Conservation of the Number of Eigenvalues of Finite Dimensional and Compact Operators Inside and Outside Circle

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Abstract. Let $i_{in}(A)$ and $i_{out}(A)$ be the numbers of the eigenvalues of a finite matrix $A$ lying inside and outside the unit circle, respectively. Let $A$ be a perturbed matrix. We obtain the conditions under which $i_{in}(A) = i_{in}(\tilde{A})$ and $i_{out}(A) = i_{out}(\tilde{A})$. Our main tool is the norm estimates for resolvents of operators on the tensor product of Euclidean spaces. The results for finite matrices are particularly generalized to Hilbert-Schmidt operators.

1. Introduction and statement of the main result

Let $\mathbb{C}^n$ be an $n$-dimensional Euclidean space with a scalar product $<.;.>_n = \sqrt{<.;.>}$ and the unit operator $I$; $\mathbb{C}^{n\times n}$ means the set of all complex $n \times n$ matrices. For an $A \in \mathbb{C}^{n\times n}$, $\lambda_k(A)$ ($k = 1, ..., n$) are the eigenvalues of $A$ enumerated in an arbitrary order with their multiplicities, $||A|| = ||A||_F = \sup_{\|x\|_2 = 1} \|Ax\|_2/\|x\|_2$ is the (operator) spectral norm, $\sigma(A)$ denotes the spectrum of $A$, $A^*$ is the adjoint to $A$, $A^T$ is the transposed one and $A^{-1}$ is the inverse to $A$; $||A||_F = \sqrt{\text{trace}(AA^*)}$ is the Hilbert-Schmidt (Frobenius) norm of $A$, $\rho(A)$ is the spectral radius, $\rho(A, \lambda) = \min_{i=1,...,n} |\lambda - \lambda_i(A)|$ is the distance between $\sigma(A)$ and a point $\lambda \in \mathbb{C}$.

The inertia $\text{In}(A)$ of a matrix $A$ with respect to the imaginary axis is defined as a triple of nonnegative integers $(\pi(A), v(A), \delta(A))$, where $\pi(A)$ is the number of eigenvalues of $A$ with positive real parts, $v(A)$ is the number of eigenvalues of $A$ with negative real parts and $\delta(A)$ is the number of eigenvalues of $A$ on the imaginary axis. For a Hermitian matrix $S$, the inequality $\text{In}(S) > 0$ (\text{In}(S) < 0) means that $S$ is positive (negative) definite.

The classical theorem of Ostrowski and Schneider [13] asserts that for a given matrix $A$, there exists a Hermitian matrix $H$, such that $\mathcal{R}(AH) = ((AH)^* + AH)/2 > 0$ if and only if $\delta(A) = 0$. If $\mathcal{R}(AH) > 0$, then $\text{In}(A) = \text{In}(H)$. In the paper [3] the just mentioned result has been reduced to the semi-definite case.

The inertia, $\text{In}_{\Omega}(A)$, of a matrix $A$ with respect to the unit circle $\Omega = \{z \in \mathbb{C} : |z| = 1\}$ is defined as a triple of nonnegative integers $(i_{in}(A), i_{out}(A), i_{\Omega}(A))$, where $i_{in}(A)$ is the number of the eigenvalues of $A$ taken with their multiplicities, lying inside $\Omega$, $i_{out}(A)$ is the number of the eigenvalues of $A$ outside $\Omega$, and $i_{\Omega}(A)$ is...
Let condition (1.1) hold. Then below we prove the following:

\[ y \otimes h; y_1 \otimes h_1 >_E := < y, y_1 >_{E_1} < h, h_1 >_{E_2} \quad (y, y_1 \in E_1; \ h, h_1 \in E_2) \]

where \(< \cdot, \cdot >_E\) is the scalar products in \(E\). In addition, \(I_E = I \otimes I\) is the unit operators on \(E\). It is clear that \(\dim E = n^2\). Assuming that

\[
\theta(A) := \min_{j,k=1,\ldots,n} |1 - \lambda_j(A)\bar{\lambda}_k(A)| > 0, \quad (1.1)
\]

put

\[
\chi(A) := \|(I_E - A^T \otimes A)^{-1}\|_{II^2}.
\]

Now we are in a position to formulate our main result.

**Theorem 1.1.** Let \(A, \bar{A} \in C^{n \times n}\), and the conditions (1.1) and

\[
\chi(A)(2\|A\|_F\|A - \bar{A}\|_F + \|A - \bar{A}\|_{II^2}) < 1
\]

hold. Then \(i_{in}(\bar{A}) = i_{in}(A)\) and \(i_{out}(\bar{A}) = i_{out}(A)\).

This theorem is proved in the next two sections.

To formulate our next result introduce the quantity (the departure from normality of \(A\))

\[
g(A) := (\|A\|_F^2 - \tau^2(A))^{1/2}, \quad \tau(A) := \left(\sum_{k=1}^n |\lambda_k(A)|^2\right)^{1/2}.
\]

In Sections 2.1 and 2.2 of the book [7] it is checked that

\[
g^2(A) \leq \|A\|_F^2 - |\text{trace } A^2|, \quad g^2(A) \leq \|A - A^*\|_F^2/2 \quad \text{and} \quad g(\alpha^*A + zI) = g(A)
\]

for all \(t \in \mathbb{R}\) and \(z \in C\). If \(A\) is a normal matrix: \(AA^* = A^*A\), then \(g(A) = 0\). In addition, if \(A_1\) and \(A_2\) are commuting \(n \times n\) matrices, then \(g(A_1 + A_2) \leq g(A_1) + g(A_2)\). By the inequality between geometric and arithmetic mean values,

\[
\left(\frac{1}{n} \sum_{k=1}^n |\lambda_k(A)|^2\right)^n \geq \prod_{k=1}^n |\lambda_k(A)|^2. \quad \text{So} \quad g^2(A) \leq \|A\|_F^2 - n(\det A)^2/n.
\]

Below we prove the following

**Lemma 1.2.** Let condition (1.1) hold. Then \(\chi(A) \leq \zeta(A)\), where

\[
\zeta(A) := \sum_{j=0}^{3} \frac{(2\tau(A)g(A) + g^2(A))^{j}}{j!g^{j+1}(A)}.
\]
From the definition of $g(A)$ it follows that $2\tau(A)g(A) \leq \tau^2(A) + g^2(A) = \|A\|_F^2$. Since $g(A) \leq \|A\|_F$, we have

$$
\zeta(A) \leq \sum_{j=0}^{3n-3} \frac{2^j\|A\|_F^j}{\sqrt{j!}\theta^{j+1}(A)}.
$$

(1.3)

Theorem 1.1 and Lemma 1.2 imply

**Corollary 1.3.** Let $A, \tilde{A} \in \mathbb{C}^{n \times n}$. Let the conditions (1.1) and

$$
\zeta(A)(2\|A\|_F\|A - \tilde{A}\|_F + \|A - \tilde{A}\|_F^2) < 1
$$

(1.4)

hold. Then $i_{in}(\tilde{A}) = i_{in}(A)$ and $i_{out}(\tilde{A}) = i_{out}(A)$.

A matrix $A$ is said to be **Schur-Cohn stable** if its spectral radius is less than one. It is said to be **Schur-Cohn unstable**, if $|\lambda_i(A)| > 1$ for at least one eigenvalue of $A$. Making use of Corollary 1.3 we arrive at

**Corollary 1.4.** Let $A, \tilde{A} \in \mathbb{C}^{n \times n}$. If $A$ is Schur-Cohn stable and condition (1.4) holds, then $\tilde{A}$ is also Schur-Cohn stable.

Moreover, if conditions (1.1) and (1.4) hold, and $A$ is Schur-Cohn unstable, then $\tilde{A}$ is also Schur-Cohn unstable.

2. Proof of Theorem 1.1

We need the following theorem proved in [16].

**Theorem 2.1.** Let $A, Q \in \mathbb{C}^{n \times n}$ and $Q$ be Hermitian strongly positive definite. If $X$ is a Hermitian solution of the equation

$$
X - AXA^* = Q, \tag{2.1}
$$

then $X$ is nonsingular and the number $i_+(X)$ of positive (the number $i_-(X)$ of negative) eigenvalues of $X$ is equal to $i_{in}(A)$ ($i_{out}(A)$).

The operator $G : \mathbb{C}^{n \times n} \to \mathbb{C}^{n \times n}$ defined by $X = GQ$ will be called the Green operator to (2.1). Put

$$
\|G\| = \sup_{Q \in \mathbb{C}^{n \times n}} \frac{\|GQ\|}{\|Q\|}.
$$

**Lemma 2.2.** Let $A, \tilde{A} \in \mathbb{C}^{n \times n}$, and the conditions (1.1) and

$$
\gamma(A, \tilde{A}) := \|A - \tilde{A}\| \|G\|(2\|A\| + \|A - \tilde{A}\|) < 1 \tag{2.2}
$$

hold. Then the equation

$$
\tilde{X} - \tilde{A}\tilde{X}\tilde{A}^* = Q, \tag{2.3}
$$

has a unique solution $\tilde{X}$ satisfying

$$
\|\tilde{X}\| \leq \frac{\|G\|\|Q\|}{1 - \gamma(A, \tilde{A})} \tag{2.4}
$$

and

$$
\|X - \tilde{X}\| \leq \frac{\|G\|\|Q\|\gamma(A, \tilde{A})}{1 - \gamma(A, \tilde{A})}, \tag{2.5}
$$

where $X$ is the solution to (2.1).
Therefore, and therefore (2.7) implies
\[ A \]
where
\[ Q_X := AXB^* + BXA^* + BXB^* \]
and
\[ Q_Y := (A_i B_i^* + B_i A_i^* + B_i B_i^*) Y, \]
where \( A_i, A_r \) are defined by \( A_i Y = AY, A_r Y = YA. \) Thus,
\[ Y = G(Q_X + Q_Y) = GQ_X + WY, \]
where \( W := (A_i B_i^* + B_i A_i^* + B_i B_i^*) \). So due to (2.2),
\[ \|W\| \leq \|G\|\|2\|A\||B|| + \|B\|^2 = \gamma(A, \bar{A}) < 1 \]
and therefore (2.7) implies
\[ Y = (I - W)^{-1} GQ_X = \sum_{k=0}^{\infty} W^k GQ_X \]
and
\[ \|Y\| = (1 - \gamma(A, \bar{A}))^{-1} \|GQ_X\|. \]
Taking into account that
\[ \|Q_X\| \leq \|X\|\|2\|A\||B|| + \|B\|^2 \leq \|G\|\|\|Q\|\|\|2\|A\||B|| + \|B\|^2 = \gamma(A, \bar{A}) \|Q\|. \]
Therefore,
\[ \|Y\| \leq \frac{\gamma(A, \bar{A})\|G\|\|\|Q\|\|}{1 - \gamma(A, \bar{A})} \]
and
\[ \|\bar{X}\| \leq \|X\| + \|Y\| \leq \|G\|\|\|Q\|\| + \frac{\gamma(A, \bar{A})\|G\|\|\|Q\|\|}{1 - \gamma(A, \bar{A})} = \frac{\|G\|\|\|Q\|\|}{1 - \gamma(A, \bar{A})}. \]
The proof is complete. □

Lemma 2.3. Let conditions (1.1) and (2.2) hold. Then \( i_{in}(A) = i_{in}(\bar{A}) \) and \( i_{out}(A) = i_{out}(\bar{A}). \)

Proof. Lemma 2.2 implies the existence of a unique solution to equation (2.3) and by Theorem 2.1 \( \bar{A} \) does not have unitary eigenvalues, i.e. the eigenvalues whose absolute values are equal to one. Now consider the matrix \( \bar{A}_t = A + t(\bar{A} - A) \) \((0 \leq t \leq 1)\). Taking into account that
\[ \|G\|\|2\|A\||A - A_t\| + \|A - A_t\|^2 \leq \gamma(A, \bar{A}) < 1, \]
we prove that \( \bar{A}_t \) does not have unitary eigenvalues. Furthermore, assume in contrary that the lemma is false: \( i_{in}(A) \neq i_{in}(\bar{A}). \) Since the eigenvalues depend continuously on \( t, \bar{A}_t \) for some \( t \) should have unitary eigenvalues. This contradiction proves the required result. □

Proof of Theorem 1.1: Following [10], with each matrix \( A = (a_{ij}) \in \mathbb{C}^{n \times n} \) associate the vector \( \text{vec} (A) \in \mathbb{C}^{n^2} \) defined by
\[ \text{vec} (A) := \text{column} (a_{11}, ..., a_{1t}, a_{12}, ..., a_{s2}, ..., a_{ln}, ..., a_{nt}). \]
For a solution $X$ of (2.1) we have $(I - A^T \otimes A')^n vec(X) = vec(Q)$, cf. [10, p. 255]. Consequently,

$$vec(X) = (I - A^T \otimes A')^{-1} vec(Q).$$

Thus

$$||X||_{\mathcal{F}} = ||vec(X)||_{\mathcal{F}}^* = ||(I - A^T \otimes A')^{-1} vec(Q)||_{\mathcal{F}}^* \leq \chi(A)||vec(Q)||_{\mathcal{F}} = \chi(A)||Q||_{\mathcal{F}}.$$

Therefore, $||G||_{\mathcal{F}} := \sup_{Q \in C^{n \times n}} \frac{||GQ||_{\mathcal{F}}}{||Q||_{\mathcal{F}}} \leq \chi(A)$. Hence, replacing in Lemma 2.3 the spectral norm by the Frobenious one, we prove the theorem. \quad \Box

3. Proof of Lemma 1.2

As it is well-known, by Schur’s theorem [10], for any $C \in C^{n \times n}$, there is an orthogonal normal basis (Schur’s basis) $\{e_k\}_{k=1}^{n}$ in which $C$ is represented by a triangular matrix:

$$Ce_k = \sum_{j=1}^{k} c_{jk} e_j \text{ with } c_{jk} = \langle Ce_k, e_j \rangle \quad (k = 1, \ldots, n),$$

and $c_{jj} = \lambda_j(C)$. So $C = D_C + V_C (\sigma(C) = \sigma(D_C))$ with a normal (diagonal) matrix $D_C$ defined by $D_C e_j = \lambda_j(C) e_j (j = 1, \ldots, n)$ and a nilpotent (strictly upper-triangular) matrix $V_C$ defined by

$$V_C e_k = \sum_{j=1}^{k-1} c_{jk} e_j \quad (k = 2, \ldots, n), V_C e_1 = 0.$$ 

$D_C$ and $V_C$ will be called the diagonal part and nilpotent part of $C$, respectively.

Below $|C| = |C|_{S^{0}}$ means the operator, whose entries in some its Schur basis $\{e_k\}$ are the absolute values of the entries of operator $C$ in that basis. That is,

$$|C| e_k = \sum_{j=1}^{n} |c_{jk}| e_j \quad (k = 1, \ldots, n).$$

We will call $|C|$ the absolute value of $C$ with respect to its Schur basis $\{e_k\}$. It can be directly checked that $||C||_{\mathcal{F}} = |||C|||_{\mathcal{F}}$ and $g(C) = ||V_C||_{\mathcal{F}}$, cf. [7, Lemma 2.3.2]. The smallest integer $\nu_C \leq n$, such that $|V_C|^n = 0$ will be called the nilpotency index of $C$. By [8, Lemma 2.2.1], for any $C \in C^{n \times n}$,

$$||(|C - \lambda I|^{-1})|| \leq \sum_{j=0}^{n-1} \frac{g^j(C)}{\sqrt{j! \theta^{j+1}(C, \lambda)}} \quad (\lambda \notin \sigma(C)).$$

Put $K = A^T \otimes A^*$. Due to Lemma 3.2 from [8], we have $\nu_K \leq 3n - 2$. Since $\sigma(A^T \otimes A^*) = \sigma(A^T) \times \sigma(A^*)$, we can write $\rho(K, 1) = \theta(A)$. Thus,

$$|||K - I||| \leq \sum_{j=0}^{3n-3} \frac{g^j(K)}{\sqrt{j! \theta^{j+1}(A)}}. \quad (3.1)$$

Since $A = D_A + V_A$, where $D_A$ is the diagonal part of $A$, and $V_A$ is the nilpotent part of $A$, we have $K = D_K + V_K$, where $D_K = D_A \otimes D_A^{*}$ is the diagonal part of $K$,

$$V_K = D_A \otimes V_A^* + V_A^T \otimes D_A^* + V_A^T \otimes V_A^*.$$
is the nilpotent part of $K$.
Since $\tau(A) = \|D_A\|$, we obtain

$$g(K) = \|V_K\|_F \leq 2\|D_A\|_F\|V_A\|_F + \|V_A\|_F^2 = 2\tau(A)g(A) + g^2(A).$$

This and (3.1) prove Lemma 1.2. \qed

Note that due to [9, Corollary 5.5] the sharper but more complicated estimate

$$\|(K - \lambda I)^{-1}\| \leq \sum_{j=0}^{3n-3} \frac{1}{\theta^{j+1}(A)} \sum_{0 \leq k_1 + k_2 \leq j} n_{k_1,k_2}(j) r_{i_1+k_1} \rho_j(\lambda)^{i_1+j-k_1-k_2} \|A\|$$

is valid, where

$$n_{k_1,k_2}(j) = \frac{j!}{(k_1!k_2!)^{3/2}(j-k_1-k_2)!^2}.$$

4. Perturbation of triangular matrices

For a matrix $A = (a_{jk})_{j,k=1}^n$, put

$$V_+ = \begin{pmatrix} 0 & a_{12} & \cdots & a_{1n} \\ 0 & 0 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \quad V_- = \begin{pmatrix} 0 & \cdots & 0 & 0 \\ a_{21} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{n,n-1} & 0 \end{pmatrix}$$

and $\hat{D} = \text{diag} (a_{11}, a_{22}, \ldots, a_{nn})$. It is assumed that

$$\theta(\hat{D}) := \min_{j,k=1,\ldots,n} |a_{jk}\hat{D}_{jk} - 1| > 0. \quad (4.1)$$

Put $A_+ = \hat{D} + V_+$. We have $\theta(\hat{D}) = \theta(A_+)$ and $g(A_+) = \|V_+\|_F$. In addition, $i_{in}(\hat{D}) = i_{in}(A_+)$ is the number of $a_{jj}$ such that $|a_{jj}| < 1$. Similarly, $i_{out}(\hat{D}) = i_{out}(A_+)$ is the number of $a_{jj}$ such that $|a_{jj}| > 1$. We have

$$\zeta(A_+) = \sum_{j=0}^{3n-3} \frac{2\|\hat{D}\|_F\|V_+\|_F + \|V_+\|_F^2}{\sqrt{j}\theta^{j+1}(\hat{D})} \leq \sum_{j=0}^{3n-3} \frac{2\|A_+\|_F^2}{\sqrt{j}\theta^{j+1}(\hat{D})}.$$

Besides,

$$\|\hat{D}\|_F^2 = \sum_{k=1}^n |a_{kk}|^2.$$

Applying Corollary 1.3 with $A_+$ instead of $A$ and $A$ instead of $\hat{A}$, we obtain

**Corollary 4.1.** Let condition (4.1) hold and

$$\zeta(A_+)(2\|A_+\|_F\|V_-\|_F + \|V_-\|_F^2) < 1. \quad (4.2)$$

Then $i_{in}(A) = i_{in}(\hat{D})$ and $i_{out}(A) = i_{out}(\hat{D})$.

In particular, if $\max_k |a_{kk}| < 1$ and (4.2) holds, then $A$ is Schur-Cohn stable.

Moreover, if conditions (4.1) and (4.2) hold, and $\max_k |a_{kk}| > 1$, then $A$ is Schur-Cohn unstable.
5. Compact operators

In this section we are going to extend Theorem 1.1 to Hilbert-Schmidt operators in a Hilbert space $\mathcal{H}$. Our results below can be easily extended to Schatten-von Neumann operators, since for any Schatten-von Neumann operator $B$ there is a natural number $m$, such that $B^m$ is a Hilbert-Schmidt operator. Note that the inertia index with respect to the imaginary axis in the infinite dimensional case was investigated, in particular, in [11, 14].

Let $A$ and $\tilde{A}$ be Hilbert-Schmidt operators in $\mathcal{H}$:

$$||A||_f = \sqrt{\text{trace } AA^*} < \infty, ||\tilde{A}||_f < \infty. \quad (5.1)$$

Again put

$$g(A) := (||A||_f^2 - \tau^2(A))^{1/2}, \quad \tau(A) := \sum_{k=1}^{\infty} |\lambda_k(A)|^2)^{1/2}.$$

Here $\lambda_k(A)$ ($k = 1, 2, \ldots$) are the eigenvalues of $A$ enumerated with the multiplicities in the non-increasing order. In the infinite dimensional case $g(A)$ has the same properties as in the finite dimensional one, cf. [7, Section 6.4]. As it is well-known [6], under condition (5.1), $A$ is a limit in the norm $||A||_f$ of $n$-dimensional operators $A_n$ ($n < \infty$). Besides $\lambda_k(A_n) \rightarrow \lambda_k(A)$. Hence we have $g(A_n) \rightarrow g(A)$. Again denote by $i_{in}(A)$ ($i_{out}(A)$) the number of the eigenvalues of $A$ taken with their multiplicities, lying inside (outside) the unit circle and assume that

$$\theta(A) = \inf_{j,k=1,2,\ldots} |1 - \lambda_j(A)\lambda_k(A)| > 0. \quad (5.2)$$

Certainly, under consideration it can be $i_{in}(A) = \infty$. Put

$$\zeta(A) = \sum_{j=0}^{\infty} \frac{(2\tau(A)g(A) + g^2(A))j}{\sqrt{j!}\theta^{j+1}(A)}.$$

Now Corollary 1.3 implies

**Corollary 5.1.** Let conditions (5.1), (5.2) and (1.4) hold. Then $i_{out}(\tilde{A}) = i_{out}(A)$.

Certainly, the result similar to Corollary 1.4 is also valid with the replacement of $n$ by infinity.

As in the finite dimensional case we have

$$\zeta(A) \leq \sum_{j=0}^{\infty} \frac{2||A||_f^2j}{\sqrt{j!}\theta^{j+1}(A)}.$$

By the Schwarz inequality

$$\zeta(A) \leq \sum_{j=0}^{\infty} \frac{2^{j/2}||A||_f^{2j}}{\sqrt{j!}\theta^{j+1}(A)} \leq \left( \sum_{j=0}^{\infty} \frac{1}{2^j} \sum_{j=0}^{\infty} \frac{8^{j}||A||_f^{4j}}{\sqrt{j!}\theta^{2j+1}(A)} \right)^{1/2} = \frac{\sqrt{2}}{\theta(A)} \exp \left( \frac{4||A||_f^4}{\theta^2(A)} \right).$$

If $A = V$ is quasinilpotent, then by the Schwarz inequality

$$\zeta(V) = \sum_{j=0}^{\infty} \frac{||V||_f^{2j}}{\sqrt{j!}} = \sum_{j=0}^{\infty} \frac{2^{j/2}||V||_f^{2j}}{\sqrt{j!}} \leq \left( \sum_{j=0}^{\infty} \frac{2^{j}||V||_f^{4j}}{\sqrt{j!}} \sum_{j=0}^{\infty} \frac{1}{2^j} \right)^{1/2} = \sqrt{2} \exp \left( ||V||_f^{4} \right). \quad (5.3)$$

**Example 5.2.** Let $V$ be a quasi-nilpotent Hilbert-Schmidt operator and $\tilde{A} = V + B$, where $B$ is an arbitrary Hilbert-Schmidt operator.

Since $i_{out}(V) = 0$, if

$$\sqrt{2} (||V||_f ||B||_f + ||B||_f^2) \exp \left( ||V||_f^4 \right) < 1,$$

then by (5.3) and Corollary 5.1 we have $i_{out}(\tilde{A}) = 0$. 
References