin-Riesz invertible operators, then there exist $S \in \mathcal{B}(X)$ and $R \in \mathcal{B}(X)$ such that

$$AS = SA \quad S^2A = S \quad \text{and} \quad A - ASA \quad \text{is Riesz},$$

and

$$BR = RB \quad R^2B = R \quad \text{and} \quad B - BRB \quad \text{is Riesz}.$$

We will prove that $S + R$ is a generalized Drazin-Riesz inverse of $A + B$.
Since $AB = BA = 0$, we have $AR = RA = 0$, $BS = SB = 0$ and $RS = SR = 0$. Then

$$(A + B)(S + R) = (S + R)(A + B)$$

and

$$(A + B)(R + S)(R + S) = (A + B)(R^2 + RS + RS + S^2) = AS^2 + BR^2 = S + R$$

Now, we have

$$\begin{aligned}(A + B) - (A + B)(A + B)(S + R) &= (A + B) - (A^2 + AB + AB + B^2)(S + R)\\ &= (A + B) - (A^2 + B^2)(S + R)\\ &= (A + B) - (A^2S + B^2R)\\ &= A - A^2S + B - B^2R\end{aligned}$$

Since $A - A^2S$ and $B - B^2R$ are Riesz and commute, by [1, Theorem 3.112] $(A + B) - (A + B)(A + B)(S + R)$ is Riesz. □

Acknowledgements:

We wish to thank the referee for his valuable comments and suggestions.

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