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Determinant spectrum: A generalization of eigenvalues

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Abstract. In this paper we introduce a generalization of eigenvalues called determinant spectrum for an element in the matrix algebra, $\mathbb{C}^{N\times N}$. For $\epsilon > 0$, the ϵ-determinant spectrum of $A \in \mathbb{C}^{N\times N}$ is denoted by $d_\epsilon(A)$ and is defined as

$$d_\epsilon(A) := \{z \in \mathbb{C} : |\det(zI - A)| \leq \epsilon\}.$$

The importance of determinant spectrum is reflected from the application of lemniscates. Determinant spectrum is also useful in various other fields of mathematics, especially in the numerical solution of matrix equations. Determinant spectrum shares some properties of eigenvalues, at the same time, it has many properties that are different from the properties of eigenvalues. In this paper we study about the linear map preserving determinant spectrum on $\mathbb{C}^{N\times N}$. We prove that the linear map preserving determinant spectrum on $\mathbb{C}^{N\times N}$ preserves eigenvalues and their multiplicity. We also prove an analogue of the Spectral Mapping Theorem for determinant spectrum in the matrix algebra. The usual Spectral Mapping Theorem is proved as a special case of this result. The results developed are illustrated with examples and pictures using matlab.

1. Introduction

Let $\mathbb{C}^{N\times N}$ denote the algebra of all $N \times N$ complex matrices and I denote the $N \times N$ identity matrix. The set of all eigenvalues (spectrum) of $A \in \mathbb{C}^{N\times N}$ is denoted by $\sigma(A)$ and is defined as

$$\sigma(A) := \{z \in \mathbb{C} : zI - A \text{ is not invertible}\}.$$

The spectrum is generalized for various important and necessary reasons and there are several generalizations of spectrum. Some well known generalizations of spectrum are Ransford spectrum [14], pseudospectrum [17] and condition spectrum [9]. In [14], the author introduced Ransford spectrum as a generalization of spectrum in a normed linear space.
Let $A \in \mathbb{C}^{N\times N}$ and $s_1(A) \geq s_2(A) \geq \cdots \geq s_N(A)$ be the singular values of A. For $\epsilon > 0$, the ϵ-pseudospectrum of A is denoted by $\Lambda_\epsilon(A)$ and is defined as ([17])

$$\Lambda_\epsilon(A) := \{z \in \mathbb{C} : s_N(zI - A) \leq \epsilon\}.$$

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Pseudospectra provide an analytical and graphical alternative for investigating non-normal matrices and operators, gives a quantitative estimate of departure from non-normality. It also give information about stability of a linear system. For more information on various applications of pseudospectrum, one may refer to [17].
For $0 < \epsilon < 1$, the ϵ-condition spectrum of A is denoted by $\sigma_\epsilon(A)$ and is defined as ([9])

$$\sigma_\epsilon(A) := \left\{ z \in \mathbb{C} : \frac{s_N(zI - A)}{s_1(zI - A)} \leq \epsilon \right\}.$$

Condition spectrum is expected to be useful for solving operator equations. For more information on condition spectrum, one may refer to [7, 9].
The eigenvalues and the generalized eigenvalues of a matrix are studied to get information on various aspects of the matrix. It has been proved that the eigenvalues, pseudospectra and condition spectra are failing to characterize norm related properties of the matrix, [16, 17]. Since that time the researchers have been trying to determine the precise conditions which guarantees identical norm behavior of matrices, see [15–17]. At the time of this writing we don't have an answer to it and the problem appears to be still open. From the definition of pseudospectrum and condition spectrum of an element $A \in \mathbb{C}^{N\times N}$ it follows that we are only using the information about the smallest and largest singular values of the matrix $zI - A$ to define the same. This motivates us to introduce a generalized eigenvalue of A using all the singular values of $zI - A$. This generalized eigenvalue is expected to give more information about A than eigenvalues, pseudospectrum and condition spectrum of A.

Definition 1.1. *Let $A \in \mathbb{C}^{N\times N}$ and $\epsilon > 0$. The ϵ-determinant spectrum of A is denoted by $d_\epsilon(A)$ and is defined as*

$$d_\epsilon(A) := \{z \in \mathbb{C} : s_1(zI - A)\, s_2(zI - A) \cdots s_N(zI - A) \leq \epsilon\}.$$

Since $d_\epsilon(A)$ use all the singular values of $zI - A$ to get defined, it is expected to give more information about A than eigenvalues, pseudospectrum and condition spectrum. Since we have

$$|\det(A)| = s_1(A)\, s_2(A) \cdots s_N(A)$$

the following is an equivalent definition of determinant spectrum.

Definition 1.2. *Let $A \in \mathbb{C}^{N\times N}$ and $\epsilon > 0$. Then*

$$d_\epsilon(A) := \{z \in \mathbb{C} : |\det(zI - A)| \leq \epsilon\}.$$

Since the definition use idea of "determinant" the generalization of eigenvalues defined above is named as determinant spectrum. The definition of determinant spectrum gives, for each $A \in \mathbb{C}^{N\times N}$ and $\epsilon > 0$, $\sigma(A) \subseteq d_\epsilon(A)$ and $\sigma(A) = d_0(A)$.

Let $T : \mathbb{C}^N \to \mathbb{C}^N$ be a linear map and $x, y \in \mathbb{C}^N$. For $\lambda \in \mathbb{C}$, consider the matrix equation $Tx - \lambda x = y$. Then

- $\lambda \notin \sigma(T) \Longrightarrow$ the above linear system is solvable.
- $\lambda \notin d_\epsilon(T) \Longrightarrow$ the linear system is solvable and have a stable solution for $\epsilon > 0$.

In view of this, the ϵ-determinant spectrum is expected to be a useful tool in the numerical solution of system of linear equations.

Let $\mathbb{P}_n$ denote the set of all monic polynomials of degree n. For $\epsilon > 0$, the ϵ-lemniscate of $p \in \mathbb{P}_n$ is denoted by $L_\epsilon(p)$ and is defined as, [6],

$$L_\epsilon(p) = \{z \in \mathbb{C} : |p(z)| \leq \epsilon\}.$$

Thus for $\epsilon > 0$ the ϵ-determinant spectrum of $A \in \mathbb{C}^{N\times N}$ is the ϵ-lemniscate of the characteristic polynomial of A. The importance of determinant spectrum is reflected through the application of lemniscate on various fields, [6, 18, 19]. For $\epsilon > 0$, the ϵ-pseudozero set of $p \in \mathbb{P}_n$ is denoted by $Z_\epsilon(p)$ and is defined as, [6],

$$Z_\epsilon(p) = \{z \in \mathbb{C} : \exists\, q \in \mathbb{P}_n, q(z) = 0 \text{ with } \|p - q\| \leq \epsilon\}.$$

The following is an equivalent definition of pseudospectra of an element $A \in \mathbb{C}^{N\times N}$. For $\epsilon > 0$,

$$\Lambda_\epsilon(A) = \{z \in \mathbb{C} : zI - A \text{ is not invertible or } \|(zI - A)^{-1}\| \geq \epsilon^{-1}\}.$$

In [6], the authors proved the relation connecting $L_\epsilon(p), Z_\epsilon(p)$ and $\Lambda_\epsilon(A_p)$; where A_p denote the companion matrix associated with the monic polynomial p.

Linear preserver problems (LPP) is an active research area in matrix and operator theory. A brief discussion on LPP can be found in [10]. The most popular among these is the problem of characterizing spectrum and spectrum related linear preserving maps. This has been studied by many authors [1, 2, 5, 7, 10, 13]. In [7], the authors characterized the linear map preserving pseudospectrum between Banach algebras. It is shown that the linear map preserving pseudospectrum also preserves spectrum. [1] and [2] study about the preservers of pseudospectrum on $\mathbb{C}^{N\times N}$ and $BL(H)$; for a Hilbert space H. In this paper we study the linear map preserving determinant spectrum on $\mathbb{C}^{N\times N}$. It turns out that the linear map preserving determinant spectrum also preserves spectrum. This result leads to many interesting corollaries.

The Spectral Mapping Theorem is a fundamental result in functional analysis of great importance. Let $\mathcal{A}$ be a complex unital Banach algebra and $a \in \mathcal{A}$, the Spectral Mapping Theorem says that if f is an analytic function on an open set containing $\sigma(a)$, then

$$f(\sigma(a)) = \sigma(f(a)).$$

It is natural to ask whether there are any results similar to the Spectral Mapping Theorem for the generalized spectrum. An analogue of the Spectral Mapping Theorem for pseudospectrum is given in [11, 12]. An analogue of the Spectral Mapping Theorem for condition spectrum is done in [8]. In this paper we give an analogue of the Spectral Mapping Theorem for determinant spectrum.

The following is the outline of the paper. In section 2, we develop some basic properties of the determinant spectrum and justifies the connection between determinant spectrum and algebraic multiplicity of the eigenvalues (Theorem 2.3, Theorem 2.4). In Section 3, we prove various results on linear map preserving determinant spectrum. We prove that any linear map on $\mathbb{C}^{N\times N}$ which preserves ϵ-determinant spectrum for some $\epsilon > 0$ also preserves eigenvalues and their algebraic multiplicity (Theorem 3.5). We give an analogue of the Gleason-Kahane-Zelazko theorem for determinant spectrum (Theorem 3.9). In section 4, the Spectral Mapping Theorem for determinant spectrum is stated and proved in the form of two set inclusions (Theorem 4.2). It is shown that the usual Spectral Mapping Theorem is a special cases of this result (Remark 4.4). It is also shown that the set inclusions reduce to an equality if the mapping is an affine function (Remark refth4). The results proved are illustrated with examples and pictures in section 5. The computations are done using matlab.

2. Basic Properties

This section gives some basic properties of the determinant spectrum. For $\epsilon > 0$, $A \in \mathbb{C}^{N\times N}$ is said to be invertible with respect to the ϵ-determinant spectrum, if $0 \notin d_\epsilon(A)$, that is, A is invertible and $|\det(A)| > \epsilon$. It is easily seen that

$$\Omega := \{A \in \mathbb{C}^{N\times N} : |\det(A)| > \epsilon\}$$

is not a Ransford set [14]. Hence the determinant spectrum is not a Ransford spectrum. The following theorem gives some properties of the determinant spectrum that follow in a straightforward manner from Definition 1.2.

Theorem 2.1. *Let $A \in \mathbb{C}^{N\times N}$ and $\epsilon > 0$. Then the following holds.*

1. *$d_\epsilon(A)$ is a nonempty compact subset of $\mathbb{C}$.*
2. *$d_\epsilon(\alpha I) = \{z \in \mathbb{C} : |z - \alpha| \leq \epsilon^{1/N}\}$ for all $\alpha \in \mathbb{C}$.*
3. *If $0 < \epsilon_1 \leq \epsilon_2$ then $d_{\epsilon_1}(A) \subseteq d_{\epsilon_2}(A)$.*

4. $\sigma(A) \subseteq d_\epsilon(A)$ *for all* $\epsilon > 0$ *and* $\sigma(A) = \bigcap\limits_{0<\epsilon<1} d_\epsilon(A)$.
5. *The map* $A \mapsto d_\epsilon(A)$ *is an upper semicontinuous function from* $\mathbb{C}^{N\times N}$ *to compact subsets of* $\mathbb{C}$.
6. *Let* $A = SBS^{-1}$ *for some* $B, S \in \mathbb{C}^{N\times N}$*: then* $d_\epsilon(A) = d_\epsilon(B)$.
7. $d_\epsilon(\alpha I + \beta A) = \alpha + \beta\, d_{\frac{\epsilon}{|\beta|^N}}(A)$ *for all* $\alpha, \beta \in \mathbb{C}$.

Proof. Since

$$z \mapsto |\det(zI - A)|$$

is a continuous map from $\mathbb{C}$ to $[0, \infty)$, $d_\epsilon(A)$ is a compact set in the complex plane containing the eigenvalues of A. (2), (3), (4) and (5) follows from the definition of $d_\epsilon(A)$. Since $|\det(zI-A)| = |\det(zI-SBS^{-1})| = |\det(zI-B)|$, (6) follows. Finally we have

$$\begin{aligned} d_\epsilon(\alpha I + \beta A) &= \{z \in \mathbb{C} : |\det(zI - \alpha I - \beta A)| \le \epsilon\} \\ &= \left\{z \in \mathbb{C} : |\beta|^N \left|\det\left(\frac{zI - \alpha I}{\beta} - A\right)\right| \le \epsilon\right\} \\ &= \left\{z \in \mathbb{C} : \left|\det\left(\frac{zI - \alpha I}{\beta} - A\right)\right| \le \frac{\epsilon}{|\beta|^N}\right\}. \end{aligned}$$

Hence

$$z \in d_\epsilon(\alpha I + \beta A) \Longleftrightarrow \frac{z-\alpha}{\beta} \in d_{\frac{\epsilon}{|\beta|^N}}(A) \Longleftrightarrow z \in \alpha + \beta\, d_{\frac{\epsilon}{|\beta|^N}}(A).$$

This proves (7). □

Example 3.1 gives matrices having same spectrum but different ϵ-determinant spectrum for all $\epsilon > 0$. It is also true that determinant spectrum of similar matrices coincide, Theorem 2.1. The following example shows that the converse of the result is not true.

Example 2.2. *Let* $A = I_2$ *and* $B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$. *Then A and B are not similar and for* $\epsilon > 0$

$$d_\epsilon(A) = d_\epsilon(B) = \{z \in \mathbb{C} : |z - 1| \le \sqrt{\epsilon}\}.$$

The following theorems justifies the connection between determinant spectrum and algebraic multiplicity of eigenvalues. We show that the algebraic multiplicity of an eigenvalue of a matrix can be identified from the determinant spectrum of the matrix.

Theorem 2.3. *Let* $A \in \mathbb{C}^{N\times N}$, $\epsilon > 0$ *and* $\lambda_1, \cdots, \lambda_k$ *be the distinct eigenvalues of A with algebraic multiplicity* $r_1, \cdots, r_k$ *respectively. Then*

$$d_\epsilon(A) = \{z \in \mathbb{C} : |z - \lambda_1|^{r_1} \cdots |z - \lambda_k|^{r_k} \le \epsilon\}.$$

Proof. By Schur decomposition there exist an upper triangular matrix $U \in \mathbb{C}^{N\times N}$ with diagonal entries as eigenvalues of A and a unitary matrix Q such that $A = QUQ^{-1}$. From Theorem 2.1,

$$\begin{aligned} d_\epsilon(A) &= d_\epsilon(U) = \{z \in \mathbb{C} : |\det(zI - U)| \le \epsilon\} \\ &= \{z \in \mathbb{C} : |z - \lambda_1|^{r_1} \cdots |z - \lambda_k|^{r_k} \le \epsilon\}. \end{aligned}$$

□

Theorem 2.4. *Let* $A, B \in \mathbb{C}^{N\times N}$ *and* $\epsilon > 0$. *Suppose* $d_\epsilon(A) = d_\epsilon(B)$. *Then the characteristic polynomial of A and B are same.*

Proof. Let $p(A), p(B)$ be the characteristic polynomials of A, B respectively. Then $p(A), p(B)$ are of same degree equal to N. Since $d_\epsilon(A) = d_\epsilon(B)$ we have

$$\{z \in \mathbb{C} : |p(A)| \leq \epsilon\} = \{z \in \mathbb{C} : |p(B)| \leq \epsilon\}.$$

This is true if and only if $p(A) = p(B)$, [3, 4]. □

3. Linear maps preserving determinant spectrum

In this section we discuss about the linear map on $\mathbb{C}^{N\times N}$ preserving determinant spectrum. We begin by giving sufficient conditions for a map on $\mathbb{C}^{N\times N}$ to preserve ϵ-determinant spectrum for all $\epsilon > 0$. We prove that every ϵ-determinant spectrum preserving map on $\mathbb{C}^{N\times N}$ for some $\epsilon > 0$ preserves eigenvalues and their algebraic multiplicity. This result leads to many interesting corollaries. This section also contain an analogue of the Gleason-Kahane-Zelazko theorem for determinant spectrum. The following example shows that; for matrices, the eigenvalues coincide does not imply that the determinant spectrum also coincides.

Example 3.1. *Let* $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ *and* $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. *Then* $\sigma(A) = \sigma(B) = \{1, 2\}$ *and for all* $\epsilon > 0$,

$$d_\epsilon(A) = \{z \in \mathbb{C} : |z-1|^2|z-2| \leq \epsilon\} \neq d_\epsilon(B) = \{z \in \mathbb{C} : |z-1||z-2|^2 \leq \epsilon\}.$$

Theorem 3.2. *Let* $A, B \in \mathbb{C}^{N\times N}$ *preserves eigenvalues and their algebraic multiplicity. Then* A, B *have same* ϵ-*determinant spectrum for all* $\epsilon > 0$.

Proof. Since A, B preserves eigenvalues and their algebraic multiplicity, the characteristic polynomial of A and B are same. The result follows from the definition of determinant spectrum. □

Theorem 3.3. *Suppose* $\Phi : \mathbb{C}^{N\times N} \to \mathbb{C}^{N\times N}$ *be linear, unital and determinant preserving map. Then* Φ *preserves* ϵ-*determinant spectrum for all* $\epsilon > 0$.

Proof. Let $A \in \mathbb{C}^{N\times N}$ and $\epsilon > 0$. If $\lambda \in d_\epsilon(A)$, then

$$|\det(\lambda I - A)| \leq \epsilon.$$

Since Φ preserves determinant

$$|\det[\Phi(\lambda I - A)]| \leq \epsilon.$$

Since Φ is linear and unital

$$|\det(\lambda I - \Phi(A))| \leq \epsilon.$$

Hence $\lambda \in d_\epsilon(\Phi(A))$ and $d_\epsilon(A) \subseteq d_\epsilon(\Phi(A))$. By symmetry we can show that $d_\epsilon(\Phi(A)) \subseteq d_\epsilon(A)$. □

Theorem 3.4. *Let* $\Phi : \mathbb{C}^{N\times N} \to \mathbb{C}^{N\times N}$ *be a spectrum preserving linear onto map. Then* Φ *preserves* ϵ-*determinant spectrum for all* $\epsilon > 0$.

Proof. Since Φ preserves spectrum, $\Phi(A) = TAT^{-1}$ or $\Phi(A) = TA^tT^{-1}$ for some $T \in \mathbb{C}^{N\times N}$ [13]. In both the cases Φ is linear, unital and determinant preserving. Hence the result follows from Theorem 3.3. □

The following theorem shows that the converse of this result is also true.

Theorem 3.5. *Let* $\Phi : \mathbb{C}^{N\times N} \to \mathbb{C}^{N\times N}$ *be an* ϵ-*determinant spectrum preserving linear onto map for some* $\epsilon > 0$. *Then* Φ *preserves the eigenvalues.*

Proof. We have

$$d_\epsilon(\Phi(A)) = d_\epsilon(A) \text{ for all } A \in \mathbb{C}^{N\times N}.$$

Let $\lambda \notin \sigma(A)$, choose $t > 0$ such that

$$t^N > \frac{\epsilon}{|\det(\lambda I - A)|}.$$

Then $|\det(t\lambda I - tA)| > \epsilon$. Thus

$$t\lambda \notin d_\epsilon(tA) = d_\epsilon(\Phi(tA)) \supseteq \sigma((\Phi(tA))) = t\sigma(\Phi(A))$$

Thus $\lambda \notin \sigma(\Phi(A))$ and $\sigma(\Phi(A)) \subseteq \sigma(A)$. In a similar way we can prove that $\sigma(A) \subseteq \sigma(\Phi(A))$. Hence $\sigma(\Phi(A)) = \sigma(A)$. □

Corollary 3.6. *Let $\Phi : \mathbb{C}^{N\times N} \to \mathbb{C}^{N\times N}$ be an ϵ-determinant spectrum preserving linear onto map for some $\epsilon > 0$. Then $\Phi(A) = TAT^{-1}$ or $\Phi(A) = TA^tT^{-1}$.*

Proof. Since Φ preserves ϵ-determinant spectrum for some $\epsilon > 0$, Φ preserves eigenvalues (Theorem 3.5). The result follows from [13]. □

Theorem 3.7. *Let $\Phi : \mathbb{C}^{N\times N} \to \mathbb{C}^{N\times N}$ be an ϵ-determinant spectrum preserving linear map for some $\epsilon > 0$. Then Φ preserves the determinant of matrices.*

Proof. Suppose there exist $A \in \mathbb{C}^{N\times N}$ such that $\det(\Phi(A)) \neq \det(A)$. Assume $|\det(A)| < |\det(\Phi(A))|$. Choose $t > 0$ such that

$$\frac{|\det(A)|}{\epsilon} < \frac{1}{t^N} < \frac{|\det(\Phi(A))|}{\epsilon}.$$

Then $|\det(tA)| < \epsilon$ and $|\det(\Phi(tA))| > \epsilon$, i.e, $0 \in d_\epsilon(tA)$ and $0 \notin d_\epsilon(\Phi(tA))$. This contradicts the fact that Φ preserves ϵ-determinant spectrum. □

The following example shows that the converse of the result is not true. i.e, there exist a linear map preserving determinant which may not preserve ϵ-determinant spectrum for all $\epsilon > 0$.

Example 3.8. *Let $T = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$ and define $\Phi : \mathbb{C}^{2\times 2} \to \mathbb{C}^{2\times 2}$ such that $\Phi(A) = TA$. Then Φ is linear, bijective and determinant preserving map. Let $A = \begin{bmatrix} a_1 & a_3 \\ 0 & a_2 \end{bmatrix}$. Then $\Phi(A) = TA = \begin{bmatrix} 2a_1 & \frac{a_3}{2} \\ 0 & \frac{a_2}{2} \end{bmatrix}$. Thus $\sigma(A) = \{a_1, a_2\}$ and $\sigma(\Phi(A)) = \{2a_1, \frac{a_2}{2}\}$. Hence Φ is not preserving spectrum and so the ϵ-determinant spectrum for all $\epsilon > 0$.*

The following is an analogue to the classical Gleason-Kahane-Zelazko theorem for determinant spectrum.

Theorem 3.9. *Let $\phi : \mathbb{C}^{N\times N} \to \mathbb{C}$ be linear such that $\phi(I) = 1$ and $\phi(A) \in d_\epsilon(A)$ for all $A \in \mathbb{C}^{N\times N}$ and for some $\epsilon > 0$. Then ϕ is multiplicative.*

Proof. We claim that $\phi(A) \in \sigma(A)$ for all $A \in \mathbb{C}^{N\times N}$. Let $\phi(A) = \lambda$ and suppose $\lambda \notin \sigma(A)$. Choose $t > 0$ such that

$$t^N > \frac{\epsilon}{|\det(\lambda I - A)|}.$$

Then $|\det(t\lambda I - tA)| > \epsilon$. Thus $t\lambda = t\phi(A) = \phi(tA) \notin d_\epsilon(tA)$. This gives a contradiction. Now the conclusion follows from the Gleason-Kahane-Zelazko theorem, [20]. □

Theorem 3.10. *Let $\Phi : \mathbb{C}^{N\times N} \to \mathbb{C}^{N\times N}$ be a linear map. Then the following are equivalent*

1. *Φ preserves ϵ- determinant spectrum for some $\epsilon > 0$.*
2. *Φ preserves the eigenvalues and their algebraic multiplicity.*
3. *Φ preserves the determinant of matrices.*
4. *Φ preserves ϵ- determinant spectrum for all $\epsilon > 0$.*

Proof. (1)$\Longrightarrow$(2), by Theorem 3.5. (2)$\Longrightarrow$(3) by [13]. (3) $\Longrightarrow$ (4), by Theorem 3.3, (4)$\Longrightarrow$(1) is trivial. □

4. Determinant Spectral Mapping Theorem

In this section we give an analogous result on Spectral Mapping Theorem for determinant spectrum. The usual Spectral Mapping Theorem is proved as a special case of this result. We also illustrate the result developed with the help of some examples. We begin by an example shows that the Spectral Mapping Theorem fails for determinant spectrum.

Example 4.1. *Let* $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ *and* $f(z) = z^2$. *We have* $f(A) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ *and*

$$d_\epsilon(f(A)) = \{z : |z-1| \le \epsilon^{1/2}\},$$
$$f(d_\epsilon(A)) = \{z^2 : |z-1| \le \epsilon^{1/2}\}.$$

Thus $f(d_\epsilon(A)) \neq d_\epsilon(f(A))$ *for all* $\epsilon \neq 1$.

The following is the determinant spectral mapping theorem for complex analytic functions. It is sharp in the sense that the pair of functions defining the sizes of the determinant spectra are optimal. Actually, the theorem is an easy consequence of the definition of these functions.

Theorem 4.2. *Let* $A \in \mathbb{C}^{N\times N}$ *and* f *be an analytic function defined on* Ω, *an open set containing* $\sigma(A)$. *For* $\epsilon > 0$, *define*

$$\phi(\epsilon) = \max_{z\in d_\epsilon(A)} |\det[f(z)I - f(A)]|.$$

Then $\phi(\epsilon)$ *is well defined,* $\lim_{\epsilon\to 0} \phi(\epsilon) = 0$ *and* $f(d_\epsilon(A)) \subseteq d_{\phi(\epsilon)}(f(A))$. *Further suppose there exists* ϵ' *with* $d_{\epsilon'}(f(A)) \subseteq f(\Omega)$. *For* $0 < \epsilon < \epsilon'$ *define*

$$\psi(\epsilon) = \max_{\mu\in f^{-1}(d_\epsilon(f(A)))\cap\Omega} |\det(\mu I - A)|.$$

Then $\psi(\epsilon)$ *is well defined,* $\lim_{\epsilon\to 0} \psi(\epsilon) = 0$ *and* $d_\epsilon(f(A)) \subseteq f(d_{\psi(\epsilon)}(A))$.

Proof. First we show that $\phi(\epsilon)$ is well defined. Define $g : \mathbb{C} \to \mathbb{R}$ by

$$g(z) = |\det[f(z)I - f(A)]|.$$

Then g is continuous. Next for $\epsilon > 0$, $d_\epsilon(A)$ is a compact subset of $\mathbb{C}$ and $\phi(\epsilon) = \max\{g(z) : z \in d_\epsilon(A)\}$. Hence $\phi(\epsilon)$ is well defined, that is, finite. It is easy to observe that ϕ is a monotonically non-decreasing function and $\phi(\epsilon)$ goes to zero as ϵ goes to zero. Now let $z \in d_\epsilon(A)$. Then $g(z) \le \phi(\epsilon)$. Hence

$$|\det(f(z)I - f(A))| = g(z) \le \phi(\epsilon).$$

This means that $f(z) \in d_{\phi(\epsilon)}(f(A))$. Thus

$$f(d_\epsilon(A)) \subseteq d_{\phi(\epsilon)}(f(A)).$$

Next assume that there exists ϵ' such that $d_{\epsilon'}(f(A)) \subseteq f(\Omega)$. We show that for each ϵ with $0 < \epsilon \le \epsilon'$, $\psi(\epsilon)$ is well defined. Define $h : \mathbb{C} \to \mathbb{R}$ by,

$$h(\mu) = |\det(\mu I - A)|.$$

Then h is continuous and hence $\psi(\epsilon)$ is well defined. It is also observed that ψ is a monotonically non-decreasing function and $\psi(\epsilon)$ goes to zero as ϵ goes to zero. Let $z \in d_\epsilon(f(A)) \subseteq d_{\epsilon'}(f(A)) \subseteq f(\Omega)$. Consider $\mu \in \Omega$ such that $z = f(\mu)$. Then $\mu \in f^{-1}(d_\epsilon(f(A)))$, hence $h(\mu) \le \psi(\epsilon)$, that is

$$|\det(\mu I - A)| \le h(\mu) \le \psi(\epsilon).$$

Thus $\mu \in d_{\psi(\epsilon)}(A)$. Hence $z = f(\mu) \in f(d_{\psi(\epsilon)}(A))$. This proves

$$d_\epsilon(f(A)) \subseteq f(d_{\psi(\epsilon)}(A)).$$

□

Remark 4.3. *Combining the two inclusions, we get*

$$f(d_\epsilon(A)) \subseteq d_{\phi(\epsilon)}(f(A)) \subseteq f(d_{\psi(\phi(\epsilon))}(A)),$$

and

$$d_\epsilon(f(A)) \subseteq f(d_{\psi(\epsilon)}(A)) \subseteq d_{\phi(\psi(\epsilon))}(f(A)).$$

Remark 4.4. *Since for every $A \in \mathbb{C}^{N\times N}$, $\lim_{\epsilon\to 0} \phi(\epsilon) = 0 = \lim_{\epsilon\to 0} \psi(\epsilon)$, we have*

$$\sigma(f(A)) = f(\sigma(A)).$$

Thus the usual Spectral Mapping Theorem can be deduced from Theorem 4.2. It is to be noted that the determinant spectral mapping theorem uses the usual Spectral Mapping Theorem.

Remark 4.5. *Let $A \in \mathbb{C}^{N\times N}$, $\epsilon > 0$ and $f(z) = \alpha + \beta z$ where α, β are complex numbers with $\beta \neq 0$. Then*

$$\begin{aligned} \phi(\epsilon) &= \max_{z\in d_\epsilon(A)} |\det[(\alpha+\beta z)I - \alpha I - \beta A]| \\ &= \max_{z\in d_\epsilon(A)} |\det[\beta(zI - A)]| \\ &= |\beta|^N \epsilon. \end{aligned}$$

We also have

$$\begin{aligned} \psi(\epsilon) &= \max_{\mu\in d_{\frac{\epsilon}{|\beta|^N}}(A)} |\det(\mu I - A)| \\ &= \frac{\epsilon}{|\beta|^N} \end{aligned}$$

Thus $\phi(\psi(\epsilon)) = \epsilon$ and $\psi(\phi(\epsilon)) = \epsilon$. Hence $f(d_\epsilon(A)) = d_\epsilon(f(A))$. This leads to the following question.

Question 4.6. *Let f be a non constant analytic function defined on a nonempty open set Ω containing $\sigma(A)$. Suppose $f(\sigma_\epsilon(A)) = \sigma_{\phi(\epsilon)}(f(A))$ for all $A \in \mathbb{C}^{N\times N}$. Then does it follow that $\phi(\epsilon) = \epsilon$ and $f(z) = \alpha + \beta z$.*

In the following we consider a 2×2 matrix and give estimates for the functions ϕ and ψ in Theorem 4.2.

Example 4.7. *Let $A = \begin{bmatrix} 1 & a \\ 0 & -1 \end{bmatrix}$ where $a \in \mathbb{C}$ and $f(z) = z^2$. The matrix is non-normal and everything can be worked out analytically. The eigenvalues of A are $\{1, -1\}$. For $\epsilon > 0$,*

$$\begin{aligned} \phi(\epsilon) &= \max_{z\in d_\epsilon(A)} |\det(z^2 I - A^2)| \\ &= \max_{|z^2-1|\leq\epsilon} |z^2 - 1|^2 = \epsilon^2. \end{aligned}$$

$$\begin{aligned} \psi(\epsilon) &= \max_{z\in f^{-1}(d_\epsilon(f(A)))} |\det(zI - A)| \\ &= \max_{|z^2-1|\leq\sqrt{\epsilon}} |z^2 - 1| = \sqrt{\epsilon}. \end{aligned}$$

Thus we have

$$d_\epsilon(A)^2 \subseteq d_{\epsilon^2}(A^2).$$

$$d_\epsilon(A^2) \subseteq d_{\sqrt{\epsilon}}(A)^2.$$

In the following we consider a 5×5 Toeplitz matrix and give estimates for the functions ϕ and ψ in Theorem 4.2.

Example 4.8. *Let us considered the following* 5×5 *Toeplitz matrix.*

$$T = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}_{5\times5}.$$

(1) Let $f(z) = z^2$*. Then* $f(T) = T^2$ *is also a Toeplitz matrix and*

$$T^2 = \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}_{5\times5}.$$

For $\epsilon > 0$

$$d_\epsilon(T) = \{z \in \mathbb{C} : |z-1| \le \epsilon^{1/5}\}.$$

From Theorem 4.2,

$$\begin{aligned} \phi(\epsilon) &= \max_{z\in D(1,\epsilon^{1/5})} |\det(z^2 I - T^2)|. \\ &= \max_{z\in D(1,\epsilon^{1/5})} |z^2-1|^5. \\ &= \epsilon(2+\epsilon^{1/5})^5. \end{aligned}$$

$$\begin{aligned} \psi(\epsilon) &= \max_{\mu^2\in D(1,\epsilon^{1/5})} |\det(\mu I - T)|. \\ &= \max_{\mu^2\in D(1,\epsilon^{1/5})} |\mu-1|^5. \\ &= [(1+\epsilon^{1/5})^{1/2}-1]^5. \end{aligned}$$

For $\epsilon > 0$*, we have*

$$d_\epsilon(T)^2 \subseteq d_{\epsilon(2+\epsilon^{1/5})^5}(T^2).$$

$$d_\epsilon(T^2) \subseteq d_{[(1+\epsilon^{1/5})^{1/4}-1]^5}(T)^2.$$

(2) Let $f(z) = e^z$*. Then* $\tilde{f}(T) = exp(T)$ *is also a Toeplitz matrix and*

$$exp(T) = \begin{bmatrix} e & e & 1.35914 & 0.45305 & 0.11326 \\ 0 & e & e & 1.35914 & 0.45305 \\ 0 & 0 & e & e & 1.35914 \\ 0 & 0 & 0 & e & e \\ 0 & 0 & 0 & 0 & e \end{bmatrix}_{5\times5}.$$

For $\epsilon > 0$

$$d_\epsilon(T) = \{z \in \mathbb{C} : |z-1| \le \epsilon^{1/5}\}.$$

From Theorem 4.2,

$$\begin{aligned}\phi(\epsilon) &= \max_{z\in D(1,\epsilon^{1/5})} |\det(e^z I - \exp(T))|. \\ &= \max_{z\in D(1,\epsilon^{1/5})} |e^z - e|^5. \\ &= e^5(e^{\epsilon^{\frac{1}{5}}} - 1)^5.\end{aligned}$$

$$\begin{aligned}\psi(\epsilon) &= \max_{e^\mu\in D(e,\epsilon^{1/5})} |\det(\mu I - T)|. \\ &= \max_{e^\mu\in D(e,\epsilon^{1/5})} |\mu - 1|^5. \\ &= [\ln(e+\epsilon^{1/5}) - 1]^5.\end{aligned}$$

For $\epsilon > 0$, we have

$$d_\epsilon(T)^2 \subseteq d_{e^5(e^{\epsilon^{\frac{1}{5}}}-1)^5}(T^2).$$

$$d_\epsilon(T^2) \subseteq d_{[\ln(e+\epsilon^{1/5})-1]^5}(T)^2.$$

5. Numerical results

In this section we report the results of some numerical experiments done using matlab. The determinant spectrum of a matrix A can be computed as follows. Since for each $\epsilon > 0$, $d_\epsilon(A)$ is a compact subset of $\mathbb{C}$, we can consider certain number of uniformly distributed points from a bounded rectangular disc enclosing $d_\epsilon(A)$, evaluate $|\det(zI-A)|$ in the disc and include and save those points in the disc satisfying $|\det(zI-A)| \le \epsilon$. This gives $d_\epsilon(A)$. Since we aim at illustrating the results developed in the previous sections we do not make any claim about the efficiency of the algorithm used for this. In the following we consider the 10×10 Grcar matrix and find the ϵ-determinant spectrum for different values of ϵ.

Example 5.1. *Let* $A = \begin{bmatrix} 1 & 1 & 1 & 1 & & & & \\ -1 & 1 & 1 & 1 & 1 & & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & -1 & 1 & 1 & 1 & 1 & \\ & & & -1 & 1 & 1 & 1 & \\ & & & & -1 & 1 & 1 & \\ & & & & & -1 & 1 & \end{bmatrix}_{10\times 10}.$

In the following, Figure 1 represents $d_1(A)$, Figure 2 represents $d_2(A)$, Figure 3 represents $d_3(A)$ and Figure 4 represents $d_4(A)$.

Next we consider a 3×3 upper triangular matrix and give approximate estimates for $\phi(\epsilon)$ and $\psi(\epsilon)$ in Theorem 4.2. To calculate approximate value of $\phi(\epsilon)$ in Theorem 4.2 we choose a certain number of uniformly distributed points in $d_\epsilon(A)$, compute $|\det[f(z)I - f(A)]|$ at each of these points and take the maximum of these values as an approximation of $\phi(\epsilon)$. Similarly $\psi(\epsilon)$ is also computed.

Example 5.2. *Let* $A = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 2 \end{bmatrix}$ *where $a, b, c \in \mathbb{C}$, $f(z) = z^3$ and $\epsilon > 0$. We have*

$$d_\epsilon(A) = \{z \in \mathbb{C} : |z-1|^2|z-2| \le \epsilon\}.$$

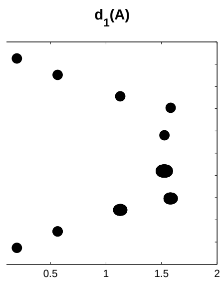

Figure 1

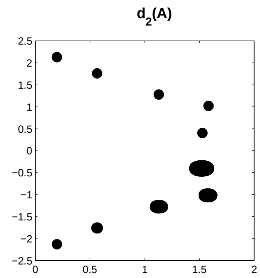

Figure 2

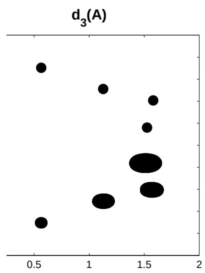

Figure 3

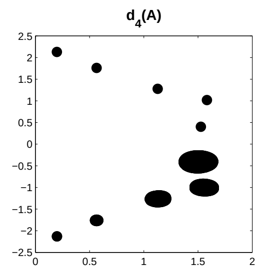

Figure 4

From Theorem 4.2,

$$\begin{aligned}
\phi(\epsilon) &= \max_{|z-1|^2|z-2|\leq\epsilon} |\det(z^3I - A^3)|. \\
&= \max_{|z-1|^2|z-2|\leq\epsilon} |z^3-1|^2\,|z^3-8|. \\
&= \epsilon \max_{|z-1|^2|z-2|\leq\epsilon} |z^2+z+1|^2\,|z^2+2z+4|.
\end{aligned}$$

We also have

$$\begin{aligned}
\psi(\epsilon) &= \max_{\mu^3\in d_\epsilon(A^3)} |\det(\mu I - A)|. \\
&= \max_{|\mu^3-1|^2|\mu^3-8|\leq\epsilon} |\det(\mu I - A)|. \\
&= \max_{|\mu^3-1|^2|\mu^3-8|\leq\epsilon} |\mu-1|^2|\mu-2|.
\end{aligned}$$

The following table gives $\phi(\epsilon)$ and $\psi(\epsilon)$ for various values of ϵ.

ϵ	$\phi(\epsilon)$	$\psi(\epsilon)$
0.001	0.5886	$1.5296 \times e^{-5}$
0.005	2.9656	$8.0641 \times e^{-5}$
0.01	5.9886	$1.6225 \times e^{-4}$
0.05	32.079	$8.5186 \times e^{-5}$
0.1	69.0789	0.0018
0.5	509.744	0.0102
1	1367.3	0.023
5	16992	0.2446

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