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$\mathcal{F}$-hypercyclic extensions and disjoint $\mathcal{F}$-hypercyclic extensions of binary relations over topological spaces

Marko Kostić[a]

[a]*Faculty of Technical Sciences, University of Novi Sad, Trg D. Obradovića 6, 21125 Novi Sad, Serbia*

Abstract. The main aim of this paper is to provide the basic information about $\mathcal{F}$-hypercyclic extensions of binary relations over topological spaces and disjoint $\mathcal{F}$-hypercyclic extensions of binary relations over topological spaces. Special attention is paid to the case that the topological space under our consideration has a linear vector structure, when we also analyze $\mathcal{F}$-hypercyclic multivalued linear extensions of binary relations and disjoint $\mathcal{F}$-hypercyclic multivalued linear extensions of binary relations. We provide several illustrative examples and results for simple graphs, digraphs and tournaments.

1. Introduction and preliminaries

Let E be a separable Fréchet space. Then a linear operator T on E is said to be hypercyclic iff there exists an element $x \in D_\infty(T) \equiv \bigcap_{n\in\mathbb{N}} D(T^n)$ whose orbit $\{T^n x : n \in \mathbb{N}_0\}$ is dense in E; T is said to be topologically transitive, resp. topologically mixing, iff for every pair of open non-empty subsets U, V of E, there exists $n \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$, resp. iff for every pair of open non-empty subsets U, V of E, there exists $n_0 \in \mathbb{N}$ such that, for every $n \in \mathbb{N}$ with $n \geq n_0$, we have $T^n(U) \cap V \neq \emptyset$.

It would be very difficult to summarize here all relevant results obtained recently in the field of linear topological dynamics. For further information on the subject, we refer the reader to the monographs [1] by F. Bayart, E. Matheron and [10] by K.-G. Grosse-Erdmann, A. Peris. Hypercyclic and topologically mixing properties of various classes of abstract Volterra integro-differential equations have been analyzed in [11] and [12].

Let $\mathcal{A}$ be a multivalued linear operator on E. As a linear submanifold of $E \times E$, $\mathcal{A}$ is always contained in some hypercyclic (chaotic, topologically transitive) multivalued linear operator on E. The analysis of hypercyclic and topologically transitive multivalued linear extensions has been initiated in [5], where the authors have also considered disjoint hypercyclic and disjoint topologically transitive multivalued linear extensions. On the other hand, in the recent research study of C.-C. Chen, J. A. Conejero, M. Kostić and M. Murillo-Arcila [6], we have analyzed dynamics on binary relations over topological spaces, focusing

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Email address: marco.s@verat.net (Marko Kostić)

special attention to the finite structures like simple graphs, digraphs and tournaments. This paper has been recently continued by M. Kostić, M. Murillo-Arcila and Y. Puig [13]-[14], where the authors have analyzed $\mathcal{F}$-hypercyclic properties of binary relations over topological spaces and their disjoint analogues; here, $\mathcal{F}$ denotes a non-empty collection of certain subsets of $\mathbb{N}$. For more details about $\mathcal{F}$-hypercyclicity and $\mathcal{F}$-topological transitivity of linear continuous operators on Fréchet spaces, we refer the reader to [2], [9], [17] and references cited therein.

As mentioned in the abstract, the main aim of this paper is to provide the basic information on $\mathcal{F}$-hypercyclic extensions of binary relations over topological spaces and disjoint $\mathcal{F}$-hypercyclic extensions of binary relations over topological spaces. The results are completely new for binary relations over topological spaces which do not have a linear vector structure, especially, for binary relations over finite topological spaces. We analyze $\mathcal{F}$-hypercyclic extensions and disjoint $\mathcal{F}$-hypercyclic extensions for simple graphs, digraphs and tournaments as well as $\mathcal{F}$-hypercyclic multivalued linear operator extensions and disjoint $\mathcal{F}$-hypercyclic multivalued linear operator extensions for linear continuous operators on Fréchet spaces, providing a great number of illustrative examples.

The organization of paper, consisting of four separate sections, is briefly described as follows. After repeating some elementary facts about binary relations and multivalued linear operators, we provide a basic information about simple graphs, digraphs and tournaments in Subsection 1.1. In Section 2, we remind ourselves of recently introduced definitions of various $\mathcal{F}$-hypercyclic and disjoint $\mathcal{F}$-hypercyclic properties of binary relations over topological spaces ([13]-[14]). Our main contributions are given in Section 3 and Section 4, where we analyze $\mathcal{F}$-hypercyclic (MLO) extensions and disjoint $\mathcal{F}$-topologically transitive (MLO) extensions of binary relations.

We use the standard notation henceforth. Suppose that X, Y, Z and T are given non-empty sets. Let us recall that a binary relation between X into Y is any subset $\rho \subseteq X \times Y$. If $\rho \subseteq X \times Y$ and $\sigma \subseteq Z \times T$ with $Y \cap Z \neq \emptyset$, then we define $\rho^{-1} \subseteq Y \times X$ and $\sigma \circ \rho \subseteq X \times T$ by $\rho^{-1} := \{(y, x) \in Y \times X : (x, y) \in \rho\}$ and

$$\sigma \circ \rho := \big\{(x, t) \in X \times T : \exists y \in Y \cap Z \text{ such that } (x, y) \in \rho \text{ and } (y, t) \in \sigma\big\},$$

respectively. Domain and range of ρ are defined by $D(\rho) := \{x \in X : \exists y \in Y \text{ such that } (x, y) \in X \times Y\}$ and $R(\rho) := \{y \in Y : \exists x \in X \text{ such that } (x, y) \in X \times Y\}$, respectively; $\rho(x) := \{y \in Y : (x, y) \in \rho\}$ $(x \in X)$, $x \, \rho \, y \Leftrightarrow (x, y) \in \rho$. Assuming ρ is a binary relation on X and $n \in \mathbb{N}$, we define ρ^n inductively; $\rho^{-n} := (\rho^n)^{-1}$ and $\rho^0 := \Delta_X := \{(x, x) : x \in X\}$. Set $D_\infty(\rho) := \bigcap_{n \in \mathbb{N}} D(\rho^n)$, $\rho(X') := \{y : y \in \rho(x) \text{ for some } x \in X'\}$ $(X' \subseteq X)$ and $\mathbb{N}_n := \{1, \cdots, n\}$ $(n \in \mathbb{N})$. As it is well-known, for any set A we define $P(A) := \{B \,|\, B \subseteq A\}$.

Any topological space under our examination will be assumed to be non-trivial. If X is a topological space equipped with a linear vector structure, then we say that any linear subspace $\mathcal{A}$ of $X \times X$ is a multivalued linear operator (MLO) on X. In particular, $\mathcal{A}$ is a binary relation on X, so that the notions of $D(\mathcal{A})$, $D_\infty(\mathcal{A})$, $R(\mathcal{A})$, $\mathcal{A}x$ $(x \in D(\mathcal{A}))$ and the MLO $\mathcal{A}^n$ $(n \in \mathbb{C})$ are clear. By $MLO(X)$ we denote the class of all MLOs on X. For more details about MLOs and their applications to the abstract degenerate Volterra integro-differential equations, the reader may consult the monographs [7] by R. Cross, [8] by A. Favini, A. Yagi, and [12] by the author.

1.1. Graphs, digraphs and tournaments

Let $X = G$ be finite and equipped with discrete topology, and let ρ be a symmetric relation on G such that, for every $g \in G$, we have $(g, g) \notin \rho$. As it is well-known, (G, ρ) is said to be a (simple) graph.

A digraph is any pair (G, ρ), where G is a finite non-empty set and $\rho \subseteq (G \times G) \setminus \Delta_G$; hence, in our definition, we do not allow G to contain any loop. The elements in G and ρ are called points (vertices) and arcs respectively; if arc $(x, y) \in \rho$, then we say that x is adjacent to y and write xy for arc(x, y). Two vertices x and y of a digraph G are said to be nonadjacent iff $(x, y) \notin \rho$ and $(y, x) \notin \rho$. If we replace each arc(x, y) in G by symmetric pairs (x, y) and (y, x) of arcs, we obtain the underlying simple graph G associated to G. The notions of outdegree $d^+(x)$, indegree $d^-(x)$ and degree $d(x) := d^+(x) + d^-(x)$ of a vertex $x \in G$ as well as the notions of a semi-walk and a walk in digraph G are defined usually ([4]). Let us recall that a digraph (G, ρ) is called strongly connected iff for any two different points x and y from G there is an oriented $x - y$ walk, while (G, ρ) is said to be weakly connected iff for any two different points x and y from G there is an

$x - y$ semi-walk, which is equivalent to say that the underlying simple graph G associated to G is connected ([4]). For various generalizations, see [6].

Let (G, ρ) be a given digraph, $G = \{x_1, x_2, \cdot \cdot \cdot, x_n\}$ being equipped with discrete topology, and let $[A(G)]_{1 \leq i,j \leq n}$ be its adjacency matrix (defined by $a_{ij} := 1$ if x_i is adjacent to x_j and $a_{ij} := 0$, otherwise). Denote, for every $k \in \mathbb{N}$, $A(G)^k = [a^k_{i,j}]_{1 \leq i,j \leq n}$. As it is well known, the element $a^k_{i,j}$ of matrix $A(G)^k$ represents the exact number of $x_i - x_j$ walks of length k in digraph (G, ρ). This fact enables one to simply reformulate the notion introduced in Definition 2.1 below in terms of appropriate conditions on the adjacency matrix $[A(G)]_{1 \leq i,j \leq n}$; see [13] for more details.

The reader may consult the monographs by J. A. Bondy and U. S. R. Murty [3], G. Chartrand and L. Lesniak [4], and V. Petrović [16] for more details about the theory of graphs and digraphs. Without any doubt, tournaments are the best studied class of digraphs (for a survey of not updated results on tournaments, the reader may consult the monograph [15] by J. W. Moon). Let us recall that a tournament $T = (G, \rho)$ is a digraph in which any pair of different vertices (sometimes also called nodes) $x,\ y \in X$ are connected by exactly one arc.

2. $\mathcal{F}$-hypercyclic and disjoint $\mathcal{F}$-hypercyclic properties of binary relations on topological spaces

Throughout this section, we assume that X and Y are two given topological spaces as well as that X^N and Y^N are equipped with the usual product space topologies ($N \in \mathbb{N}$). Suppose that $\mathcal{F}$ is a non-empty collection of certain subsets of $\mathbb{N}$, i.e., $\mathcal{F} \in P(P(\mathbb{N}))$ and $\mathcal{F} \neq \emptyset$. Observe that we do not require here that $\mathcal{F}$ satisfies the following property:

(I) $A \in \mathcal{F}$ and $A \subseteq B$ imply $B \in \mathcal{F}$.

If $\mathcal{F}$ satisfies (I), then it is said that $\mathcal{F}$ is a Furstenberg family; furthermore, we say that $\mathcal{F}$ is a proper Furstenberg family iff, in addition to the above, $\emptyset \notin \mathcal{F}$. For more details about Furstenberg families and their importance in the theory of linear topological dynamics, we refer the reader to [9], [2] and references cited therein.

We need to recall the following definition from [13]:

Definition 2.1. Let $(\rho_n)_{n \in \mathbb{N}}$ be a sequence of binary relations between the spaces X and Y, let ρ be a binary relation on X, and let $x \in X$. Suppose that $\mathcal{F} \in P(P(\mathbb{N}))$ and $\mathcal{F} \neq \emptyset$. Then we say that:

(i) x is a strong $\mathcal{F}$-hypercyclic element of the sequence $(\rho_n)_{n \in \mathbb{N}}$ iff $x \in \bigcap_{n \in \mathbb{N}} D(\rho_n)$ and for each $n \in \mathbb{N}$ there exists an element $y_n \in \rho_n(x)$ such that for each open non-empty subset V of Y we have that $\{n \in \mathbb{N} : y_n \in V\} \in \mathcal{F}$; $(\rho_n)_{n \in \mathbb{N}}$ is said to be strongly $\mathcal{F}$-hypercyclic iff there exists a strong $\mathcal{F}$-hypercyclic element of $(\rho_n)_{n \in \mathbb{N}}$;

(ii) ρ is strong $\mathcal{F}$-hypercyclic iff the sequence $(\rho^n)_{n \in \mathbb{N}}$ is strong $\mathcal{F}$-hypercyclic; x is said to be a strong $\mathcal{F}$-hypercyclic element of ρ iff x is a strong $\mathcal{F}$-hypercyclic element of the sequence $(\rho^n)_{n \in \mathbb{N}}$;

(iii) x is an $\mathcal{F}$-hypercyclic element of the sequence $(\rho_n)_{n \in \mathbb{N}}$ iff $x \in \bigcap_{n \in \mathbb{N}} D(\rho_n)$ and for each open non-empty subset V of Y we have that

$$S(x, V) := \big\{n \in \mathbb{N} : \rho_n x \cap V \neq \emptyset\big\} \in \mathcal{F};$$

$(\rho_n)_{n \in \mathbb{N}}$ is said to be $\mathcal{F}$-hypercyclic iff there exists an $\mathcal{F}$-hypercyclic element of $(\rho_n)_{n \in \mathbb{N}}$;

(iv) ρ is $\mathcal{F}$-hypercyclic iff the sequence $(\rho^n)_{n \in \mathbb{N}}$ is $\mathcal{F}$-hypercyclic; x is said to be an $\mathcal{F}$-hypercyclic element of ρ iff x is an $\mathcal{F}$-hypercyclic element of the sequence $(\rho^n)_{n \in \mathbb{N}}$;

(v) $(\rho_n)_{n \in \mathbb{N}}$ is said to be strongly $\mathcal{F}$-topologically transitive iff for every open non-empty subset $U \subseteq X$ and for every integer $n \in \mathbb{N}$ there exists an element $y_n \in \rho_n(U)$ such that for each open non-empty subset V of Y we have that $\{n \in \mathbb{N} : y_n \in V\} \in \mathcal{F}$;

(vi) ρ is strongly $\mathcal{F}$-topologically transitive iff the sequence $(\rho^n)_{n\in\mathbb{N}}$ is strongly $\mathcal{F}$-topologically transitive;

(vii) $(\rho_n)_{n\in\mathbb{N}}$ is said to be $\mathcal{F}$-topologically transitive iff for every two open non-empty subsets $U \subseteq X$ and $V \subseteq Y$ we have that

$$S(U,V) := \{n \in \mathbb{N} : \rho_n(U) \cap V \neq \emptyset\} \in \mathcal{F};$$

(viii) ρ is $\mathcal{F}$-topologically transitive iff the sequence $(\rho^n)_{n\in\mathbb{N}}$ is $\mathcal{F}$-topologically transitive.

In any case set out above, the validity of (I) for $\mathcal{F}$ yields that the strong $\mathcal{F}$-hypercyclicity (topological transitivity) implies, in turn, the $\mathcal{F}$-hypercyclicity (topological transitivity) of considered sequence of binary relations (binary relation, element). This condition also ensures that, for every dynamical property introduced above, say $\mathcal{F}$-hypercyclicity, any extension of an $\mathcal{F}$-hypercyclic binary relation ρ is $\mathcal{F}$-hypercyclic (a similar statement holds for sequences of binary relations). Furthermore, the following holds ([13]):

(i) The validity of (i), resp. (iii), [(ii), resp. (iv)] implies that $\bigcap_{n\in\mathbb{N}} D(\rho_n) \neq \emptyset$ $[D_\infty(\rho) \neq \emptyset]$, while the validity of (v) [(vi)] implies that $D(\rho_n) \neq \emptyset$ for all $n \in \mathbb{N}$ $[D(\rho^n) \neq \emptyset$ for all $n \in \mathbb{N}]$ but not $\bigcap_{n\in\mathbb{N}} D(\rho_n) \neq \emptyset$ $[D_\infty(\rho) \neq \emptyset]$.

(ii) In the case of consideration parts (vii) and (viii), we do not need to have that $D(\rho_n) = \emptyset$ for all $n \in \mathbb{N}$ $[D(\rho^n) \neq \emptyset$ for all $n \in \mathbb{N}]$; if $\emptyset \notin \mathcal{F}$, then the validity of (vii) [(viii)] implies that $D(\rho_n) \neq \emptyset$ for all $n \in \mathbb{N}$ $[D_\infty(\rho) \neq \emptyset]$.

(iii) If $X = Y$ and $(\rho_n)_{n\in\mathbb{N}}$ is a sequence of symmetric binary relations on X, then for every two open non-empty subsets $U \subseteq X$ and $V \subseteq X$, we have $S(U,V) = S(V,U)$. Especially, if ρ is a symmetric binary relation on X, then ρ^n is symmetric as well ($n \in \mathbb{N}$) and therefore $S(U,V) = S(V,U)$ for the sequence $(\rho_n \equiv \rho^n)_{n\in\mathbb{N}}$.

The following is a disjoint analogue of Definition 2.1; see [14] for more details:

Definition 2.2. Suppose that $\mathcal{F} \in P(P(\mathbb{N}))$, $\mathcal{F} \neq \emptyset$, $N \geq 2$, $(\rho_{j,n})_{n\in\mathbb{N}}$ is a sequence of binary relations between the spaces X and Y $(1 \leq j \leq N)$, ρ_j is a binary relation on X $(1 \leq j \leq N)$ and $x \in X$. Then we say that:

(i) x is a strong $d\mathcal{F}$-hypercyclic element of the sequences $(\rho_{1,n})_{n\in\mathbb{N}}, \cdots, (\rho_{N,n})_{n\in\mathbb{N}}$ iff for each $n \in \mathbb{N}$ there exist elements $y_{j,n} \in \rho_{j,n}(x)$ $(1 \leq j \leq N)$ such that for every open non-empty subsets $V_1, \cdots, V_N$ of Y, we have $\{n \in \mathbb{N} : y_{1,n} \in V_1,\ y_{2,n} \in V_2, \cdots,\ y_{N,n} \in V_N\} \in \mathcal{F}$; the sequences $(\rho_{1,n})_{n\in\mathbb{N}}, \cdots, (\rho_{N,n})_{n\in\mathbb{N}}$ are called strongly $d\mathcal{F}$-hypercyclic iff there exists a strong $d\mathcal{F}$-hypercyclic element of $(\rho_{1,n})_{n\in\mathbb{N}}, \cdots, (\rho_{N,n})_{n\in\mathbb{N}}$;

(ii) x is a strong $d\mathcal{F}$-hypercyclic element of the binary relations $\rho_1, \cdots, \rho_N$ iff x is a strong $d\mathcal{F}$-hypercyclic element of the sequences $(\rho_1^n)_{n\in\mathbb{N}}, \cdots, (\rho_N^n)_{n\in\mathbb{N}}$; the binary relations $\rho_1, \cdots, \rho_N$ are called strongly $d\mathcal{F}$-hypercyclic iff there exists a strong $d\mathcal{F}$-hypercyclic element of $\rho_1, \cdots, \rho_N$;

(iii) x is a $d\mathcal{F}$-hypercyclic element of the sequences $(\rho_{1,n})_{n\in\mathbb{N}}, \cdots, (\rho_{N,n})_{n\in\mathbb{N}}$ iff $x \in \bigcap_{1\leq j\leq N, n\in\mathbb{N}} D_\infty(\rho_{j,n})$ and for every open non-empty subsets $V_1, \cdots, V_N$ of Y, we have $(\mathrm{V} = (V_1,\ V_2, \cdots, V_N))$

$$S(x,\mathrm{V}) := \{n \in \mathbb{N} : \rho_{1,n}x \cap V_1 \neq \emptyset,\ \rho_{2,n}x \cap V_2 \neq \emptyset, \cdots,\ \rho_{N,n}x \cap V_N \neq \emptyset\} \in \mathcal{F};$$

the sequences $(\rho_{1,n})_{n\in\mathbb{N}}, \cdots, (\rho_{N,n})_{n\in\mathbb{N}}$ are called $d\mathcal{F}$-hypercyclic iff there exists a $d\mathcal{F}$-hypercyclic element of $(\rho_{1,n})_{n\in\mathbb{N}}, \cdots, (\rho_{N,n})_{n\in\mathbb{N}}$;

(iv) x is a $d\mathcal{F}$-hypercyclic element of the binary relations $\rho_1, \cdots, \rho_N$ iff x is a $d\mathcal{F}$-hypercyclic element of the sequences $(\rho_1^n)_{n\in\mathbb{N}}, \cdots, (\rho_N^n)_{n\in\mathbb{N}}$; the binary relations $\rho_1, \cdots, \rho_N$ are called $d\mathcal{F}$-hypercyclic iff there exists a $d\mathcal{F}$-hypercyclic element of $\rho_1, \cdots, \rho_N$.

Definition 2.3. Suppose that $\mathcal{F} \in P(P(\mathbb{N}))$, $\mathcal{F} \neq \emptyset$, $N \geq 2$, $(\rho_{j,n})_{n\in\mathbb{N}}$ is a sequence of binary relations between the spaces X and Y $(1 \leq j \leq N)$, and ρ_j is a binary relation on X $(1 \leq j \leq N)$. Then we say that:

(i) the sequences $(\rho_{1,n})_{n\in\mathbb{N}}, \cdot\cdot\cdot, (\rho_{N,n})_{n\in\mathbb{N}}$ are strongly $d\mathcal{F}$-topologically transitive iff for every open non-empty subset $U \subseteq X$ and and for every open non-empty subsets $V_1, \cdot\cdot\cdot, V_N$ of Y, there exists an element $x \in U$ such that, for every integers $n \in \mathbb{N}$ and $j \in \mathbb{N}_N$, there exists an element $y_{j,n} \in \rho_{j,n}x$ so that $\{n \in \mathbb{N} : y_{j,n} \in V_j \text{ for all } j \in \mathbb{N}_N\} \in \mathcal{F}$;

(ii) the binary relations $\rho_1, \cdot\cdot\cdot, \rho_N$ are called strongly $d\mathcal{F}$-topologically transitive iff the sequences $(\rho_1^n)_{n\in\mathbb{N}}, \cdot\cdot\cdot, (\rho_N^n)_{n\in\mathbb{N}}$ are strongly $d\mathcal{F}$-topologically transitive;

(iii) the sequences $(\rho_{1,n})_{n\in\mathbb{N}}, \cdot\cdot\cdot, (\rho_{N,n})_{n\in\mathbb{N}}$ are $d\mathcal{F}$-topologically transitive iff for every open non-empty subset $U \subseteq X$ and and for every open non-empty subsets $V_1, \cdot\cdot\cdot, V_N$ of Y, we have that $\{n \in \mathbb{N} : (\exists x \in U)\, \rho_{j,n}x \cap V_j \neq \emptyset \text{ for all } j \in \mathbb{N}_N\} \in \mathcal{F}$;

(iv) the binary relations $\rho_1, \cdot\cdot\cdot, \rho_N$ are $d\mathcal{F}$-topologically transitive iff the sequences $(\rho_1^n)_{n\in\mathbb{N}}, \cdot\cdot\cdot, (\rho_N^n)_{n\in\mathbb{N}}$ are $d\mathcal{F}$-topologically transitive.

If the binary relations $\rho_1, \cdot\cdot\cdot, \rho_N$ are $d\mathcal{F}$-hypercyclic ($d\mathcal{F}$-topologically transitive), then we also say that the tuple $(\rho_1, \cdot\cdot\cdot, \rho_N)$ is $d\mathcal{F}$-hypercyclic ($d\mathcal{F}$-topologically transitive) and vice versa. For our later purposes, it will be necessary to recall the following fact from [14]:

(D) Suppose that the binary relations $\rho_1, \rho_2, \cdot\cdot\cdot, \rho_N$ on X are $d\mathcal{F}$-hypercyclic ($d\mathcal{F}$-topologically transitive). Then any of them is $\mathcal{F}$-hypercyclic ($\mathcal{F}$-topologically transitive) provided that $\rho_1 = \rho_2 = \cdot\cdot\cdot = \rho_N$ or that condition (I) holds for $\mathcal{F}$.

3. $\mathcal{F}$-hypercyclic (MLO) extensions and $\mathcal{F}$-topologically transitive (MLO) extensions

The main aim of this section is to provide the basic information on $\mathcal{F}$-hypercyclic ($\mathcal{F}$-topologically transitive) extensions of binary relations and $\mathcal{F}$-hypercyclic ($\mathcal{F}$-topologically transitive) multivalued linear operator (MLO) extensions of binary relations. For anything that follows, the next example will be crucially important:

Example. Suppose that $\mathbb{N} \in \mathcal{F}$, $\rho := X \times X$ and X is equipped with arbitrary topology. Then ρ is $\mathcal{F}$-hypercyclic with any element $x \in X$ being the hypercyclic vector of ρ, and ρ is $\mathcal{F}$-topologically transitive, as well. On the other hand, there exists a great number of concrete situations in which $\rho = X \times X$ cannot be strongly $\mathcal{F}$-hypercyclic; for example, let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{a, b\}, \{c, d\}, \{a, b, c, d\}\}$ and $\mathcal{F} = \{\mathbb{N}, 2\mathbb{N}\}$. Then there is no strong $\mathcal{F}$-hypercyclic vector x for ρ because, if we suppose the contrary, then for the sequence (y_n) in X satisfying the requirements prescribed in Definition 2.1(i) we need to have $\{n \in \mathbb{N} : y_n = a \text{ or } y_n = b\} \in \mathcal{F}$ and $\{n \in \mathbb{N} : y_n = c \text{ or } y_n = d\} \in \mathcal{F}$, which immediately leads to the fact that y_{2n} needs to be simultaneously an element of the set $\{a, b\}$ and an element of the set $\{c, d\}$, for any $n \in \mathbb{N}$; this is a contradiction. A similar line of reasoning shows that $\rho = X \times X$ cannot be strongly $\mathcal{F}$-topologically transitive for $\mathcal{F} = \{\mathbb{N}, 2\mathbb{N}\}$.

Based on the conclusions obtained, in the sequel of this section we will always assume that X is a topological space and $\mathbb{N} \in \mathcal{F}$. We will limit ourselves to the study of $\mathcal{F}$-hypercyclicity and $\mathcal{F}$-topological transitivity.

Let ρ be a binary relation on X. Define

$$S(\rho) := \{\rho' \subseteq X \times X : \rho \subseteq \rho'\}.$$

Then it is said that an element ρ' of $S(\rho)$ is an $\mathcal{F}$-hypercyclic ($\mathcal{F}$-topologically transitive) extension of ρ iff $\rho' \in S(\rho)$ and ρ' is $\mathcal{F}$-hypercyclic ($\mathcal{F}$-topologically transitive). The set consisting of all $\mathcal{F}$-hypercyclic ($\mathcal{F}$-topologically transitive) extensions of ρ, denoted by $S_h^{\mathcal{F}}(\rho)$ ($S_{tt}^{\mathcal{F}}(\rho)$), is non-empty since it contains the full relation $X \times X$ (see the previous example). In the case that $\mathcal{F}$ is a collection of all non-empty subsets of $\mathbb{N}$, then we say that ρ' is a hypercyclic (topologically transitive) extension of ρ and denote by $S_h(\rho)$ ($S_{tt}(\rho)$) the set consisting of all such binary relations. Similar agreement and notation will be used for any (quasi, disjoint) extension defined below, so that, e.g., quasi $\mathcal{F}$-hypercyclic extension will be quasi hypercyclic extension with $\mathcal{F}$ being the collection of all non-empty subsets of $\mathbb{N}$.

Consider now the following extensions of a binary relation ρ on X :

$$\tilde{\rho}_{\mathcal{F}} := \bigcap \bigl\{\rho' \in S(\rho) : \rho' \text{ is } \mathcal{F}\text{-hypercyclic}\bigr\}$$

and

$$\hat{\rho}_{\mathcal{F}} := \bigcap \bigl\{\rho' \in S(\rho) : \rho' \text{ is } \mathcal{F}\text{-topologically transitive}\bigr\}.$$

We call $\tilde{\rho}_{\mathcal{F}}$, resp. $\hat{\rho}_{\mathcal{F}}$, the quasi $\mathcal{F}$-hypercyclic, resp. the quasi $\mathcal{F}$-topologically transitive, extension of ρ.

Assume also that X is a linear vector space. Then we define

$$\mathrm{S}(\rho) := \bigl\{\mathcal{A}' \in MLO(X) : \rho \subseteq \mathcal{A}'\bigr\}.$$

Then it is said that an element $\mathcal{A}'$ of $\mathrm{S}(\rho)$ is an $\mathcal{F}$-hypercyclic ($\mathcal{F}$-topologically transitive) MLO extension of ρ iff $\mathcal{A}' \in \mathrm{S}(\rho)$ and $\mathcal{A}'$ is $\mathcal{F}$-hypercyclic ($\mathcal{F}$-topologically transitive). It is clear that the validity of condition (I) implies that, if ρ' is an $\mathcal{F}$-hypercyclic ($\mathcal{F}$-topologically transitive, resp. $\mathcal{A}' \in MLO(X)$ is an $\mathcal{F}$-hypercyclic MLO, $\mathcal{A}' \in MLO(X)$ is an $\mathcal{F}$-topologically transitive MLO) extension of ρ and $\rho' \subseteq \rho''$ (resp., $\mathcal{A}'' \in MLO(X)$ and $\mathcal{A}' \subseteq \mathcal{A}''$), then ρ'' is an $\mathcal{F}$-hypercyclic ($\mathcal{F}$-topologically transitive, resp. $\mathcal{A}''$ is an $\mathcal{F}$-hypercyclic MLO, $\mathcal{A}''$ is an $\mathcal{F}$-topologically transitive MLO) extension of ρ, as well. The set consisting of all $\mathcal{F}$-hypercyclic MLO ($\mathcal{F}$-topologically transitive MLO) extensions of ρ, denoted by $S^{\mathcal{F}}_{h,MLO}(\rho)$ ($S^{\mathcal{F}}_{tt,MLO}(\rho)$), is non-empty. The quasi $\mathcal{F}$-hypercyclic MLO and quasi $\mathcal{F}$-topologically transitive MLO extension of ρ are defined respectively through

$$\tilde{\rho}_{\mathcal{F},MLO} := \bigcap \bigl\{\mathcal{A}' \in \mathrm{S}(\rho) : \mathcal{A}' \text{ is } \mathcal{F}\text{-hypercyclic}\bigr\}$$

and

$$\hat{\rho}_{\mathcal{F},MLO} := \bigcap \bigl\{\mathcal{A}' \in \mathrm{S}(\rho) : \mathcal{A}' \text{ is } \mathcal{F}\text{-topologically transitive}\bigr\}.$$

It is clear that any $\mathcal{F}$-hypercyclic ($\mathcal{F}$-topologically transitive) MLO extension of ρ is already $\mathcal{F}$-hypercyclic ($\mathcal{F}$-topologically transitive) extension of ρ. Furthermore, we have:

Proposition 3.1. *(i)* $span(\tilde{\rho}_{\mathcal{F}}) \subseteq \tilde{\rho}_{\mathcal{F},MLO}$ *and* $span(\hat{\rho}_{\mathcal{F}}) \subseteq \hat{\rho}_{\mathcal{F},MLO}$.

(ii) $\bigcap_{\rho' \in S_h(\rho)} span(\rho') \subseteq \tilde{\rho}_{\mathcal{F},MLO}$ *and* $\bigcap_{\rho' \in S_{tt}(\rho)} span(\rho') \subseteq \hat{\rho}_{\mathcal{F},MLO}$.

(iii) Suppose that $\mathcal{F}$ *satisfies* (I). *Then* $\bigcap_{\rho' \in S_h(\rho)} span(\rho') = \tilde{\rho}_{\mathcal{F},MLO}$ *and* $\bigcap_{\rho' \in S_{tt}(\rho)} span(\rho') = \hat{\rho}_{\mathcal{F},MLO}$.

Proof. For (i), it suffices to observe that $\tilde{\rho}_{\mathcal{F}} \subseteq \tilde{\rho}_{\mathcal{F},MLO}$ and $\hat{\rho}_{\mathcal{F}} \subseteq \hat{\rho}_{\mathcal{F},MLO}$ by definition as well as that $\tilde{\rho}_{\mathcal{F},MLO}$ and $\hat{\rho}_{\mathcal{F},MLO}$ are vector subspaces of $X \times X$. The proof of (ii) is trivial and therefore omitted. Suppose now that $\mathcal{F}$ satisfies (I). For the proofs of inclusions $\tilde{\rho}_{\mathcal{F},MLO} \subseteq \bigcap_{\rho' \in S_h(\rho)} span(\rho')$ and $\hat{\rho}_{\mathcal{F},MLO} \subseteq \bigcap_{\rho' \in S_{tt}(\rho)} span(\rho')$, it suffices to observe that for each $\mathcal{F}$-hypercyclic ($\mathcal{F}$-topologically transitive) relation ρ' containing ρ, we have that $span(\rho')$ is an $\mathcal{F}$-hypercyclic MLO ($\mathcal{F}$-topologically transitive MLO) extension of ρ due to condition (I). □

We continue by stating a few illustrative examples and remarks about $\mathcal{F}$-hypercyclic ($\mathcal{F}$-topologically transitive) extensions of simple graphs; unless stated otherwise, we will denote by $x_1, x_2, \cdots, x_n$ the nodes of G. In [6], we have shown that the graph G is connected iff G is hypercyclic (topologically transitive) iff G is (Devaney) chaotic; if this is the case, then any element of G is a hypercyclic element of ρ. As the next example shows, the situation is far from being clear and so simple for $\mathcal{F}$-hypercyclicity:

Example. Let $G = \{x_1, x_2, x_3, x_4\}$ be equipped with discrete topology, let G be the unoriented square $x_1x_2x_3x_4$, and let $\mathcal{F}_0$ contain, besides the set $\mathbb{N}$, the collection of all non-empty subsets of $\mathbb{N}$ containing only odd numbers. Then, for every $i \in \mathbb{N}_4$ and $n \in 2\mathbb{N}+1$, we have that $x_i \notin \rho^n x_i$, which simply implies that G do not possess any of the introduced $\mathcal{F}_0$-dynamical properties from Definition 2.1; furthermore, G is $\mathcal{F}$-hypercyclic ($\mathcal{F}$-topologically transitive) iff $\{2\mathbb{N}, 2\mathbb{N}+1\} \subseteq \mathcal{F}$. Assume now that $2\mathbb{N} \notin \mathcal{F}$ or $2\mathbb{N}+1 \notin \mathcal{F}$. Then there exist two subcases: $\mathbb{N} \setminus \{1\} \in \mathcal{F}$ or $\mathbb{N} \setminus \{1\} \notin \mathcal{F}$. In the first subcase, we can add the unoriented arc x_1x_3 or x_2x_4 to G; then the resulted graph G' will be both $\mathcal{F}$-hypercyclic and $\mathcal{F}$-topologically transitive, so that the quasi $\mathcal{F}$-hypercyclic (quasi $\mathcal{F}$-topologically transitive) extension of G will be the graph G itself. In the second subcase, the complete graph K_4 is not $\mathcal{F}$-hypercyclic ($\mathcal{F}$-topologically transitive) and therefore there is no $\mathcal{F}$-hypercyclic ($\mathcal{F}$-topologically transitive) extension of G within the class of simple graphs; it can be easily seen that any $\mathcal{F}$-hypercyclic ($\mathcal{F}$-topologically transitive) extension of G contains the binary relation ρ' obtained as the union of G and four loops $x_i \mapsto x_i$ ($i \in \mathbb{N}_4$), which is a quasi $\mathcal{F}$-hypercyclic (quasi $\mathcal{F}$-topologically transitive) extension of G.

Concerning the complete graph K_n, where $n \geq 2$, we have the following elementary observations:

Example. Let K_n be equipped with discrete topology. We consider two possible cases:

(i) $n = 2$: Then K_n is $\mathcal{F}$-hypercyclic ($\mathcal{F}$-topologically transitive) iff $\{2\mathbb{N}, 2\mathbb{N}+1\} \subseteq \mathcal{F}$. If $2\mathbb{N} \notin \mathcal{F}$ or $2\mathbb{N}+1 \notin \mathcal{F}$, then the situation is quite similar to that one examined in the former example: if $\mathbb{N} \setminus \{1\} \in \mathcal{F}$, then adding any of two loops $x_1 \mapsto x_1$ or $x_2 \mapsto x_2$ to G has as a result that the obtained binary relation ρ', with the meaning clear, is $\mathcal{F}$-hypercyclic ($\mathcal{F}$-topologically transitive). If $\mathbb{N}\setminus\{1\} \notin \mathcal{F}$, then the full relation is the only $\mathcal{F}$-topologically transitive extension of G, which is not the case for $\mathcal{F}$-hypercyclic extensions: adding the loop $x_1 \mapsto x_1$ to K_n has as a result that the obtainted binary relation ρ' is $\mathcal{F}$-hypercyclic with x_1 being its $\mathcal{F}$-hypercyclic vector.

(ii) $n \geq 3$: Then K_n is $\mathcal{F}$-hypercyclic ($\mathcal{F}$-topologically transitive) iff $\mathbb{N} \setminus \{1\} \in \mathcal{F}$. In the case that $\mathbb{N} \setminus \{1\} \notin \mathcal{F}$, the smallest $\mathcal{F}$-topologically transitive binary relation containing K_n is the full relation, while adding the loop $x_1 \mapsto x_1$ to K_n has as a result that the obtainted binary relation ρ' is $\mathcal{F}$-hypercyclic with x_1 being its $\mathcal{F}$-hypercyclic vector.

Let G be equipped with discrete topology. If $\emptyset \notin \mathcal{F}$, then the $\mathcal{F}$-hypercyclicity ($\mathcal{F}$-topological transitivity) of G implies that G is connected. On the other hand, if $\emptyset \in \mathcal{F}$ and G is not connected, then G cannot be $\mathcal{F}$-hypercyclic ($\mathcal{F}$-topologically transitive) and any $\mathcal{F}$-hypercyclic ($\mathcal{F}$-topologically transitive) extension of G contains at least one unoriented arc connected two nodes belonging to different components of G.

In the remaining part of this section, we will examine $\mathcal{F}$-hypercyclic ($\mathcal{F}$-topologically transitive) MLO extensions. For this, we need to recall the following facts from [5]. Suppose that $A \in L(X)$, where X is a Fréchet space whose topology is induced by the fundamental system $(p_n)_{n\in\mathbb{N}}$ of increasing seminorms and $L(X)$ denotes the space consisting of all linear continuous mappings from X into X. We endow X with the F-norm $\|\cdot\| := d(0,\cdot)$, where the translation invariant metric $d : X \times X \to [0,\infty)$ is defined by

$$d(x,y) := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{p_n(x-y)}{1+p_n(x-y)}, \ x, \ y \in X.$$

Let us recall that this metric satisfies, among many other properties, the following ones: $d(x+u, y+v) \leq d(x,y) + d(u,v)$ and $d(cx, cy) \leq (|c|+1)d(x,y)$, $c \in \mathbb{K}$, x, y, u, $v \in X$. Set $L(x,\epsilon) := \{y \in X : d(x,y) < \epsilon\}$.

We know that any MLO extension $\mathcal{A}$ of A has the form $\mathcal{A}x = Ax + W$, $x \in X$, where $W = \mathcal{A}0$ is a linear subspace of X ([5]). Inductively, we can show that

$$\mathcal{A}^n x = A^n x + \sum_{j=0}^{n-1} A^j(W), \quad n \in \mathbb{N}, \ x \in X. \tag{3.1}$$

This formula enables one to profile the quasi $\mathcal{F}$-hypercyclic MLO extension of A in the following manner: Denote by $\mathcal{T}$ the set consisting of all linear submanifolds W of X such that, for every open non-empty subset V of X, we have that

$$S_W(x,V) := \left\{ n \in \mathbb{N} : (\exists \omega_0, \ \omega_1, \cdots, \ \omega_{n-1} \in W)\, A^n x + \sum_{j=0}^{n-1} A^j \omega_j \in V \right\} \in \mathcal{F}.$$

Then $\tilde{A}_{\mathcal{F},MLO} = A + \bigcap \mathcal{T}$. We can similarly profile the quasi $\mathcal{F}$–topologically transitive MLO extension of A.

Calculating precisely $\mathcal{F}$-hypercyclic ($\mathcal{F}$-topologically transitive) extensions and $\mathcal{F}$-hypercyclic ($\mathcal{F}$-topologically transitive) MLO extensions of binary relations, even certain multiples of the identity operators, is a non-trivial problem:

Example. (i) Suppose that X is a finite-dimensional Banach space and $A \in L(X)$. Then we have seen in [5, Example 4.1(a)] that A cannot be hypercyclic (topologically transitive) as well as that the only hypercyclic (topologically transitive) MLO extension of A is $X \times X$. If $\emptyset \notin \mathcal{F}$, this immediately implies that A cannot be $\mathcal{F}$-hypercyclic ($\mathcal{F}$-topologically transitive) as well as that the only $\mathcal{F}$-hypercyclic ($\mathcal{F}$-topologically transitive) MLO extension of A is $X \times X$. The situation is quite different in the case that $\emptyset \in \mathcal{F}$: then any continuous linear operator on a Fréchet space is $\mathcal{F}$-hypercyclic with zero vector being its $\mathcal{F}$-hypercyclic vector.

(ii) Suppose that X is a Fréchet space, $A \in L(X)$, W is a dense linear subspace of X and $\mathcal{A}x := Ax + W$, $x \in X$. Then we know that for each $n \in \mathbb{N}$ and for each pair $U,\ V$ of open non-empty subsets of X, we have $\mathcal{A}^n U \cap V \neq \emptyset$; see [5, Example 4.2]. Hence, our standing assumption $\mathbb{N} \in \mathcal{F}$ yields that $\mathcal{A}$ is $\mathcal{F}$-topologically transitive.

(iii) Suppose that X is a Fréchet space and $A \in L(X)$. Then A may or may not be $\mathcal{F}$-topologically transitive and the most simplest example is provided by the operator $A = cI$, where I denotes the identity operator on X and $c \in \mathbb{K}$, with $\mathbb{K}$ being the field of scalars. Then we have the following:

(a) If $c = 0$ or $c = 1$, then A is $\mathcal{F}$-topologically transitive iff $\emptyset \in \mathcal{F}$. In the case that $\emptyset \notin \mathcal{F}$, we can use the part (ii) and an elementary line of reasoning to see that any $\mathcal{F}$-topologically transitive MLO extension $\mathcal{A}$ of A has the form $\mathcal{A}x := Ax + W$, $x \in X$, where W is a dense linear subspace of X (see also [5, Example 4.1(iii)]).

(b) Suppose $|c| = 1$ and $c \neq 1$. If $\mathbb{K} = \mathbb{R}$, then $c = -1$ and A is $\mathcal{F}$-topologically transitive iff $\{\emptyset, 2\mathbb{N}, 2\mathbb{N}+1\} \subseteq \mathcal{F}$. Suppose now $\mathbb{K} = \mathbb{C}$. Then the result is same for $c = -1$, when any $\mathcal{F}$-topologically transitive MLO extension $\mathcal{A}$ of A has the form $\mathcal{A}x := Ax + W$, $x \in X$, where W is a dense linear subspace of X, provided that $\emptyset \notin \mathcal{F}$ or $2\mathbb{N} \notin \mathcal{F}$ or $2\mathbb{N}+1 \notin \mathcal{F}$. Speaking-matter-of-factly, if $\emptyset \notin \mathcal{F}$ and W is not dense in X, then there exists $v \in X$ and $\epsilon > 0$ such that $B(v, \epsilon) := \{x \in X : d(x, v) \leq \epsilon\} \subseteq X \setminus \overline{W}$. Then, for a small ball U around zero and $V = L(v, \epsilon/3) = \{x \in X : d(x, v) < \epsilon/3\}$, we have $[(-1)^n U + W] \cap V = \emptyset$ for all $n \in \mathbb{N}$ so that $\{n \in \mathbb{N} : \mathcal{A}^n U \cap V \neq \emptyset\} = \emptyset$ and the MLO $\mathcal{A}$ defined above cannot be $\mathcal{F}$-topologically transitive. If $2\mathbb{N} \notin \mathcal{F}$, then we can take $U = V = L(v, \delta/3)$, where $\delta = d(2v, \overline{W}) := \inf_{w \in \overline{W}} d(2v, w)$, so that $\{n \in \mathbb{N} : \mathcal{A}^n U \cap V \neq \emptyset\} = 2\mathbb{N}$ and $\mathcal{A}$ cannot be $\mathcal{F}$-topologically transitive; to verify the last equality, it is enough to show that $(U + W) \cap V \neq \emptyset$ and $(-U + W) \cap V = \emptyset$. The first equality is trivial while the second one can be simply deduced by assuming the contraposition. Indeed, if we assume that $(-U + W) \cap V \neq \emptyset$, then there exist $\omega \in W$ and two elements $v_1,\ v_2 \in V$ such that $\omega = v_1 + v_2$, wich is a contradiction since $\|2v - (v_1 + v_2)\| \leq \|v - v_1\| + \|v - v_2\| < 2\delta/3$. If $2\mathbb{N}+1 \notin \mathcal{F}$, then we can take $V = L(v, \delta/3)$, and $U = -V$, with δ being defined in the same manner. Otherwise, we recognize the following subcases:

(b1) there exist natural numbers p and q such that $p < q$, $(p, q) = 1$ and $c = e^{i\alpha}$ with $\alpha = 2\pi p/q$. Then A is $\mathcal{F}$-topologically transitive iff $\mathcal{F}$ contains the sets $\emptyset$, $q\mathbb{N}$, $q\mathbb{N}+1, \cdots, q\mathbb{N}+q-1$ and their finite unions. Assume now that A is not $\mathcal{F}$-topologically transitive. Then a similar argumentation as in the case $c = -1$ shows that any $\mathcal{F}$-topologically transitive MLO extension $\mathcal{A}$ of A has the form $\mathcal{A}x := Ax + W$, $x \in X$, where W is a dense linear subspace of X; to see this, it is enough to find, for every numbers $0 \leq n_1 < \cdots < n_k \leq q-1$ ($k \in \mathbb{N}$), two open non-empty subsets $U,\ V$ of X such that $\{n \in \mathbb{N} : [e^{2\pi i p n/q} U + W] \cap V \neq \emptyset\} = \bigcup_{1 \leq j \leq k} (q\mathbb{N} + n_j)$. We can take $U = L(v, \nu)$ and $V = \bigcup_{1 \leq j \leq k} e^{2\pi i p n_j/q} U$, where $\nu > 0$ is sufficiently small and $v \notin \overline{W}$.

(b2) $c = e^{2\pi i \alpha}$ for some irrational number $\alpha \in (0, 1)$. Then $\{c^n : n \in \mathbb{N}\}$ is a dense subset of the unit circle of complex plane and it is not so simple to profile the form of family $\mathcal{F}$ for which

A is $\mathcal{F}$-topologically transitive, as well as to prove that any $\mathcal{F}$-topologically transitive MLO extension $\mathcal{A}$ of A has the same form as in the parts (a) and (b1).

(c) If $0 < |c| \neq 1$, then A is $\mathcal{F}$-topologically transitive iff $\mathcal{F} = P(P(\mathbb{N}))$. Assume, indeed, that $|c| > 1$ ($|c| < 1$). To see that A is $\mathcal{F}$-topologically transitive iff $\mathcal{F} = P(P(\mathbb{N}))$, it is sufficient to construct, for any given subset D of $\mathbb{N}$, two open non-empty subsets U, V of X such that $\{n \in \mathbb{N} : c^n U \cap V \neq \emptyset\} = D$. This is clear for $D = \emptyset$, when we can take $V = \{x \in X : \|x\| < 1\}$ ($V = \{x \in X : \|x\| > 1\}$) and $U = \{x \in X : \|x\| > 1\}$ ($U = \{x \in X : \|x\| < 1\}$). If $D = \{n_1,\ n_2, \cdots,\ n_k, \cdots\}$, where (n_k) is an increasing sequence of natural numbers, we can take $U = \{x \in X : 1 < \|x\| < c\}$ ($U = \{x \in X : c < \|x\| < 1\}$) and $V = \bigcup_{k\in\mathbb{N}}\{x \in X : c^{n_k} < \|x\| < c^{n_k+1}\}$ ($V = \bigcup_{k\in\mathbb{N}}\{x \in X : c^{n_k+1} < \|x\| < c^{n_k}\}$). In the case that X is finite-dimensional, we can easily show that the only $\mathcal{F}$-topologically transitive MLO extension of A is the full relation $X \times X$, provided that $\mathcal{F} \neq P(P(\mathbb{N}))$. If X is infinite-dimensional and the last condition holds, then, in the present situation, we cannot tell whether any $\mathcal{F}$-topologically transitive MLO extension $\mathcal{A}$ of A is of the same form as in the parts (a) and (b1).

4. Disjoint $\mathcal{F}$-hypercyclic (MLO) extensions and disjoint $\mathcal{F}$-topologically transitive (MLO) extensions

In this section, we will analyze disjoint $\mathcal{F}$-hypercyclic (MLO) extensions of binary relations and disjoint $\mathcal{F}$-topologically transitive (MLO) extensions of binary relations. As before, by X we denote a general non-trivial topological space.

The following example is very similar to the first example of previous section:

Example. Suppose that $\mathbb{N} \in \mathcal{F}$, $N \in \mathbb{N} \setminus \{1\}$, $\rho_j := X \times X$ for $j \in \mathbb{N}_N$ and X is equipped with arbitrary topology. Then $\rho_1, \cdots, \rho_N$ are disjoint $\mathcal{F}$-hypercyclic with any element $x \in X$ being its d-hypercyclic vector; furthermore, $\rho_1, \cdots, \rho_N$ are disjoint $\mathcal{F}$-topologically transitive, as well. A class of very elementary counterexamples shows that these binary relations cannot be strongly disjoint $\mathcal{F}$-hypercyclic or strongly disjoint $\mathcal{F}$-topologically transitive.

Because of that, we assume henceforth that $\mathbb{N} \in \mathcal{F}$ and focus our attention only to disjoint $\mathcal{F}$-hypercyclicity and disjoint $\mathcal{F}$-topological transitivity.

Let $N \geq 2$, and let $\rho_1, \cdots, \rho_N$ be given binary relations on X. Then the binary relations $X \times X, \cdots, X \times X$, totally counted N times, are both disjoint $\mathcal{F}$-hypercyclic and disjoint $\mathcal{F}$-topologically transitive. Set

$$S(\rho_1, \cdots, \rho_N) := \Big\{ (\sigma_1, \cdots, \sigma_N) : \sigma_i \subseteq X \times X \text{ and } \rho_i \subseteq \sigma_i \text{ for all } i \in \mathbb{N}_N \Big\}.$$

We say that the tuple $(\sigma_1, \cdots, \sigma_N)$ of binary relations on X is a disjoint $\mathcal{F}$-hypercyclic (disjoint $\mathcal{F}$-topologically transitive) extension of $(\rho_1, \cdots, \rho_N)$ iff $(\sigma_1, \cdots, \sigma_N) \in S(\rho_1, \cdots, \rho_N)$ and the binary relations $\sigma_1, \cdots, \sigma_N$ are disjoint $\mathcal{F}$-hypercyclic (disjoint $\mathcal{F}$-topologically transitive). By $S^{\mathcal{F}}_{dh}(\rho_1, \cdots, \rho_N)$ and $S^{\mathcal{F}}_{dtt}(\rho_1, \cdots, \rho_N)$ we denote the sets consisting of all disjoint $\mathcal{F}$-hypercyclic extensions and all disjoint $\mathcal{F}$-topologically transitive extensions of tuple $(\rho_1, \cdots, \rho_N)$, respectively. These sets are non-empty, clearly. Disjoint $\mathcal{F}$-hypercyclic (disjoint $\mathcal{F}$-topologically transitive) extension will be shortly called $d\mathcal{F}$-hypercyclic ($d\mathcal{F}$-topologically transitive) extension; the same terminology will be accepted for disjoint $\mathcal{F}$-hypercyclic (disjoint $\mathcal{F}$-topologically transitive) MLO extensions introduced below.

The notions of a disjoint quasi $\mathcal{F}$-hypercyclic extension and a disjoint quasi $\mathcal{F}$-topologically transitive extension of tuple $(\rho_1, \cdots, \rho_N)$ is introduced by

$$\widetilde{(\rho_1, \cdots, \rho_N)}_{\mathcal{F}} := \bigcap \big\{ (\sigma_1, \cdots, \sigma_N) \in S(\rho_1, \cdots, \rho_N) : \sigma_1, \cdots, \sigma_N \text{ are } d\mathcal{F}\text{-hypercyclic} \big\}$$

and

$$\widehat{(\rho_1, \cdots, \rho_N)}_{\mathcal{F}} := \bigcap \big\{ (\sigma_1, \cdots, \sigma_N) \in S(\rho_1, \cdots, \rho_N) : \sigma_1, \cdots, \sigma_N \text{ are } d\mathcal{F}\text{-topologically transitive} \big\},$$

respectively. Here, the intersection is taken with respect to the components.

Assume also that X is a linear vector space. Then we define

$$\mathrm{S}(\rho_1,\cdots,\rho_N) := \Big\{(\mathcal{A}_1',\cdots,\mathcal{A}_N') : \mathcal{A}_i' \in MLO(X) \text{ and } \rho_i \subseteq \mathcal{A}_i' \text{ for all } i \in \mathbb{N}_N\Big\}.$$

Then it is said that an element $(\mathcal{A}_1',\cdots,\mathcal{A}_N')$ of $\mathrm{S}(\rho_1,\cdots,\rho_N)$ is a disjoint $\mathcal{F}$-hypercyclic (disjoint $\mathcal{F}$-topologically transitive) MLO extension of tuple $(\rho_1,\cdots,\rho_N)$ iff $\mathcal{A}_1',\cdots,\ \mathcal{A}_N'$ are disjoint $\mathcal{F}$-hypercyclic (disjoint $\mathcal{F}$-topologically transitive). If the condition (I) holds and $(\sigma_1,\cdots,\sigma_N)$ is a $d\mathcal{F}$-hypercyclic extension ($d\mathcal{F}$-topologically transitive extension) of tuple $(\rho_1,\cdots,\rho_N)$, resp. $(\mathcal{A}_1,\cdots,\mathcal{A}_N)\in \mathrm{S}(\rho_1,\cdots,\rho_N)$ is a $d\mathcal{F}$-hypercyclic MLO extension ($d\mathcal{F}$-topologically transitive MLO extension) of $(\rho_1,\cdots,\rho_N)$ and $\sigma_i \subseteq \sigma_i'$ for all $i\in\mathbb{N}_n$, resp. $(\mathcal{A}_1',\cdots,\mathcal{A}_N')\in \mathrm{S}(\mathcal{A}_1,\cdots,\mathcal{A}_N)$, then $(\sigma_1',\cdots,\sigma_N')$ is a $d\mathcal{F}$-hypercyclic extension ($d\mathcal{F}$-topologically transitive extension) of tuple $(\rho_1,\cdots,\rho_N)$, resp. $(\mathcal{A}_1',\cdots,\mathcal{A}_N')$ is a $d\mathcal{F}$-hypercyclic MLO extension ($d\mathcal{F}$-topologically transitive MLO extension) of $(\rho_1,\cdots,\rho_N)$.

The set consisting of all disjoint $\mathcal{F}$-hypercyclic MLO (disjoint $\mathcal{F}$-topologically transitive MLO) extensions of $(\rho_1,\cdots,\rho_N)$, denoted by $S^{\mathcal{F}}_{dh,MLO}(\rho_1,\cdots,\rho_N)$ ($S^{\mathcal{F}}_{dtt,MLO}(\rho_1,\cdots,\rho_N)$), is non-empty. The disjoint quasi $\mathcal{F}$-hypercyclic MLO extension and disjoint quasi $\mathcal{F}$-topologically transitive MLO extension of $(\rho_1,\cdots,\rho_N)$, are defined respectively through

$$\big(\widetilde{\rho_1,\cdots,\rho_N}\big)_{\mathcal{F},MLO} := \bigcap\big\{\big(\mathcal{A}_1',\cdots,\mathcal{A}_N'\big)\in \mathrm{S}\big(\rho_1,\cdots,\rho_N\big) : \mathcal{A}_1,\cdots,\mathcal{A}_N \text{ are } d\mathcal{F}\text{-hypercyclic}\big\}$$

and

$$\big(\widehat{\rho_1,\cdots,\rho_N}\big)_{\mathcal{F},MLO} := \bigcap\big\{\big(\mathcal{A}_1',\cdots,\mathcal{A}_N'\big)\in \mathrm{S}\big(\rho_1,\cdots,\rho_N\big) : \mathcal{A}_1,\cdots,\mathcal{A}_N \text{ are } d\mathcal{F}\text{-topologically transitive}\big\}.$$

By our definitions, we have that $(\widetilde{\rho_1,\cdots,\rho_N})_{\mathcal{F}} \in (P(X\times X))^N$ and $(\widehat{\rho_1,\cdots,\rho_N})_{\mathcal{F}} \in (P(X\times X))^N$ as well as $(\widetilde{\rho_1,\cdots,\rho_N})_{\mathcal{F},MLO} \in (MLO(X))^N$ and $(\widehat{\rho_1,\cdots,\rho_N})_{\mathcal{F},MLO} \in (MLO(X))^N$. Denote by $\mathrm{P}_i : (P(X\times X))^N \to P(X\times X)$ the i-th projection defined by $\mathrm{P}_i(\rho_1,\cdots,\rho_N) := \rho_i$, $(\rho_1,\cdots,\rho_N)\in (P(X\times X))^N$ $(i\in\mathbb{N}_N)$.

The following disjoint analogue of Proposition 3.1, stated here without a simple proof, holds good:

Proposition 4.1. *(i) We have* $span(\mathrm{P}_i((\widetilde{\rho_1,\cdots,\rho_N})_{\mathcal{F}})) \subseteq \mathrm{P}_i((\widetilde{\rho_1,\cdots,\rho_N})_{\mathcal{F},MLO})$ *and* $span(\mathrm{P}_i(((\widehat{\rho_1,\cdots,\rho_N})_{\mathcal{F}}))) \subseteq \mathrm{P}_i((\widehat{\rho_1,\cdots,\rho_N})_{\mathcal{F},MLO})$ *for all* $i\in\mathbb{N}_N$.

(ii) We have $\bigcap_{(\sigma_1,\cdots,\sigma_N)\in S^{\mathcal{F}}_{dh}(\rho_1,\cdots,\rho_N)}(span(\sigma_1),\cdots,span(\sigma_N)) \subseteq (\widetilde{\rho_1,\cdots,\rho_N})_{\mathcal{F},MLO}$ *and* $\bigcap_{(\sigma_1,\cdots,\sigma_N)\in S^{\mathcal{F}}_{dtt}(\rho_1,\cdots,\rho_N)}(span(\sigma_1),\cdots,span(\sigma_N)) \subseteq (\widehat{\rho_1,\cdots,\rho_N})_{\mathcal{F},MLO}$.

(iii) Suppose that $\mathcal{F}$ *satisfies* (I). *Then* $\bigcap_{(\sigma_1,\cdots,\sigma_N)\in S^{\mathcal{F}}_{dh}(\rho_1,\cdots,\rho_N)}(span(\sigma_1),\cdots,span(\sigma_N)) = (\widetilde{\rho_1,\cdots,\rho_N})_{\mathcal{F},MLO}$ *and* $\bigcap_{(\sigma_1,\cdots,\sigma_N)\in S^{\mathcal{F}}_{dtt}(\rho_1,\cdots,\rho_N)}(span(\sigma_1),\cdots,span(\sigma_N)) = (\widehat{\rho_1,\cdots,\rho_N})_{\mathcal{F},MLO}$.

It would be interesting to construct an example in which we have that the inclusions $span(\tilde{\rho}_{\mathcal{F}}) \subseteq \tilde{\rho}_{\mathcal{F},MLO}$ and $span(\hat{\rho}_{\mathcal{F}}) \subseteq \hat{\rho}_{\mathcal{F},MLO}$ are strict (see Proposition 3.1). A similar question can be posed in the case of consideration of Proposition 4.1.

Concerning disjoint $\mathcal{F}$-hypercyclic ($\mathcal{F}$-topologically transitive) extensions of N copies of the complete graph K_n, where $n,\ N\geq 2$, we have the following:

Example. Let K_n be equipped with discrete topology, and let ρ be the associated binary relation. Then the following holds:

(i) $n=2$: Then the graphs $K_n,\cdots,K_n$, totally counted N times, are $\mathcal{F}$-hypercyclic ($\mathcal{F}$-topologically transitive) iff $\{\emptyset, 2\mathbb{N}, 2\mathbb{N}+1\}\subseteq\mathcal{F}$; this simply follows from the statement (D) and an elementary argumentation. If this is not the case, then the analysis of possible subcases is, more or less, not difficult to be handle out, and we would like to note the following, only: If $\mathbb{N}\setminus\{1\}\in\mathcal{F}$, then adding any of two loops $x_1 \mapsto x_1$ or $x_2 \mapsto x_2$ to the first component K_n, and repeating this procedure for all other components (it is not prohibited to add the loop $x_1 \mapsto x_1$ to the first component and the loop $x_2 \mapsto x_2$ to the second component), has as a result that the obtained tuple of binary relations is a disjoint $\mathcal{F}$-hypercyclic (disjoint $\mathcal{F}$-topologically transitive) extension of $(K_n, \cdots, K_n)$. If $\mathbb{N} \setminus \{1\} \notin \mathcal{F}$, then the tuple consisting of full relations is the only disjoint $\mathcal{F}$-topologically transitive extension of tuple $(K_n, \cdots, K_n)$, which is not the case for disjoint $\mathcal{F}$-hypercyclic extensions; strictly speaking, adding the loop $x_1 \mapsto x_1$ to any component has as a result that the obtainted binary relations are disjoint $\mathcal{F}$-hypercyclic with x_1 being its disjoint $\mathcal{F}$-hypercyclic vector.

(ii) $n \geq 3$: Then $\rho^k x_i = \{x_1, x_2, \cdots, x_n\}$, $k \geq 2$, $1 \leq i \leq n$ and, because of that, the graphs $K_n, \cdots, K_n$, totally counted N times, are $\mathcal{F}$-hypercyclic ($\mathcal{F}$-topologically transitive) iff $\mathbb{N} \setminus \{1\} \in \mathcal{F}$. In the case that $\mathbb{N} \setminus \{1\} \notin \mathcal{F}$, the smallest disjoint $\mathcal{F}$-topologically transitive tuple containing $(K_n, \cdots, K_n)$ is that one consisting of full relations, while adding the loop $x_1 \mapsto x_1$ to any component has as a result that the obtainted binary relations are disjoint $\mathcal{F}$-hypercyclic with x_1 being its disjoint $\mathcal{F}$-hypercyclic vector.

In our previous analyses, we have seen that for a given digraph G with $n \geq 3$ nodes, the validity of condition $\mathbb{N} \setminus \{1\} \in \mathcal{F}$ ensures one to see that there is an $\mathcal{F}$-hypercyclic ($\mathcal{F}$-topologically transitive) extension of G within the class of simple graphs. By the previous example, for a given tuple $\mathrm{G} = (G_1, G_2, \cdots, G_N)$ of digraphs, each of which has $n \geq 3$ nodes, the validity of condition $\mathbb{N} \setminus \{1\} \in \mathcal{F}$ ensures the existence of a disjoint $\mathcal{F}$-hypercyclic (disjoint $\mathcal{F}$-topologically transitive) extension of G satisfying that each component of this extension belongs to the class of simple graphs.

Suppose now that T is a tournament with n vertices. Denote by $E(T)$ the set consisting of all arcs in T. Then it is well known that the changing of orientation of just one arc in T can turn T into a strongly connected tournament (equivalently, Hamiltonian or topologically transitive tournament; see [16]). Since $|E(T)| = \frac{n(n-1)}{2}$, this immediately implies the first part of the following result; combining the above fact with [6, Theorem 5.9], we can deduce the second part of the following theorem:

Theorem 4.2. *(i) Let T be a tournament with n vertices, and let T be equipped with discrete topology. Then there exists a topologically transitive extension of T with less or equal than $\frac{n(n-1)}{2} + 1$ arcs.*

(ii) Let $T_1,\ T_2, \cdots,\ T_N$ be tournaments with $n \geq 4$ vertices, and let each of them be equipped with discrete topology. Then there exists a d-topologically extension of $(T_1,\ T_2, \cdots,\ T_N)$ which has less than or equal to $\frac{n(n-1)N}{2} + N$ arcs totally counted.

Example. It is a well known fact that there exist only four non-isomorphic tournaments of order four. Only two of them, T_1 and T_2, defined as explained below, are hypercyclic: T_1 is the union of the Hamiltonian circle $x_2 \mapsto x_3 \mapsto x_4$ and oriented segments $x_2 \mapsto x_1$, $x_3 \mapsto x_1$, $x_4 \mapsto x_1$, while T_2 is the union of the Hamiltonian contour $x_1 \mapsto x_2 \mapsto x_3 \mapsto x_4$ and oriented segments $x_1 \mapsto x_3$, $x_2 \mapsto x_4$. Furthermore, T_1 is not topologically transitive while T_2 is; see [6, Example 5.10] for more details. As a consequence, we have that the tuple $(T_1, T_1, \cdots, T_1)$ containing N components is not d-topologically transitive. Adding exactly one of oriented arcs $x_1 \mapsto x_2$, $x_1 \mapsto x_3$ or $x_1 \mapsto x_4$ to T_1 leads us to a new strongly connected digraph T_1' satisfying that the tuple $(T_1', T_1', \cdots, T_1')$ is d-topologically transitive (we can apply here [6, Theorem 5.9] again because T_1' contains a topologically transitive tournament as a subgraph). This d-topologically extension of $(T_1,\ T_2, \cdots,\ T_N)$ has exactly $\frac{n(n-1)N}{2} + N$ arcs totally counted ($n = 4$). If $n \geq 5$, then there exist a topologically transitive tournament and a hypercyclic, non-Hamiltonian tournament with n nodes (the existence of such a tournament having a node with the outdegree equals to zero can be proved inductively). Summa summarum, for any given numbers $n \geq 4$, $N \geq 2$ and $j \in \mathbb{N}_N$, we can simply construct examples of tuples $(T_1,\ T_2, \cdots,\ T_N)$ such that one of their d-topologically transitive extensions having exactly $\frac{n(n-1)N}{2} + j$ arcs totally counted.

Let $n \leq 3$, and let $T_1,\ T_2, \cdots,\ T_N$ be tournaments equipped with discrete topologies. By [6, Theorem 5.8], $T_1,\ T_2, \cdots,\ T_N$ cannot be d-topologically transitive (d-hypercyclic). The interested reader will probably find some relief in computation of their d-topologically transitive (d-hypercyclic) extensions.

Concerning disjoint $\mathcal{F}$-hypercyclic (disjoint $\mathcal{F}$-topologically transitive) MLO extensions of linear continuous operators on Fréchet spaces, we want to note only the following:

(i) Suppose that X is a finite-dimensional Banach space and $A_1, A_2, \cdot\cdot\cdot, A_N \in L(X)$. If $\emptyset \notin \mathcal{F}$, then the tuple $(A_1, A_2, \cdot\cdot\cdot, A_N)$ cannot be disjoint $\mathcal{F}$-hypercyclic (disjoint $\mathcal{F}$-topologically transitive) and the only disjoint $\mathcal{F}$-hypercyclic (disjoint $\mathcal{F}$-topologically transitive) MLO extension of this tuple is that one consisting of full relations on X. If $\emptyset \in \mathcal{F}$, then the tuple $(A_1, A_2, \cdot\cdot\cdot, A_N)$ is already disjoint $\mathcal{F}$-hypercyclic (disjoint $\mathcal{F}$-topologically transitive) with zero being its disjoint $\mathcal{F}$-hypercyclic vector.

(ii) Suppose that X is a Fréchet space, $A_1, A_2, \cdot\cdot\cdot, A_N \in L(X)$, $W_1, W_2, \cdot\cdot\cdot, W_N$ are dense linear subspaces of X and $\mathcal{A}_i x := A_i x + W_i$, $x \in X$, $i \in \mathbb{N}_N$. Then the tuple $(\mathcal{A}_1, \mathcal{A}_2, \cdot\cdot\cdot, \mathcal{A}_N)$ is disjoint $\mathcal{F}$-topologically transitive (see also [5, Example 4.6(iv)]).

The precise characterizations of families $\mathcal{F}$ for which the tuple $(c_1 I, c_2 I, \cdot\cdot\cdot, c_N I)$, where $c_i \in \mathbb{K}$ for $1 \leq i \leq N$, is disjoint $\mathcal{F}$-topologically transitive could be also leave to our readers. The situation in which there exists a number $i \in \mathbb{N}_N$ such that $|c_i| = 1$ and $c_i \notin e^{2\pi i \mathbb{Q}}$ is delicate, as well as the corresponding one in which X is infinite-dimensional and there exists a number $i \in \mathbb{N}_N$ such that $0 < |c_i| \neq 1$.

We deeply believe that the study of $\mathcal{F}$-hypercyclic MLO extensions and disjoint $\mathcal{F}$-hypercyclic MLO extensions of binary relations over topological spaces will receive certain attention and find some worthwhile applications in the field of linear topological dynamics soon. Many intriguing questions and problems can be proposed, so that further analysis is completely without scope of this paper.

References

[1] F. Bayart, E. Matheron, *Dynamics of Linear Operators,* vol. 179 of Cambridge Tracts in Mathematics, Cambridge University Press, Cambridge, UK, 2009.

[2] J. Bès, Q. Menet, A. Peris, Y. Puig, *Strong transitivity properties for operators,* preprint.

[3] J. A. Bondy, U. S. R. Murty, *Graph Theory with Applications,* MacMillan, London, 1976.

[4] G. Chartrand, L. Lesniak, *Graphs and Digraphs,* First CRC Press reprint, Boca Raton, Fl., 2000.

[5] C.-C. Chen, J. A. Conejero, M. Kostić, M. Murillo-Arcila, *Dynamics of multivalued linear operators*, Open Math. **15** (2017), 948–958.

[6] C.-C. Chen, J. A. Conejero, M. Kostić, M. Murillo-Arcila, *Dynamics on binary relations over topological spaces*, Appl. Anal. Discrete Math., submitted.

[7] R. Cross, *Multivalued Linear Operators,* Marcel Dekker Inc., New York, 1998.

[8] A. Favini, A. Yagi, *Degenerate Differential Equations in Banach Spaces,* Chapman and Hall/CRC Pure and Applied Mathematics, New York, 1998.

[9] H. Furstenberg, *Recurrence in Ergodic Theory and Combinatorial Number Theory,* Princeton University Press, Princeton, N. J., 1981.

[10] K.-G. Grosse-Erdmann, A. Peris, *Linear Chaos (Universitext),* Springer-Verlag, London, 2011.

[11] M. Kostić, *Abstract Volterra Integro-Differential Equations*, CRC Press, Boca Raton, Fl., 2015.

[12] M. Kostić, *Abstract Degenerate Volterra Integro-Differential Equations: Linear Theory and Applications*, Book Manuscript, 2016.

[13] M. Kostić, *$\mathcal{F}$-Hypercyclic operators on Frechet spaces,* preprint.

[14] M. Kostić, *Disjoint $\mathcal{F}$-Hypercyclic and disjoint $\mathcal{F}$-Hypercyclic properties of binary relations over topological spaces,* preprint.

[15] J. W. Moon, *Topics on Tournaments,* Holt, Rinehart and Winston, New York, 1967.

[16] V. Petrović, *Graph Theory,* University of Novi Sad, Novi Sad, 1998 (in Serbian).

[17] Y. Puig de Dios, *Linear dynamics and recurrence properties defined via essential idempotents of* $\beta\mathbb{N}$, Ergodic Theory Dynam. Systems **38** (2018), 285–300.