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Antisymmetric relations of operators which satisfy specified conditions

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Abstract. This paper is provided a condition that two relations $\triangle$ and $\square$ are equivalent. It is shown that if $T\square S$, then $(TS^\dagger)^\dagger \triangle TS^\dagger$ and $(S^\dagger T)^\dagger \triangle S^\dagger T$. Also, it is presented a condition that $T^\dagger \triangle T$ establishes.

1. Introduction

Rao, Mitra and Bhimasankaram [6] studied one of the most striking results of generalized inverses which states that for complex $m \times n$ matrices, A and B,

$$AB^\dagger A = A, \;\; BA^\dagger B = B \Longrightarrow A = B, \tag{1}$$

in which $(.)^\dagger$ denotes the Moore-Penrose inverse of the matrix (.)[1]. The proof given in [6] was based on the singular value decomposition theory, which is essentially nonalgebraic and finite dimensional in nature.

Hartwig [3] extracted the purely algebraic conditions, which make this result valid, and to extend the class of objects for which (1) remains true in terms of ring elements.

Hartwig [3] proved that the relations

$$T \triangle S \quad \text{iff} \quad TT^*T = TS^*T, \tag{2}$$

and

$$T \square S \quad \text{iff} \quad T = TS^\dagger T, \tag{3}$$

are both antisymmetric as well as reflexive.

The purpose of this paper is to provide a condition that two relations $\triangle$ and $\square$ are equivalent. Also, we show that if $T\square S$, then $(TS^\dagger)^\dagger \triangle TS^\dagger$ and $(S^\dagger T)^\dagger \triangle S^\dagger T$. Moreover, we prove that if T is an idempotent operator, then $T^\dagger \triangle T$.

Xu and Sheng in [7] showed that a bounded adjointable operator between two Hilbert C^*-modules admits a bounded Moore–Penrose inverse if and only if the operator has closed range.

Throughout the paper $\mathcal{A}$ is a C*-algebra (not necessarily unital). A (right) pre-Hilbert module over a C^*-algebra $\mathcal{A}$ is a complex linear space $\mathcal{X}$, which is an algebraic right $\mathcal{A}$-module and $\lambda(xa) = (\lambda x)a = x(\lambda a)$

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equipped with an $\mathcal{A}$-valued inner product $\langle .,. \rangle : \mathcal{X} \times \mathcal{X} \to \mathcal{A}$ satisfying,
(i) $\langle x, x\rangle \geq 0$, and $\langle x, x\rangle = 0$ iff $x = 0$,
(ii)$\langle x, y + \lambda z\rangle = \langle x, y\rangle + \lambda\langle x, z\rangle$,
(iii)$\langle x, ya\rangle = \langle x, y\rangle a$,
(iv) $\langle y, x\rangle = \langle x, y\rangle^*$.
for each $x, y, z \in \mathcal{X}$, $\lambda \in \mathbb{C}$, $a \in \mathcal{A}$. A pre-Hilbert $\mathcal{A}$-module $\mathcal{X}$ is called a Hilbert $\mathcal{A}$-module if it is complete with respect to the norm $\|x\| = \|\langle x, x\rangle\|^{\frac{1}{2}}$. Left Hilbert $\mathcal{A}$-modules are defined in a similar way. For example every C^*-algebra $\mathcal{A}$ is a Hilbert $\mathcal{A}$-module with respect to the inner product $\langle x, y\rangle = x^*y$, and every inner product space is a left Hilbert $\mathbb{C}$-module. Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are Hilbert $\mathcal{A}$-modules. Then $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ is the set of all maps $T : \mathcal{X} \to \mathcal{Y}$ for which there is a map $T^* : \mathcal{Y} \to \mathcal{X}$ such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for each $x \in \mathcal{X}$, $y \in \mathcal{Y}$. It is known that any element T of $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ must be a bounded linear operator, which is also $\mathcal{A}$-linear in the sense that $T(xa) = (Tx)a$ for $x \in \mathcal{X}$ and $a \in \mathcal{A}$ [4, Page 8]. We use the notations $\mathcal{L}(\mathcal{X})$ in place of $\mathcal{L}(\mathcal{X}, \mathcal{X})$, and ker(·) and ran(·) for the kernel and the range of operators, respectively.

Theorem 1.1. *[4, Theorem 3.2]. Suppose that $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ has closed range. Then*

- $\ker(T)$ *is orthogonally complemented in $\mathcal{X}$ with complement* $\operatorname{ran}(T^*)$.
- $\operatorname{ran}(T)$ *is orthogonally complemented in $\mathcal{Y}$ with complement* $\ker(T^*)$.
- *The map $T^* \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ has closed range.*

Definition 1.2. *Let $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$. The Moore-Penrose inverse of T (if it exists) is an element $T^\dagger$ of $\mathcal{L}(\mathcal{Y}, \mathcal{X})$ satisfying*

$$TT^\dagger T = T, \quad T^\dagger TT^\dagger = T^\dagger, \quad (TT^\dagger)^* = TT^\dagger, \quad (T^\dagger T)^* = T^\dagger T. \tag{4}$$

Motivated by these conditions $T^\dagger$ is unique and $T^\dagger T$ and $TT^\dagger$ are orthogonal projections. Clearly, T is Moore-Penrose invertible if and only if T^* is Moore-Penrose invertible, and in this case $(T^*)^\dagger = (T^\dagger)^*$.

By Definition 4, we have

$$\begin{aligned} \operatorname{ran}(T) &= \operatorname{ran}(TT^\dagger), & \operatorname{ran}(T^\dagger) &= \operatorname{ran}(T^\dagger T) = \operatorname{ran}(T^*), \\ \ker(T) &= \ker(T^\dagger T), & \ker(T^\dagger) &= \ker(TT^\dagger) = \ker(T^*), \end{aligned}$$

and by Theorem 1.1, we have

$$\begin{aligned} \mathcal{X} &= \ker(T) \oplus \operatorname{ran}(T^\dagger) = \ker(T^\dagger T) \oplus \operatorname{ran}(T^\dagger T), \\ \mathcal{Y} &= \ker(T^\dagger) \oplus \operatorname{ran}(T) = \ker(TT^\dagger) \oplus \operatorname{ran}(TT^\dagger). \end{aligned}$$

A matrix form of a bounded adjointable operator $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ can be induced by some natural decompositions of Hilbert C^*-modules. Indeed, let $\mathcal{M}$ and $\mathcal{N}$ be closed orthogonally complemented submodules of $\mathcal{X}$ and $\mathcal{Y}$, respectively, and let $\mathcal{X} = \mathcal{M} \oplus \mathcal{M}^\perp$ and $\mathcal{Y} = \mathcal{N} \oplus \mathcal{N}^\perp$. Then T can be written as the following 2×2 matrix

$$T = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}, \tag{5}$$

where $T_1 \in \mathcal{L}(\mathcal{M}, \mathcal{N})$, $T_2 \in \mathcal{L}(\mathcal{M}^\perp, \mathcal{N})$, $T_3 \in \mathcal{L}(\mathcal{M}, \mathcal{N}^\perp)$, and $T_4 \in \mathcal{L}(\mathcal{M}^\perp, \mathcal{N}^\perp)$. In fact if $P_\mathcal{M}$ denotes the projection corresponding to $\mathcal{M}$, then $T_1 = P_\mathcal{N} T P_\mathcal{M}$, $T_2 = P_\mathcal{N} T(1 - P_\mathcal{M})$, $T_3 = (1 - P_\mathcal{N})TP_\mathcal{M}$, $T_4 = (1 - P_\mathcal{N})T(1 - P_\mathcal{M})$.

The proof of the following lemata can be found in [2] and [5].

Lemma 1.3. *Suppose that $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ has closed range. Then T has the following matrix decomposition with respect to the orthogonal decompositions of closed submodules $\mathcal{X} = \operatorname{ran}(T^*) \oplus \ker(T)$ and $\mathcal{Y} = \operatorname{ran}(T) \oplus \ker(T^*)$:*

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \operatorname{ran}(T^*) \\ \ker(T) \end{bmatrix} \mapsto \begin{bmatrix} \operatorname{ran}(T) \\ \ker(T^*) \end{bmatrix},$$

where T_1 is invertible. Moreover

$$T^\dagger = \begin{bmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \operatorname{ran}(T) \\ \ker(T^*) \end{bmatrix} \mapsto \begin{bmatrix} \operatorname{ran}(T^*) \\ \ker(T) \end{bmatrix}.$$

Lemma 1.4. *Suppose that $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ has closed range. Let $\mathcal{X}_i$ and $\mathcal{Y}_i$, respectively, be closed submodules of $\mathcal{X}$ and $\mathcal{Y}$, $i = 1, 2$, such that $\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2$ and $\mathcal{Y} = \mathcal{Y}_1 \oplus \mathcal{Y}_2$. Then the operator T has the following matrix representations with respect to the orthogonal sums of submodules $\mathcal{X} = \operatorname{ran}(T^*) \oplus \ker(T)$ and $\mathcal{Y} = \operatorname{ran}(T) \oplus \ker(T^*)$:*

- *If*

$$T = \begin{bmatrix} T_1 & T_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{X}_1 \\ \mathcal{X}_2 \end{bmatrix} \to \begin{bmatrix} \operatorname{ran}(T) \\ \ker(T^*) \end{bmatrix},$$

then $E = T_1T_1^* + T_2T_2^* \in \mathcal{L}(\operatorname{ran}(T))$ is positive and invertible. Moreover,

$$T^\dagger = \begin{bmatrix} T_1^*E^{-1} & 0 \\ T_2^*E^{-1} & 0 \end{bmatrix}. \tag{6}$$

- *If*

$$T = \begin{bmatrix} T_1 & 0 \\ T_3 & 0 \end{bmatrix} : \begin{bmatrix} \operatorname{ran}(T^*) \\ \ker(T) \end{bmatrix} \to \begin{bmatrix} \mathcal{Y}_1 \\ \mathcal{Y}_2 \end{bmatrix}, \tag{7}$$

then $F = T_1^*T_1 + T_3^*T_3 \in \mathcal{L}(\operatorname{ran}(T^*))$ is positive and invertible. Moreover,

$$T^\dagger = \begin{bmatrix} F^{-1}T_1^* & F^{-1}T_3^* \\ 0 & 0 \end{bmatrix}. \tag{8}$$

Theorem 1.5. *[5, Theorem 2.3.]. Suppose that $T \in L(\mathcal{X})$ has closed range. Then the following assertions are equivalent:*

(i) *$T = PQ$ for two projections P and Q,*
(ii) *$T^2 = TT^*T$,*
(iii) *$T^* = T^\dagger T^2 T^\dagger$,*
(iv) *$T = T(T^\dagger)^2 T$,*
(v) *$(T^\dagger)^2 = T^\dagger$,*
(vi) *$|Tx|^2 = \langle Tx, x\rangle$, for all $x \in (\ker(T))^\perp$,*
(vii) *$T^\dagger T^* = T^*$.*

2. Main results

In the section provid a condition on two relations $\triangle$ and $\square$ are equivalent. Also, we show that if $T\square S$, then $(TS^\dagger)^\dagger \triangle TS^\dagger$ and $(S^\dagger T)^\dagger \triangle S^\dagger T$. Moreover, we state a condition that $T^\dagger \triangle T$ establishes.

In the following theorem, we provide a condition that $\square$ and $\triangle$ are equivalent.

Theorem 2.1. *Suppose that $T, S \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ with closed ranges such that $S^\dagger ST^\dagger S = S^\dagger ST^*(S^\dagger)^*$ then $S\square T$ iff $S\triangle T$.*

Proof. Since T and S have closed ranges. Then T and S have the following matrices decomposition with respect to the orthogonal decompositions of submodules $\mathcal{X} = \mathrm{ran}(S^*) \oplus \ker(S)$ and $\mathcal{Y} = \mathrm{ran}(T) \oplus \ker(T^*)$

$$S = \begin{bmatrix} S_1 & 0 \\ S_2 & 0 \end{bmatrix} : \begin{bmatrix} \mathrm{ran}(S^*) \\ \ker(S) \end{bmatrix} \to \begin{bmatrix} \mathrm{ran}(T) \\ \ker(T^*) \end{bmatrix}, \tag{9}$$

$$T = \begin{bmatrix} T_1 & T_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathrm{ran}(S^*) \\ \ker(S) \end{bmatrix} \to \begin{bmatrix} \mathrm{ran}(T) \\ \ker(T^*) \end{bmatrix}. \tag{10}$$

Since $S^\dagger ST^\dagger S = S^\dagger ST^*(S^\dagger)^*$, by using Lemma 1.4, we obtain

$$\begin{aligned} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_1^*D^{-1} & 0 \\ T_2^*D^{-1} & 0 \end{bmatrix} \begin{bmatrix} S_1 & 0 \\ S_2 & 0 \end{bmatrix} &= \left(\begin{bmatrix} E^{-1}S_1^* & E^{-1}S_2^* \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_1 & T_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right)^* \\ \begin{bmatrix} T_1^*D^{-1}S_1 & 0 \\ 0 & 0 \end{bmatrix} &= \begin{bmatrix} T_1^*S_1E^{-1} & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

Hence,

$$T_1^*D^{-1}S_1 = T_1^*S_1E^{-1}, \tag{11}$$

where $D = T_1T_1^* + T_2T_2^*$ and $E = S_1^*S_1 + S_2^*S_2$ are invertible.

($\Longrightarrow$) Since $S\square T$, by using Lemma 1.4, we have

$$ST^\dagger S = S, \tag{12}$$

$$\begin{bmatrix} S_1 & 0 \\ S_2 & 0 \end{bmatrix} \begin{bmatrix} T_1^*D^{-1} & 0 \\ T_2^*D^{-1} & 0 \end{bmatrix} \begin{bmatrix} S_1 & 0 \\ S_2 & 0 \end{bmatrix} = \begin{bmatrix} S_1 & 0 \\ S_2 & 0 \end{bmatrix}, \tag{13}$$

$$\begin{bmatrix} S_1T_1^*D^{-1}S_1 & 0 \\ S_2T_1^*D^{-1}S_1 & 0 \end{bmatrix} = \begin{bmatrix} S_1 & 0 \\ S_2 & 0 \end{bmatrix}, \tag{14}$$

then

$$S_1T_1^*D^{-1}S_1 = S_1, \tag{15}$$

$$S_2T_1^*D^{-1}S_1 = S_2. \tag{16}$$

By multiplication S_1^* and S_2^* on the left of equations (15) and (16) respectively, we get

$$S_1^*S_1T_1^*D^{-1}S_1 = S_1^*S_1, \tag{17}$$

$$S_2^*S_2T_1^*D^{-1}S_1 = S_2^*S_2. \tag{18}$$

By additive obtained equations, it follows that $(S_1^*S_1 + S_2^*S_2)T_1^*D^{-1}S_1 = S_1^*S_1 + S_2^*S_2$. From invertibility of $E = S_1^*S_1 + S_2^*S_2$ we conclude that

$$T_1^*D^{-1}S_1 = 1. \tag{19}$$

Now, by combination (19) and (11) we have $T_1^*S_1E^{-1} = 1$ iff $T_1^*S_1 = E$, that is $T_1^*S_1$ is self adjoint $T_1^*S_1 = S_1^*T_1 = E$, with rewrite matrix forms, we have

$$S^\dagger ST^*S = \begin{bmatrix} T_1^*S_1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} = S^*S, \tag{20}$$

and

$$S^*TS^\dagger S = \begin{bmatrix} S_1^*T_1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} = S^*S. \tag{21}$$

By multiplication S on the left of equations (20) we get

$$ST^*S = SS^*S \Leftrightarrow S\triangle T. \tag{22}$$

($\Longleftarrow$) Since $S\triangle T$, then by matrix forms T and S in (9) and (10) we have

$$\begin{bmatrix} S_1 & 0 \\ S_2 & 0 \end{bmatrix}\begin{bmatrix} S_1^* & S_2^* \\ 0 & 0 \end{bmatrix}\begin{bmatrix} S_1 & 0 \\ S_2 & 0 \end{bmatrix} = \begin{bmatrix} S_1 & 0 \\ S_2 & 0 \end{bmatrix}\begin{bmatrix} T_1^* & 0 \\ T_2^* & 0 \end{bmatrix}\begin{bmatrix} S_1 & 0 \\ S_2 & 0 \end{bmatrix},$$
$$\begin{bmatrix} S_1E & 0 \\ S_2E & 0 \end{bmatrix} = \begin{bmatrix} S_1T_1^*S_1 & 0 \\ S_2T_1^*S_1 & 0 \end{bmatrix}.$$

Therefore,

$$S_1E = S_1T_1^*S_1, \tag{23}$$
$$S_2E = S_2T_1^*S_1. \tag{24}$$

By multiplication S_1^* and S_2^* on the left of equations (23) and (24), respectively, and additive obtained equations, it follows that

$$(S_1^*S_1 + S_2^*S_2)E = (S_1^*S_1 + S_2^*S_2)T_1^*S_1 \tag{25}$$
$$\Longrightarrow E = T_1^*S_1 \tag{26}$$

Eq. (26) implies that $T_1^*S_1E^{-1} = 1$. Since Eq. (11) holds. Then Eqs. (15) and (16) satisfy, therefore we conclude Eq. (14) holds, that is $S\square T$. □

In the following theorem, we provide conditions that adjoint plays role Moore-Penrose inverse in reverse order law.

Theorem 2.2. *Suppose that $T, S \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ with closed ranges, $T\square S$ and $S\square T$. Then $TS^\dagger = (ST^\dagger)^*$ and $ST^\dagger = (TS^\dagger)^*$*

Proof. (*i*) Since $ST^\dagger S = S$. Hence we have $ST^\dagger SS^\dagger = SS^\dagger$, then $SS^\dagger = (ST^\dagger SS^\dagger)^* = SS^\dagger(T^\dagger)^*S^*$ and thus $S^\dagger = S^\dagger(T^\dagger)^*S^*$

$$\begin{aligned} TS^\dagger &= TS^\dagger(T^\dagger)^*S^* = TS^\dagger TT^\dagger(T^\dagger)^*S^* \\ &= (TS^\dagger T)T^\dagger(T^\dagger)^*S^* = TT^\dagger(T^\dagger)^*S^* = (T^\dagger)^*S^* = (ST^\dagger)^*. \end{aligned}$$

Therefore $TS^\dagger = (ST^\dagger)^*$.

Similarly, since $TS^\dagger T = T$, then $T^\dagger = T^\dagger(S^\dagger)^*T^*$

$$\begin{aligned} ST^\dagger &= ST^\dagger(S^*)^\dagger T^* = ST^\dagger SS^\dagger(S^*)^\dagger T^* = (ST^\dagger S)S^\dagger(S^*)^\dagger T^* \\ &= SS^\dagger(S^*)^\dagger T^* = (S^*)^\dagger T^* = (TS^\dagger)^*. \end{aligned}$$

□

Theorem 2.3. *Suppose that $T, S \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ have closed ranges, $T\square S$ and $ST^\dagger = (TS^\dagger)^*$. Then $T^\dagger\square S^\dagger$.*

Proof. (*i*) Since $TS^\dagger T = T$. Hence we have $TS^\dagger TT^\dagger = TT^\dagger$. Taking adjoint, we get $TT^\dagger(TS^\dagger)^* = TT^\dagger$. Since $ST^\dagger = (TS^\dagger)^*$, we have $TT^\dagger ST^\dagger = TT^\dagger$. Now pre multiply by $T^\dagger$, we get $T^\dagger\square S^\dagger$. □

Theorem 2.4. *Suppose that $T, S \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and S has closed range. If $T\square S$, then $(TS^\dagger)^\dagger \triangle TS^\dagger$ and $(S^\dagger T)^\dagger \triangle S^\dagger T$.*

Proof. Notice that, since $TS^\dagger T = T$ holds then $TS^\dagger$ and $S^\dagger T$ are idempotent. Therefore [4, Corollary 3.3] implies that $TS^\dagger$ and $S^\dagger T$ have closed ranges.

Multiplying $TS^\dagger T = T$ by $S^\dagger$ on the right, we get $TS^\dagger TS^\dagger = TS^\dagger$ that is, $TS^\dagger$ is idempotent. Hence, condition (v) of Theorem 1.5 holds, and by (v)⇒(vii) of that lemma, we conclude that $TS^\dagger((TS^\dagger)^\dagger)^* = TS^\dagger$. Taking adjoint, we have $(TS^\dagger)^\dagger(TS^\dagger)^* = (TS^\dagger)^\dagger$. Multiplying $(TS^\dagger)^\dagger(TS^\dagger)^* = (TS^\dagger)^\dagger$ by $(TS^\dagger)^\dagger$ on the right, we get

$$(TS^\dagger)^\dagger(TS^\dagger)^*(TS^\dagger)^\dagger = (TS^\dagger)^\dagger(TS^\dagger)^\dagger = ((TS^\dagger)^\dagger)^2. \tag{27}$$

On the other hand, since $TS^\dagger$ is idempotent. By applying Theorem 1.5 part (v)⇒(ii), implies that

$$((TS^\dagger)^\dagger)^2 = (TS^\dagger)^\dagger((TS^\dagger)^\dagger)^*(TS^\dagger)^\dagger. \tag{28}$$

Equations (27) and (28) show that

$$(TS^\dagger)^\dagger((TS^\dagger)^\dagger)^*(TS^\dagger)^\dagger = (TS^\dagger)^\dagger(TS^\dagger)^*(TS^\dagger)^\dagger.$$

That is $(TS^\dagger)^\dagger \vartriangle TS^\dagger$.

Also, multiplying $TS^\dagger T = T$ by $S^\dagger$ on the left, we get $S^\dagger TS^\dagger T = S^\dagger T$ that is, $S^\dagger T$ is idempotent. Replacing $TS^\dagger$ by $S^\dagger T$ in the above argument implies that $(S^\dagger T)^\dagger \vartriangle S^\dagger T$. □

In the following theorem states a condition that $T^\dagger \vartriangle T$ establishes.

Theorem 2.5. *Suppose that $T \in \mathcal{L}(\mathcal{X})$ is idempotent, then $T^\dagger \vartriangle T$.*

Proof. Since T is an idempotent operator, [4, Corollary 3.3] implies that T has closed range.

Since T is idempotent, Theorem 1.5 part (v)⇒(ii), implies that

$$(T^\dagger)^2 = T^\dagger(T^\dagger)^*T^\dagger. \tag{29}$$

Again, condition (v) of Theorem 1.5 holds, and by (v)⇒(vii) of that lemma, we conclude that $T(T^\dagger)^* = (T^\dagger)^*$. Taking adjoint, we get $T^\dagger T^* = T^\dagger$. Post multiply by $T^\dagger$, we have

$$T^\dagger T^* T^\dagger = (T^\dagger)^2. \tag{30}$$

Thus equation (29) and (30) yields $T^\dagger \vartriangle T$ □

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