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Conservation of the Number of Eigenvalues of Finite Dimensional and Compact Operators Inside and Outside Circle

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Abstract. Let $i_{in}(A)$ and $i_{out}(A)$ be the numbers of the eigenvalues of a finite matrix A lying inside and outside the unit circle, respectively. Let $\tilde{A}$ be a perturbed matrix. We obtain the conditions under which $i_{in}(A) = i_{in}(\tilde{A})$ and $i_{out}(A) = i_{out}(\tilde{A})$. Our main tool is the norm estimates for resolvents of operators on the tensor product of Euclidean spaces. The results for finite matrices are particularly generalized to Hilbert-Schmidt operators.

1. Introduction and statement of the main result

Let $\mathbb{C}^n$ be an n-dimensional Euclidean space with a scalar product $< .,. >_{\mathbb{C}^n} =< .,. >$, the norm $\|.\|_n = \sqrt{< .,. >}$ and the unit operator I; $\mathbb{C}^{n\times n}$ means the set of all complex $n \times n$ matrices. For an $A \in \mathbb{C}^{n\times n}$, $\lambda_k(A)$ $(k = 1, ..., n)$ are the eigenvalues of A enumerated in an arbitrary order with their multiplicities, $\|A\| = \|A\|_n = \sup_{x\in\mathbb{C}^n} \|Ax\|_n/\|x\|_n$ is the (operator) spectral norm, $\sigma(A)$ denotes the spectrum of A, A^* is the adjoint to A, A^T is the transposed one and A^{-1} is the inverse to A; $\|A\|_F = \sqrt{\text{trace } AA^*}$ is the Hilbert-Schmidt (Frobenius) norm of A, $r_s(A)$ is the spectral radius, $\rho(A, \lambda) = \min_{k=1,...,n} |\lambda - \lambda_k(A)|$ is the distance between $\sigma(A)$ and a point $\lambda \in \mathbb{C}$.

The inertia $\text{In}(A)$ of a matrix A with respect to the imaginary axis is defined as a triple of nonnegative integers $(\pi(A), \nu(A), \delta(A))$, where $\pi(A)$ is the number of eigenvalues of A with positive real parts, $\nu(A)$ is the number of eigenvalues of A with negative real parts and $\delta(A)$ is the number of eigenvalues of A on the imaginary axis. For a Hermitian matrix S, the inequality $S > 0$ $(S < 0)$ means that S is positive (negative) definite.

The classical theorem of Ostrowski and Schneider [13] asserts that for a given matrix A, there exists a Hermitian matrix H, such that $\Re(AH) = ((AH)^* + AH)/2 > 0$ if and only if $\delta(A) = 0$. If $\Re(AH) > 0$, then $\text{In}(A) = \text{In}(H)$. In the paper [3] the just mentioned result has been reduced to the semi-definite case.

The inertia, $\text{In}_\Omega(A)$, of a matrix A with respect to the unit circle $\Omega = \{z \in \mathbb{C} : |z| = 1\}$ is defined as a triple of nonnegative integers $(i_{in}(A), i_{out}(A), i_\Omega(A))$, where $i_{in}(A)$ is the number of the eigenvalues of A taken with their multiplicities, lying inside Ω, $i_{out}(A)$ is the number of the eigenvalues of A outside Ω, and $i_\Omega(A)$ is

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the number of the eigenvalues of A on Ω. In the paper by Wimmer [16] , the result similar to the inertia theorem of Ostrowski and Schneider has been derived for $\mathrm{In}_\Omega(A)$ (see Theorem 2.1 below). Afterwards the Ostrowski-Schneider and Wimmer results have been generalized in various directions, cf. [1]-[5], [12, 15]. At the same time, to the best of our knowledge, perturbations of the inertia were almost not investigated in the available literature although they are important for various applications. Here we can only mention the paper [8] which contains some results devoted to perturbations of the inertia with respect to imaginary axis. In this paper we establish a condition that provides conservation of the numbers $i_{in}(A)$ and $i_{out}(A)$ under perturbations.

Let $E_1 = E_2 = \mathbb{C}^n$ and $E = E_1 \otimes E_2$, where $\otimes$ is the symbol of the tensor product. The scalar products $< .,. >_E$ in E is defined by

$$< y \otimes h, y_1 \otimes h_1 >_E := < y, y_1 >_{E_1} < h, h_1 >_{E_2} \quad (y, y_1 \in E_1;\ h, h_1 \in E_2)$$

where $< y, y_1 >_{E_l} = < y, y_1 >_n$ is the scalar products in E_l $(l = 1, 2)$. The norm in E is defined by $\|.\|_E = \sqrt{< .,. >_E}$. In addition, $I_E = I \otimes I$ is the unit operators on E. It is clear that $\dim E = n^2$. Assuming that

$$\theta(A) := \min_{j,k=1,\dots,n} |1 - \lambda_j(A)\overline{\lambda}_k(A)| > 0, \tag{1.1}$$

put

$$\chi(A) := \|(I_E - A^T \otimes A^*)^{-1}\|_{n^2}.$$

Now we are in a position to formulate our main result.

Theorem 1.1. *Let $A, \tilde{A} \in \mathbb{C}^{n\times n}$, and the conditions (1.1) and*

$$\chi(A)(2\|A\|_F\|A - \tilde{A}\|_F + \|A - \tilde{A}\|_F^2) < 1 \tag{1.2}$$

hold. Then $i_{in}(\tilde{A}) = i_{in}(A)$ and $i_{out}(\tilde{A}) = i_{out}(A)$.

This theorem is proved in the next two sections.

To formulate our next result introduce the quantity (the departure from normality of A)

$$g(A) := (\|A\|_F^2 - \tau^2(A))^{1/2}, \text{ where } \tau(A) := (\sum_{k=1}^n |\lambda_k(A)|^2)^{1/2}.$$

In Sections 2.1 and 2.2 of the book [7] it is checked that

$$g^2(A) \le \|A\|_F^2 - |\text{trace } A^2|, g^2(A) \le \|A - A^*\|_F^2/2 \text{ and } g(e^{it}A + zI) = g(A)$$

for all $t \in \mathbb{R}$ and $z \in \mathbb{C}$. If A is a normal matrix: $AA^* = A^*A$, then $g(A) = 0$. In addition, if A_1 and A_2 are commuting $n \times n$ matrices, then $g(A_1 + A_2) \le g(A_1) + g(A_2)$. By the inequality between geometric and arithmetic mean values,

$$\left(\frac{1}{n}\sum_{k=1}^n |\lambda_k(A)|^2\right)^n \ge \prod_{k=1}^n |\lambda_k(A)|^2. \text{ So } g^2(A) \le \|A\|_F^2 - n(\det\ A)^{2/n}.$$

Below we prove the following

Lemma 1.2. *Let condition (1.1) hold. Then $\chi(A) \le \zeta(A)$, where*

$$\zeta(A) := \sum_{j=0}^{3n-3} \frac{(2\tau(A)g(A) + g^2(A))^j}{\sqrt{j!}\,\theta^{j+1}(A)}.$$

From the definition of $g(A)$ it follows that $2\tau(A)g(A) \le \tau^2(A) + g^2(A) = \|A\|_F^2$. Since $g(A) \le \|A\|_F$, we have

$$\zeta(A) \le \sum_{j=0}^{3n-3} \frac{2^j \|A\|_F^{2j}}{\sqrt{j!}\theta^{j+1}(A)}. \tag{1.3}$$

Theorem 1.1 and Lemma 1.2 imply

Corollary 1.3. *Let $A, \tilde{A} \in \mathbb{C}^{n\times n}$. Let the conditions (1.1) and*

$$\zeta(A)(2\|A\|_F\|A - \tilde{A}\|_F + \|A - \tilde{A}\|_F^2) < 1 \tag{1.4}$$

hold. Then $i_{in}(\tilde{A}) = i_{in}(A)$ and $i_{out}(\tilde{A}) = i_{out}(A)$.

A matrix A is said to be *Schur-Cohn stable* if its spectral radius is less than one. It is said to be *Schur-Cohn unstable*, if $|\lambda_k(A)| > 1$ for at least one eigenvalue of A. Making use of Corollary 1.3 we arrive at

Corollary 1.4. *Let $A, \tilde{A} \in \mathbb{C}^{n\times n}$. If A is Schur-Cohn stable and condition (1.4) holds, then $\tilde{A}$ is also Schur-Cohn stable.*

Moreover, if conditions (1.1) and (1.4) hold, and A is Schur-Cohn unstable, then $\tilde{A}$ is also Schur-Cohn unstable.

2. Proof of Theorem 1.1

We need the following theorem proved in [16].

Theorem 2.1. *Let $A, Q \in \mathbb{C}^{n\times n}$ and Q be Hermitian strongly positive definite. If X is a Hermitian solution of the equation*

$$X - AXA^* = Q, \tag{2.1}$$

then X is nonsingular and the number $i_+(X)$ of positive (the number $i_-(X)$ of negative) eigenvalues of X is equal to $i_{in}(A)$ $(i_{out}(A))$.

The operator $G : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$ defined by $X = GQ$ will be called the Green operator to (2.1). Put

$$\|G\| = \sup_{Q\in\mathbb{C}^{n\times n}} \frac{\|GQ\|}{\|Q\|}.$$

Lemma 2.2. *Let $A, \tilde{A} \in \mathbb{C}^{n\times n}$, and the conditions (1.1) and*

$$\gamma(A, \tilde{A}) := \|A - \tilde{A}\| \, \|G\|(2\|A\| + \|A - \tilde{A}\|) < 1 \tag{2.2}$$

hold. Then the equation

$$\tilde{X} - \tilde{A}\tilde{X}\tilde{A}^* = Q \tag{2.3}$$

has a unique solution $\tilde{X}$ satisfying

$$\|\tilde{X}\| \le \frac{\|G\|\|Q\|}{1 - \gamma(A, \tilde{A})} \tag{2.4}$$

and

$$\|X - \tilde{X}\| \le \frac{\|G\|\|Q\|\gamma(A, \tilde{A})}{1 - \gamma(A, \tilde{A})}, \tag{2.5}$$

where X is the solution to (2.1).

Proof. Put $Y = \tilde{X} - X, B = \tilde{A} - A$. Then, subtracting (2.1) from (2.3), we can write

$$Y - (A + B)(X + Y)(A^* + B^*) + AXA^* = 0.$$

So

$$Y - AYA^* = Q_X + Q_Y, \tag{2.6}$$

where $Q_X := AXB^* + BXA^* + BXB^*$ and $Q_Y := AYB^* + BYA^* + BYB^*$. Or

$$Q_Y := (A_l B_r^* + B_l A_r^* + B_l B_r^*)Y,$$

where A_l, A_r are defined by $A_l Y = AY, A_r Y = YA$. Thus,

$$Y = G(Q_X + Q_Y) = GQ_X + WY, \tag{2.7}$$

where $W := G(A_l B_r^* + B_l A_r^* + B_l B_r^*)$. So due to (2.2),

$$\|W\| \le \|G\|(2\|A\|\|B\| + \|B\|^2) = \gamma(A, \tilde{A}) < 1 \tag{2.8}$$

and therefore (2.7) implies

$$Y = (I - W)^{-1} GQ_X = \sum_{k=0}^{\infty} W^k GQ_X$$

and

$$\|Y\| = (1 - \gamma(A, \tilde{A}))^{-1} \|GQ_X\|. \tag{2.9}$$

Take into account that

$$\|Q_X\| \le \|X\|(2\|A\|\|B\| + \|B\|^2) \le \|G\|\|Q\|(2\|A\|\|B\| + \|B\|^2) = \gamma(A, \tilde{A})\|Q\|.$$

Therefore,

$$\|Y\| \le \frac{\gamma(A, \tilde{A})\|G\|\|Q\|}{1 - \gamma(A, \tilde{A})}$$

and

$$\|\tilde{X}\| \le \|X\| + \|Y\| \le \|G\|\|Q\| + \frac{\gamma(A, \tilde{A})\|G\|\|Q\|}{1 - \gamma(A, \tilde{A})} = \frac{\|G\|\|Q\|}{1 - \gamma(A, \tilde{A})}.$$

The proof is complete. □

Lemma 2.3. *Let conditions (1.1) and (2.2) hold. Then $i_{in}(A) = i_{in}(\tilde{A})$ and $i_{out}(A) = i_{out}(\tilde{A})$.*

Proof. Lemma 2.2 implies the existence of a unique solution to equation (2.3) and by Theorem 2.1 $\tilde{A}$ does not have unitary eigenvalues, i.e. the eigenvalues whose absolute values are equal to one. Now consider the matrix $A_t = A + t(\tilde{A} - A)$ $(0 \le t \le 1)$. Taking into account that

$$\|G\|(2\|A\|\|A - A_t\| + \|A - A_t\|^2) \le \gamma(A, \tilde{A}) < 1,$$

we prove that A_t does not have unitary eigenvalues. Furthermore, assume in contrary that the lemma is false: $i_{in}(A) \ne i_{in}(\tilde{A})$. Since the eigenvalues depend continuously on t, A_t for some t should have unitary eigenvalues. This contradiction proves the required result. □

Proof of Theorem 1.1: Following [10], with each matrix $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ associate the vector $vec\ (A) \in \mathbb{C}^{n^2}$ defined by

$$vec\ (A) := column\ (a_{11}, ..., a_{s1}, a_{12}, ..., a_{s2}, ..., a_{1n}, ..., a_{sn}).$$

For a solution X of (2.1) we have $(I - A^T \otimes A^*)\mathrm{vec}(X) = \mathrm{vec}(Q)$, cf. [10, p. 255]. Consequently,

$$\mathrm{vec}(X) = (I - A^T \otimes A^*)^{-1}\mathrm{vec}(Q),$$

Thus

$$\|X\|_F = \|vec\ (X)\|_{n^2} = \|(I - A^T \otimes A^*)^{-1}\mathrm{vec}(Q)\|_{n^2}$$
$$\leq \chi(A)\|\mathrm{vec}(Q)\|_{n^2} = \chi(A)\|Q\|_F.$$

Therefore, $\|G\|_F := \sup_{Q\in\mathbb{C}^{n\times n}} \frac{\|GQ\|_F}{\|Q\|_F} \leq \chi(A)$. Hence, replacing in Lemma 2.3 the spectral norm by the Frobenious one, we prove the theorem. □

3. Proof of Lemma 1.2

As it is well-known, by Schur's theorem [10], for any $C \in \mathbb{C}^{n\times n}$, there is an orthogonal normal basis (Schur's basis) $\{e_k\}_{k=1}^n$ in which C is represented by a triangular matrix:

$$Ce_k = \sum_{j=1}^{k} c_{jk}e_j \text{ with } c_{jk} =< Ce_k, e_j > \quad (k = 1, ..., n),$$

and $c_{jj} = \lambda_j(C)$. So $C = D_C + V_C$ ($\sigma(C) = \sigma(D_C)$) with a normal (diagonal) matrix D_C defined by $D_Ce_j = \lambda_j(C)e_j$ ($j = 1, ..., n$) and a nilpotent (strictly upper-triangular) matrix V_C defined by

$$V_Ce_k = \sum_{j=1}^{k-1} c_{jk}e_j \quad (k = 2, ..., n), V_Ce_1 = 0.$$

D_C and V_C will be called the diagonal part and nilpotent part of C, respectively.

Below $|C| = |C|_{Sb}$ means the operator, whose entries in some its Schur basis $\{e_k\}$ are the absolute values of the entries of operator C in that basis. That is,

$$|C|e_k = \sum_{j=1}^{k} |a_{jk}|e_j \quad (k = 1, ..., n).$$

We will call $|C|$ the absolute value of C with respect to its Schur basis $\{e_k\}$. It can be directly checked that $\|C\|_F = \|\ |C|\ \|_F$ and $g(C) = \|V_C\|_F$, cf. [7, Lemma 2.3.2]. The smallest integer $\nu_C \leq n$, such that $|V_C|^{\nu_C} = 0$ will be called *the nilpotency index of* C. By [8, Lemma 2.2], for any $C \in \mathbb{C}^{n\times n}$,

$$\|(C - \lambda I)^{-1}\| \leq \sum_{j=0}^{\nu_C-1} \frac{g^j(C)}{\sqrt{j!}\rho^{j+1}(C, \lambda)} \quad (\lambda \notin \sigma(C)).$$

Put $K = A^T \otimes A^*$. Due to Lemma 3.2 from [8], we have $\nu_K \leq 3n - 2$. Since $\sigma(A^T \otimes A^*) = \sigma(A^T) \times \sigma(A^*)$, we can write $\rho(K, 1) = \theta(A)$. Thus,

$$\|(K - I)^{-1}\| \leq \sum_{j=0}^{3n-3} \frac{g^j(K)}{\sqrt{j!}\theta^{j+1}(A)}. \tag{3.1}$$

Since $A = D_A + V_A$, where D_A is the diagonal part of A, and V_A is the nilpotent part of A, we have $K = D_K + V_K$, where $D_K = D_A \otimes D_A^*$ is the diagonal part of K,

$$V_K = D_A \otimes V_A^* + V_A^T \otimes D_A^* + V_A^T \otimes V_A^*$$

is the nilpotent part of K.

Since $\tau(A) = \|D_A\|_F$, we obtain

$$g(K) = \|V_K\|_F \leq 2\|D_A\|_F\|V_A\|_F + \|V_A\|_F^2 = 2\tau(A)g(A) + g^2(A).$$

This and (3.1) prove Lemma 1.2. □

Note that due to [9, Corollary 5.5] the sharper but more complicated estimate

$$\|(K-\lambda I)^{-1}\| \leq \sum_{j=0}^{3n-3} \frac{1}{\theta^{j+1}(A)} \sum_{0\leq k_1+k_2\leq j} \eta^{(j)}_{k_1,k_2} r_s^{k_1+k_2}(A) g^{2j-k_1-k_2}(A),$$

is valid, where

$$\eta^{(j)}_{k_1,k_2} = \frac{j!}{(k_1!k_2!)^{3/2}[(j-k_1-k_2)!]^2}.$$

4. Perturbation of triangular matrices

For a matrix $A = (a_{jk})_{j,k=1}^n$, put

$$V_+ = \begin{pmatrix} 0 & a_{12} & \dots & a_{1n} \\ 0 & 0 & \dots & a_{2n} \\ . & \dots & . & . \\ 0 & 0 & \dots & 0 \end{pmatrix}, \quad V_- = \begin{pmatrix} 0 & \dots & 0 & 0 \\ a_{21} & \dots & 0 & 0 \\ . & \dots & . & \\ a_{n1} & \dots & a_{n,n-1} & 0 \end{pmatrix}$$

and $\hat{D} = diag\ (a_{11}, a_{22}, ..., a_{nn})$. It is assumed that

$$\theta(\hat{D}) := \min_{j,k=1,\dots,n} |a_{jj}\bar{a}_{kk} - 1| > 0. \tag{4.1}$$

Put $A_+ = \hat{D} + V_+$. We have $\theta(\hat{D}) = \theta(A_+)$ and $g(A_+) = \|V_+\|_F$. In addition, $i_{in}(\hat{D}) = i_{in}(A_+)$ is the number of a_{jj}, such that $|a_{jj}| < 1$. Similarly, $i_{out}(\hat{D}) = i_{out}(A_+)$ is the number of a_{jj}, such that $|a_{jj}| > 1$. We have

$$\zeta(A_+) = \sum_{j=0}^{3n-3} \frac{(2\|\hat{D}\|_F\|V_+\|_F + \|A_+\|_F^2)^j}{\sqrt{j!}\theta^{j+1}(\hat{D})} \leq \sum_{j=0}^{3n-3} \frac{2^j\|A_+\|_F^{2j}}{\sqrt{j!}\theta^{j+1}(\hat{D})}.$$

Besides,

$$\|\hat{D}\|_F^2 = \sum_{k=1}^n |a_{kk}|^2.$$

Applying Corollary 1.3 with A_+ instead of A and A instead of $\tilde{A}$, we obtain

Corollary 4.1. *Let condition (4.1) hold and*

$$\zeta(A_+)(2\|A_+\|_F\|V_-\|_F + \|V_-\|_F^2) < 1. \tag{4.2}$$

Then $i_{in}(A) = i_{in}(\hat{D})$ and $i_{out}(A) = i_{out}(\hat{D})$.

In particular, if $\max_k |a_{kk}| < 1$ *and (4.2) holds, then A is Schur-Cohn stable.*

Moreover, if conditions (4.1) and (4.2) hold, and $\max_k |a_{kk}|| > 1$, *then A is Schur-Cohn unstable.*

5. Compact operators

In this section we are going to extend Theorem 1.1 to Hilbert-Schmidt operators in a Hilbert space $\mathcal{H}$. Our results below can be easily extended to Schatten-von Neumann operators, since for any Schatten-von Neumann operator B there is a natural number m, such that B^m is a Hilbert-Schmidt operator. Note that the inertia index with respect to the imaginary axis in the infinite dimensional case was investigated, in particular, in [11, 14].

Let A and $\tilde{A}$ be Hilbert-Schmidt operators in $\mathcal{H}$:

$$\|A\|_F = \sqrt{\text{trace } AA^*} < \infty, \|\tilde{A}\|_F < \infty. \tag{5.1}$$

Again put

$$g(A) := (\|A\|_F^2 - \tau^2(A))^{1/2} \text{ , where } \tau(A) := (\sum_{k=1}^{\infty} |\lambda_k(A)|^2)^{1/2}.$$

Here $\lambda_k(A)$ $(k = 1, 2, ...)$ are the eigenvalues of A enumerated with the multiplicities in the non-increasing order. In the infinite dimensional case $g(A)$ has the same properties as in the finite dimensional one, cf. [7, Section 6.4]. As it is well-known [6], under condition (5.1), A is a limit in the norm $\|A\|_F$ of n-dimensional operators A_n $(n < \infty)$. Besides $\lambda_k(A_n) \to \lambda_k(A)$. Hence we have $g(A_n) \to g(A)$. Again denote by $i_{in}(A)$ $(i_{out}(A))$ the number of the eigenvalues of A taken with their multiplicities, lying inside (outside) the unit circle and assume that

$$\theta(A) = \inf_{j,k=1,2,...} |1 - \lambda_j(A)\overline{\lambda}_k(A)| > 0. \tag{5.2}$$

Certainly, under consideration it can be $i_{in}(A) = \infty$. Put

$$\zeta(A) = \sum_{j=0}^{\infty} \frac{(2\tau(A)g(A) + g^2(A))^j}{\sqrt{j!}\theta^{j+1}(A)}$$

Now Corollary 1.3 implies

Corollary 5.1. *Let conditions (5.1), (5.2) and (1.4) hold. Then* $i_{out}(\tilde{A}) = i_{out}(A)$.

Certainly, the result similar to Corollary 1.4 is also valid with the replacement of n by infinity.

As in the finite dimensional case we have

$$\zeta(A) \le \sum_{j=0}^{\infty} \frac{2^j \|A\|_F^{2j}}{\sqrt{j!}\theta^{j+1}(A)}.$$

By the Schwarz inequality

$$\zeta(A) \le \sum_{j=0}^{\infty} \frac{2^{3j/2} \|A\|_F^{2j}}{2^{j/2} \sqrt{j!}\theta^{j+1}(A)} \le \left(\sum_{k=0}^{\infty} \frac{1}{2^k} \sum_{j=0}^{\infty} \frac{8^j \|A\|_F^{4j}}{j!\theta^{2(j+1)}(A)} \right)^{1/2} = \frac{\sqrt{2}}{\theta(A)} \exp\left(\frac{4\|A\|_F^4}{\theta^2(A)}\right).$$

If $A = V$ is quasinilpotent, then by the Schwarz inequality

$$\zeta(V) = \sum_{j=0}^{\infty} \frac{\|V\|_F^{2j}}{\sqrt{j!}} = \sum_{j=0}^{\infty} \frac{2^{j/2}\|V\|_F^{2j}}{2^{j/2}\sqrt{j!}} \le (\sum_{j=0}^{\infty} \frac{2^j \|V\|_F^{4j}}{j!} \sum_{j=0}^{\infty} \frac{1}{2^j})^{1/2} = \sqrt{2} \exp(\|V\|_F^4). \tag{5.3}$$

Example 5.2. *Let V be a quasi-nilpotent Hilbert-Schmidt operator and $\tilde{A} = V + B$, where B is an arbitrary Hilbert-Schmidt operator.*

Since $i_{out}(V) = 0$, if

$$\sqrt{2}(2\|V\|_F\|B\|_F + \|B\|_F^2) \exp(\|V\|_F^4) < 1,$$

then by (5.3) and Corollary 5.1 we have $i_{out}(\tilde{A}) = 0$.

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