A Cline’s formula for the generalized Drazin-Riesz inverses

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\textbf{Abstract.} Let $X$ be Banach space, $A, B, C$ be bounded linear operators on $X$ satisfying operator equation $ABA = ACA$. In this note, we show that $AC$ is generalized Drazin-Riesz invertible if and only if $BA$ is generalized Drazin-Riesz invertible. So, we generalize Cline’s formula to the case of the generalized Drazin-Riesz invertibility.

1. Introduction and Preliminaries

Throughout, $X$ denotes a complex Banach space and $\mathcal{B}(X)$ denotes the Banach algebra of all bounded linear operators on $X$. An operator $T \in \mathcal{B}(X)$ is Riesz, if $T - \lambda I$ is Fredholm in the usual sense for every $\lambda \in \mathbb{C}\setminus\{0\}$ [1]. Recall that a bounded operator $T \in \mathcal{B}(X)$ is said to be a Drazin invertible if there exists a positive integer $k$ and an operator $S \in \mathcal{B}(X)$ such that

$$ST = TS, \quad S^2T = S \quad \text{and} \quad T^{k+1}S = T^k.$$  

The concept of Drazin invertible operators has been generalized by Koliha [6] by replacing the third condition in this definition with the condition that $TST - T$ is quasi-nilpotent. Recently, Živković-Zlatanović SC and M D. Cvetković [10] introduced and studied a new concept of pseudo-inverse to extend the Koliha concept to “generalized Drazin-Riesz invertible”. In fact, an operator $T \in \mathcal{B}(X)$ is said to be generalized Drazin-Riesz invertible, if there exists $S \in \mathcal{B}(X)$ such that

$$TS = ST, \quad STS = S \quad \text{and} \quad TST - T \text{ is Riesz}.$$  

In this case $S$ is called a generalized Drazin-Riesz inverse of $T$. Until now we will be considered that the generalized Drazin-Riesz inverse is not unique. Živković-Zlatanović SC and M D. Cvetković also showed that $T$ is generalized Drazin-Riesz invertible iff it has a direct sum decomposition $T = T_1 \oplus T_0$ with $T_1$ is invertible and $T_0$ is Riesz. The generalized Drazin-Riesz spectrum of $T \in \mathcal{B}(X)$ is defined by

$$\sigma_{DRI}(T) = \{\lambda \in \mathbb{C}, \quad T - \lambda I \text{ is not generalized Drazin-Riesz invertible}\}$$

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Jacobson’s Lemma [2] asserts that if $A, B \in \mathcal{B}(X)$, then

$$AB - I \text{ is invertible } \iff BA - I \text{ is invertible.} \quad (1)$$

As extensions of Jacobson’s lemma, Corach et al. [4] investigated (1) under the assumption $ABA = ACA$. They studied common properties of $AC$ and $BA$ in algebraic viewpoint and also obtained some nice topological analogues. For an associative ring $R$ with unit, R.E Cline [3] showed that if $a, b \in R$ such that $ab$ is Drazin invertible then so is $ba$ and in this case the Drazin inverse of $ba$ is $(ba)^D = b((ab)^D)^2a$. This formula is so-called Cline’s formula. Recently, Cline’s formula for Drazin and generalized Drazin in a ring under the condition $aba = aca$ was extended respectively by Zeng and Zhong [9] and Lian and Zeng [7]. In this note, we establish Cline’s formula for the generalized Drazin-Riesz inverse for bounded linear operators under the condition $ABA = ACA$.

2. Main Results

The following lemma will be needed in the sequel.

Lemma 2.1. Suppose that $A, B, C \in \mathcal{B}(X)$ satisfy $ABA = ACA$. Then

$$AC \text{ is Riesz } \iff BA \text{ is Riesz.}$$

Proof.

$$AC \text{ is Riesz } \iff \lambda I - AC \text{ is Fredholm for all } \lambda \in \mathbb{C} \setminus \{0\}$$

$$\iff \lambda I - BA \text{ is Fredholm for all } \lambda \in \mathbb{C} \setminus \{0\}$$

$$\iff BA \text{ is Riesz}$$

see [8, Theorem 2.8].

Theorem 2.2. If $A, B, C \in \mathcal{B}(X)$ satisfy $ABA = ACA$. Then

$$AC \text{ is generalized Drazin-Riesz invertible } \iff BA \text{ is generalized Drazin-Riesz invertible.}$$

In this case if $S$ is a generalized Drazin-Riesz inverse of $AC$ then $T = BS^2A$ is a generalized Drazin-Riesz inverse of $BA$.

Proof. Suppose that $AC$ is generalized Drazin-Riesz invertible, then there exists $S \in \mathcal{B}(X)$ such that

$$S(AC) = (AC)S, \quad S(AC)S = S \quad \text{and} \quad (AC)S(AC) - (AC) \text{ is Riesz}$$

Let $T = BS^2A$. We have

$$T(BA) = BS^2ABA = BS^2ACA = BSA$$

and

$$(BA)T = (BA)BS^2A$$

$$= BABACS^2A$$

$$= BACAC^2A$$

$$= BACS^2A = BSA.$$

Then $T(BA) = (BA)T$. 

\[
T(BA)T = BS^2A(BA)BS^2A
= BS^2ABABACS^3A
= BS^2ACACACS^3A
= BS^2ACSA
= BSSA = BS^2A = T.
\]
Hence \(T(BA)T = T\).

Now, let \(Q = I - ACS\). \(QAC = (I - ACS)AC = AC - ACSAC\) is Riesz.

We have
\[
BA - (BA)^2T = BA - BABABS^2A
= BA - BABABACS^2SA
= BA - BACACACS^2SA
= BA - BACSA
= B(I - ACS)A
= BQA
\]
and
\[
ABQA = AB(I - ACS)A
= ABA - ABACSA
= ACA - ACACSA
= AC(I - ACS)A = ACQA
\]
Then \((QA)B(QA) = QACQA = (QA)C(QA)\), and since \(QAC\) is Riesz by lemma 2.1 \(BA - (BA)^2T = BQA\) is Riesz. Consequently, \(BA\) is generalized Drazin-Riesz invertible with \(T = BS^2A\) is a generalized Drazin-Riesz inverse of \(BA\).

Conversely, if \(BA\) is generalized Drazin-Riesz invertible with a generalized Drazin-Riesz inverse \(T\), \(AC\) is generalized Drazin-Riesz invertible with \((AT)^2C\) is a generalized Drazin-Riesz inverse of \(AC\). Indeed:
\[
(AC)AT^2C = ACAT^2C = ABAT^2C = ATC.
\]
\[
(AT^2C)(AC) = AT^2CAC = A(BAT^2)TCAC
= AT^3BACAC
= AT^3BABAC
= AT^3C.
\]
Hence \((AC)(AT^2C) = (AT^2C)(AC)\).

\[
(AT^2C)(AC)(AT^2C) = AT^2CACAT^2C
= AT^3BACACAT^2C
= AT^3BABACAT^2C
= AT^3BABABAT^2C
= AT^2BABAT^2C
= AT^2C.
\]
Let $Q = I - BAT$

$BAQ = (I - BAT)BA = BA - BATBA$ is a Riesz operator.

And

\[
AC - (AC)^2(AT^2C) = AC - ACACAT^2C = AC - ACACA(BAT^2)TC = AC - ACACABAT^3C = AC - ABACABAT^3C = AC - ABABABAT^3C = AC - ABABAT^2C = AC - ABATC = A(I - BAT)C = AQC.
\]

\[
AQCA = A(I - BAT)CA = ACA - ABATCA = ABA - ATBACA = ABA - ATBABA = ABA - ABATBA = A(I - BAT)BA = AQBA.
\]

Now, we have $(AQ)B(AQ) = (AQ)C(AQ)$. Since $BAQ$ is a Riesz operator, by lemma 2.1 $AC - (AC)^2(AT^2C) = AQC$ is Riesz.

\[\square\]

In the case $B = C$, we have the following theorem.

**Theorem 2.3.** If $A, B \in B(X)$. Then

$AB$ is generalized Drazin-Riesz invertible $\iff$ $BA$ is generalized Drazin-Riesz invertible

Then from Theorem 2.2 we have

**Theorem 2.4.** If $A, B, C \in B(X)$ satisfy $ABA = ACA$. Then

\[
\sigma_{gDR}(AC) = \sigma_{gDR}(BA)
\]

**Corollary 2.5.** If $A, B \in B(X)$. Then

\[
\sigma_{gDR}(AB) = \sigma_{gDR}(BA)
\]

Let $H$ be complex Hilbert space. For $T \in B(H)$, let $T = U|T|$ be the polar decomposition of $T$, where $|T| = (T^*T)^{\frac{1}{2}}$. The Aluthge transform of $T$ is given by $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$. Set $B = |T|^{\frac{1}{2}}$ and $A = U|T|^{\frac{1}{2}}$. Then $AB = T$ and $BA = \tilde{T}$. From corollary 2.5, we have the following corollary.

**Corollary 2.6.** Let $T \in B(H)$, then

\[
\sigma_{gDR}(T) = \sigma_{gDR}(\tilde{T})
\]

**Remark 2.7.**

1) Generalized inverses are not unique in general. For example, consider a regular operator $A$ and suppose that $B$ is a generalized inverse of $A$. One can then easily verify that the operator $BAB$ is also a generalized inverse of $A$. It is well known that if a generalized Drazin inverse (Drazin inverse) exists then it is unique. A logical question to ask is whether generalized Drazin-Riesz inverses are unique provided they exist.

2) Živković-Zlatanović SC and M D. Cvetković [10] showed that $T$ is generalized Drazin-Riesz invertible iff there exists a bounded projection $P$ on $X$ which commutes with $T$ such that $T + P$ is Browder in the usual sense [1] and $TP$ is Riesz. Does it exist a unique projection satisfy previous conditions?
Now, we present an additive result concerning generalized Drazin-Riesz invertible operators.

**Proposition 2.8.** Let \( A, B \in \mathcal{B}(X) \) be generalized Drazin-Riesz invertible operators such that \( AB = BA = 0 \). Then \( A + B \) is generalized Drazin-Riesz invertible.

**Proof.** Suppose that \( A \) and \( B \) are generalized Drazin-Riesz invertible operators, then there exist \( S \in \mathcal{B}(X) \) and \( R \in \mathcal{B}(X) \) such that

\[
AS = SA \quad S^2A = S \quad \text{and} \quad A - ASA \text{ is Riesz},
\]

and

\[
BR = RB \quad R^2B = R \quad \text{and} \quad B - BRB \text{ is Riesz}.
\]

We will prove that \( S + R \) is a generalized Drazin-Riesz inverse of \( A + B \).

Since \( AB = BA = 0 \), we have \( AR = RA = 0 \), \( BS = SB = 0 \) and \( RS = SR = 0 \). Then

\[
(A + B)(S + R) = (S + R)(A + B)
\]

and

\[
\]

Now, we have

\[
(A + B) - (A + B)(A + B)(S + R) = (A + B) - (A^2 + AB + AB + B^2)(S + R)
= (A + B) - (A^2 + B^2)(S + R)
= (A + B) - (A^2S + B^2R)
= A - A^2S + B - B^2R
\]

Since \( A - A^2S \) and \( B - B^2R \) are Riesz and commute, by [1, Theorem 3.112] \( (A + B) - (A + B)(A + B)(S + R) \) is Riesz. \( \square \)

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**References**