Porosity and the weighted $L^p$–spaces

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Abstract. Let $G$ be a locally compact group, $\omega$ be a weight function on $G$ and $1 < p, q < \infty$. Recently, it has been introduced some important $\sigma-c$–lower porous subsets $L^p(G) \times L^q(G)$, where $1/p + 1/q < 1$. Using these achievements, almost all available results connected to the existence of convolution of two functions belonging to $L^q(G)$ are obtained. In the present work these results will be extended for the weighted case. In fact for $2 < p < \infty$, some important $\sigma-c$–lower porous subsets $L^p(G, \omega) \times L^p(G, \omega)$ are introduced.

1. Introduction and Preliminaries

Throughout the paper, let $G$ be a locally compact group with a fixed left Haar measure $\lambda$ and $\omega : G \to (0, \infty)$ be a weight function on $G$; that is, a Borel measurable function on $G$. The weight function $\omega$ is called submultiplicative if

$$\omega(xy) \leq \omega(x)\omega(y),$$

for all $x, y \in G$. We say $\omega$ is of moderate growth if

$$\text{ess sup}_{y \in G} \frac{\omega(xy)}{\omega(y)} < \infty,$$

(1.1)

for all $x \in G$. For $1 \leq p < \infty$, the space $L^p(G, \omega)$ with respect to $\lambda$ is the set of all complex valued measurable functions $f$ on $G$ such that $f \omega \in L^p(G)$, as defined in [10]. Let us remark that

$$\|f\|_{L^p(G, \omega)} := \left(\int_G |f(x)|^p \omega(x)^p d\lambda(x)\right)^{1/p} \quad (f \in L^p(G, \omega)),$$

defines a norm on $L^p(G, \omega)$ under which it a Banach space. We denote this space by $\ell^p(G, \omega)$ when $G$ is discrete.

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For measurable functions $f$ and $g$ on $G$, the convolution

$$(f * g)(x) = \int_G f(y) g(y^{-1}x) \, d\lambda(y)$$

is defined at each point $x \in G$ for which the function $y \mapsto f(y) g(y^{-1}x)$ is $\lambda$-integrable. If $f * g(x)$ is defined and finite for $\lambda$-almost all $x \in G$, then we say that the convolution of functions $f * g$ exists as a function. The convolution $f * g$ does not necessarily exist for all measurable functions $f$ and $g$. So, it would be interesting to know does $f * g$ exist for all functions $f$ and $g$ in a space $X$ of measurable functions on $G$. If this is the case, then it is desirable to study the closedness of $X$ under the convolution. It is well-known that $L^1(G)$ is always closed under the convolution. Saeki [13] proved that, for $1 < p < \infty$, the space $L^p(G)$ is closed under the convolution if and only if $G$ is compact. But the convolution of elements in $L^p(G)$ even does not exist in general. Several authors have been studied the existence of convolution on certain function spaces. In fact on a locally compact non-compact group $G$, the space $L^p(G)$ for $2 < p < \infty$, contains functions $f$ and $g$ whose convolution is infinite on a set of positive measure; see [12] and also [1].

In a recent work, this result was strengthened by Glab and Strobin [8] and they proved a quantitative version of this result. Indeed, for $1 < p, q < \infty$, they considered the product $L^p(G) \times L^q(G)$ as a Banach space with a norm defined as the maximum of norms of coordinates, that is

$$\|(f, g)\| = \max\{|f|_p, |g|_q\}.$$ 

As a main result, they showed that if $1/p + 1/q < 1$ then for each compact subset $K$ of locally compact non-compact group $G$, the set $E_K$ of pairs $(f, g) \in L^p(G) \times L^q(G)$ for which $f * g$ is well-defined at some point of $K$ (i.e., $f * g(x)$ is finite or equal to $\infty$ or $-\infty$), satisfies a porosity condition as the following: every ball about a point of $E_K$ contains balls that are disjoint from it. Such sets are nowhere dense and thus if $2 < p < \infty$ and $G$ is $\sigma$–compact then the pairs of functions whose convolution is nowhere defined is residual in $L^p(G) \times L^q(G)$. Then [1, Theorem 1.1] can immediately be obtained by using this result. See also [5], as a work in this direction.

Before stating the aim of this paper, we will briefly describe some notions of porosity from [14]. Let $X$ be a metric space. For positive number $R$, the open ball with a radius $R$ centered at a point $x \in X$ will be denoted by $B(x; R)$. Let $c \in (0, 1]$. We say that $M \subseteq X$ is $c$–lower porous if for each $x \in M$,

$$\liminf_{R \to 0} \frac{\gamma(x, M, R)}{R} \leq \frac{c}{2},$$

where

$$\gamma(x, M, R) = \sup\{r \geq 0 : \exists x \in X, B(z, r) \subseteq B(x; R) \setminus M\}.$$ 

Equivalently, $M$ is $c$–lower porous if and only if for each $x \in M$ and $0 < \beta < c/2$, there exits $R_0 > 0$ and $z \in X$ such that for each $0 < R < R_0$,

$$B(z, \beta R) \subseteq B(x; R) \setminus M;$$

see [14]. If $M$ is a countable union of $c$–lower porous sets then we say that $M$ is $\sigma$–c–lower porous. It can be easily seen that the $c$–lower porosity implies the nowhere density and hence the $\sigma$–lower porosity implies the meagerness. Thus if $X$ is a complete space, then $\sigma$–porous sets are small subsets of $X$.

It has been done a lot of works connected to the $L^p$–spaces so far. See for example the first authors works such as [1], [2], [3], [4], [5] and also the work due to Hao, Guo, Li Rong-lu and Wu Jun-de [9], that for Banach spaces $X$ and $Y$, it is characterized matrix transformations of $\ell^p(X)$ to $\ell^p(Y)$. Furthermore, most of the researches in the topics related to the Lebesgue spaces, have been generalized to the weighted case; see Kuznetsova’s works such as [11], that is relevant to the present subject.

Recently, we investigated the existence of $f * g$ as a function for every two function $f, g \in L^p(G, \omega)$, where $\omega$ is a submultiplicative weight function on $G$, and obtained some necessary or sufficient conditions for that the property holds [3]. Mainly, we proved that the $\sigma$–compactness of $G$ is a necessary condition for the
existence of \( f \ast g \), when \( f, g \) run into \( L^p(G, \omega) \). Our purpose of the present work is to consider the Banach space \( L^p(G, \omega) \times L^p(G, \omega) \), for \( 2 < p < \infty \), under the norm

\[
\| (f, g) \| = \max\{\| f \|_{p, \omega}, \| g \|_{p, \omega}\}.
\]

We then mix some ideas and techniques from [8] and also our paper [3] to prove the following result, as a generalization of Theorems [3, Theorem 2.2] and also [8, Theorem 1.1] whenever \( p = q \). In our debate, we use the notation \( L^p_\omega \times L^p_\omega \) rather than \( L^p(G, \omega) \times L^p(G, \omega) \), in convenience. Let us recall a symmetric weight function \( \omega \) as \( \omega(x) = \bar{\omega}(x) = \omega(x^{-1}) \), for all \( x \in G \). Every symmetric and submultiplicative weight function \( \omega \) is bounded below by the constant 1, obviously. We state here the main theorem of the present work.

**Theorem 1.1.** Let \( G \) be a locally compact non-\( \sigma \)-compact group, \( 2 < p < \infty \) and \( \omega \) be a symmetric and submultiplicative weight function on \( G \). Then for every compact subset \( K \subseteq G \), the set

\[
E^\omega_K = \{ (f, g) \in L^p_\omega \times L^p_\omega : \exists x \in K \ | f | \ast | g |(x) < \infty \}
\]

is \( \sigma - c \)-lower porous for some \( c > 0 \).

### 2. THE PROOF OF THEOREM 1.1

The proof can be done by slightly modified techniques and methods used in [8, Theorem 1.1], [5, Theorem 2.4] and also [1, Theorem 1.1], as follows. It is easy to see that we can assume \( K \) is a compact symmetric neighborhood of the identity element of \( G \) with \( \lambda(K) > 0 \). Since \( E^\omega_K = \bigcup_{n=1}^{\infty} E^\omega_n \), where

\[
E^\omega_n = \{ (f, g) \in L^p_\omega \times L^p_\omega : \exists x \in K, \ | f | \ast | g |(x) < n \},
\]

thus we only have to show that for each \( n \in \mathbb{N} \), the set \( E^\omega_n \) is \( c \)-lower porous for some \( c > 0 \). Suppose that \( n \in \mathbb{N} \) and \( (f, g) \in E^\omega_n \) and let

\[
S = \sup_{x \in K} \Delta(x). \tag{2.1}
\]

Submultiplicativity of \( \omega \) implies that it is bounded and bounded away from zero on every compact subset of \( G \) [6, Proposition 1.16]. It follows that there is the number \( M \geq 1 \) such that for each \( x \in K \), \( \omega(x) \leq M \). Since \( G = \bigcup_{m=1}^{\infty} \omega^{-1}([1, m]) \) together with the fact that \( G \) is not \( \sigma \)-compact, thus there is \( m \in \mathbb{N} \) such that \( \omega^{-1}([1, m]) \) does not contain in any compact subsets of \( G \). Take \( d \in (0, 1) \) with

\[
\left( \frac{d}{1-d} \right)^p + \left( \frac{d}{1-d} \right)^p \frac{\lambda(K^2)}{\lambda(K)} = 1
\]

and let

\[
c = \frac{d}{mM^2}.
\]

For each \( 0 < \delta < c \) we have \( 0 < \delta mM^2 < d < 1 \). It follows that

\[
\left( \frac{\delta mM^2}{1-\delta mM^2} \right)^p + \left( \frac{\delta mM^2}{1-\delta mM^2} \right)^p \frac{\lambda(K^2)}{\lambda(K)} < 1.
\]

Continuity of the function \( \theta \) defined by

\[
\theta(x) = \left( \frac{\delta mM^2}{x} \right)^p \left( 1 + \frac{\lambda(K^2)}{\lambda(K)} \right)
\]

on \((0, 1)\), together with the fact that \( \theta(1-\delta mM^2) < 1 \) yield that there is \( 0 < t < 1-\delta mM^2 \) such that \( \theta(t) < 1 \). Choosing \( t < \eta < 1-\delta \) and \( D \in (0, 1) \) such that \( t = \eta(1-D) \) we obtain

\[
P = 1 - \theta(\eta(1-D)) = 1 - \left( \frac{\delta mM^2}{\eta(1-D)} \right)^p - \left( \frac{\delta mM^2}{\eta(1-D)} \right)^p \frac{\lambda(K^2)}{\lambda(K)} > 0. \tag{2.2}
\]
We repeat the argument given in [1] to obtain the sequence \((a_k)\) of \(\omega^{-1}([1, m])\) such that for all \(l, k \in \mathbb{N}\) with \(l \neq k\)
\[
a_lK^2 \cap a_kK^2 = \emptyset, \quad Ka_l^{-1} \cap Ka_k^{-1} = \emptyset \quad \text{and} \quad \Delta(a_k) \leq 1. \tag{2.3}
\]
Indeed, take \(a_1 \in \omega^{-1}([1, m])\) with \(\Delta(a_1) \leq 1\), by the symmetricity of \(\omega\). Assume that we have already defined \(a_1, \cdots, a_k\). Since \(\omega\) is symmetric and \(\omega^{-1}([1, m])\) is not contained in any compact subsets of \(G\), so we can take
\[
a_{k+1} \in \omega^{-1}([1, m]) \setminus \bigcup_{i=1}^k (a_iK^4 \cup K^4a_i^{-1})
\]
such that \(\Delta(a_{k+1}) \leq 1\). Not that for each \(x \in \bigcup_{k=0}^\infty a_kK^2 \cup Ka_k^{-1}\) we have
\[
1 \leq \omega(x) \leq mM^2. \tag{2.4}
\]
Given \(R > 0\) and set \(Q = \frac{R}{mM^2}\), since \(f, g \in L'(G, \omega)\), there is \(n_0 \in \mathbb{N}\) such that
\[
\left(\int_{\bigcup_{k=n_0}^\infty Ka_k^{-1}} |f(x)|^p \omega(x)^p \right)^{1/p} \leq (1 - \delta - \eta)R \tag{2.5}
\]
and
\[
\left(\int_{\bigcup_{k=n_0}^\infty a_kK^2} |g(x)|^p \omega(x)^p \right)^{1/p} \leq (1 - \delta - \eta)R. \tag{2.6}
\]
Choose \(n_1 > n_0\) such that
\[
(\lambda(K)(n_1 - n_0 + 1))^{1-2/p} > n \left(D^2\eta^2Q^2S^{1/p-1} \left(\frac{\lambda(K)}{\lambda(K^2)}\right)^{1/p} \right)^{-1} \tag{2.7}
\]
and let \(A = \bigcup_{k=n_0}^{n_1} Ka_k^{-1}\) and \(B = \bigcup_{k=n_0}^{n_1} a_kK^2\). Then
\[
\lambda(A^{-1}) = (n_1 - n_0 + 1)\lambda(K) \quad \text{and} \quad \lambda(B) = (n_1 - n_0 + 1)\lambda(K^2) \tag{2.8}
\]
and so
\[
\frac{\lambda(B)}{\lambda(A^{-1})} = \frac{\lambda(K^2)}{\lambda(K)}. \tag{2.9}
\]
So by (2.7) and (2.8) we obtain
\[
\lambda(A^{-1})^{1-2/p} > n \left(D^2\eta^2Q^2S^{1/p-1} \left(\frac{\lambda(K)}{\lambda(K^2)}\right)^{1/p} \right)^{-1}. \tag{2.10}
\]
Take the positive numbers \(M_1\) and \(M_2\) such that
\[
M_1\lambda(A^{-1})^{1/p} = \eta Q \quad \text{and} \quad M_2\lambda(B)^{1/p} = \eta Q. \tag{2.11}
\]
Next we define the functions \(\tilde{f}\) and \(\tilde{g}\) the same functions in [8, Theorem 1.1]; i.e. for each \(x \in A\), set \(\tilde{f}(x) = M_1\Delta(x^{-1})^{1/p}\) and \(\tilde{f}(x) = f(x)\) otherwise. Also for each \(x \in B\), set \(\tilde{g}(x) = M_2\) and \(\tilde{g}(x) = g(x)\) otherwise.
We use (2.4), (2.5) and (2.11) to obtain
\[
\|f - \tilde{f}\|_{p, \omega} \leq \eta R + (1 - \delta - \eta)R = (1 - \delta)R.
\]
Similarly by (2.4), (2.6) and (2.11) we have
\[ \|g - \bar{g}\|_{p,\omega} \leq (1 - \delta)R. \]
It follows that
\[ B((\bar{f}, \bar{g}), \delta R) \subseteq B((f, g), R). \]
We should prove that \( B((\bar{f}, \bar{g}), \delta R) \cap E^\omega = \emptyset \). To that end, take \((h, s) \in B((\bar{f}, \bar{g}), \delta R)\) and let
\[ A_1 = \{ x \in A : |h(x)| \geq DM_1 \Delta(x^{-1})^{1/p} \} \quad \text{and} \quad A_2 = A \setminus A_1. \]
Also
\[ B_1 = \{ x \in B : |s(x)| \geq DM_2 \} \quad \text{and} \quad B_2 = B \setminus B_1. \]
Then by (2.4) we have
\[
\delta R \geq \|h - \bar{f}\|_{p,\omega} \geq \left( \int_{A_2} |h(x) - M_1 \Delta(x^{-1})^{1/p} \omega(x)\alpha \lambda(x)|^{1/p} \right) \geq \left( \int_{A_2} |M_1 \Delta(x^{-1})^{1/p} (1 - D) \omega(x)\alpha \lambda(x)|^{1/p} \right) \geq M_1 (1 - D) \lambda(A_2^{-1})^{1/p}
\]
and (2.11) implies that
\[
\lambda(A_2^{-1}) \leq \lambda(A^{-1}) \left( \frac{\delta M^2}{\eta(1 - D)} \right)^p. \tag{2.12}
\]
Similarly
\[
\lambda(B_2) \leq \lambda(A^{-1}) \left( \frac{\lambda(K)^2}{\lambda(K)} \right) \left( \frac{\delta m M^2}{\eta(1 - D)} \right)^p. \tag{2.13}
\]
Now let \( y_0 \in K \) and put \( F = (A_1^{-1} y_0) \cap B_1 \) and \( H = y_0 F^{-1} \). Clearly \( A_1^{-1} y_0 \subseteq B \), and thus \( A_1^{-1} y_0 \subseteq B \). The implications (2.12) and (2.13) yield that
\[
\lambda(H^{-1}) = \lambda(Fy_0^{-1}) = \lambda(A_1^{-1}) - \lambda((B_1 y_0^{-1}) \setminus (B_1 y_0^{-1})) \geq \lambda(A_1^{-1}) - \lambda((B_1 y_0^{-1}) \setminus (B_1 y_0^{-1})) = \lambda(A_1^{-1}) - \lambda(B_2 y_0^{-1}) = \lambda(A^{-1}) - \lambda(A_2^{-1}) - \lambda(y_0^{-1}) \lambda(B_2) \geq \lambda(A^{-1})^p,
\]
where the last inequality is obtained by (2.2), (2.12) and (2.13). Also \( H \subseteq A_1, F \subseteq B_1 \) and \( H^{-1} y_0 = F \). Furthermore, if \( y \in A \) then \( y = xa_n^{-1} \) for some \( x \in K \) and \( n \in \mathbb{N} \) with \( n_0 \leq n \leq n_1 \). Thus by (2.1) and (2.3),
\[
\Delta(y^{-1}) = \Delta(a_n) \Delta(x^{-1}) \leq S.
\]
Finally we obtain by (2.8), (2.9), (2.10) and (2.11)

\[
\int_{H} |h(y)||s(y^{-1}y_{0})|d\lambda(y) \geq D^{2}M_{1}M_{2} \int_{H} \Delta(y^{-1})^{1/p}d\lambda(y) \\
= D^{2}M_{1}M_{2} \int_{H} \Delta(y^{-1})^{1/p-1}\Delta(y^{-1})d\lambda(y) \\
\geq D^{2}M_{1}M_{2} S^{1/p-1} \Delta(y^{-1})d\lambda(y) \\
= D^{2}M_{1}M_{2}S^{1/p-1} \int_{H} \Delta(y^{-1})d\lambda(y) \\
\geq D^{2}M_{1}M_{2}S^{1/p-1} \lambda(H^{-1})P \\
= D^{2}M_{1}M_{2}S^{1/p-1} \lambda(A^{-1})P \\
> n.
\]

It follows that $|h| \cdot |s(y_{0})| > n$, for each $y_{0} \in K$ and so $(h,s) \notin E_{n}^{\ast}$, as claimed. \qed

**Remarks 2.1.** Let $G$ be a locally compact group, $\omega$ be a submultiplicative weight function on $G$ and $2 < p < \infty$.

(i) As we pointed out in the explanations before the theorem, if $\omega$ is symmetric and submultiplicative then $\omega(x) \geq 1$, for each $x \in G$. Thus $L^{p}(G,\omega) \subseteq L^{p}(G)$. It follows that Theorem 1.1 in the present paper is equivalent to [8, Corollary 2] and with some modifications in the proofs, each of them can be obtained independently from the other one.

(ii) The assumption of non $\sigma$-compactness of $G$ can not be replaced by the non compactness of $G$, whenever $\omega$ is not necessarily a constant function. Indeed, if $f \ast g$ exists as a function for all $f, g \in L^{p}(G,\omega)$, then $G$ is $\sigma$-compact and not necessarily compact. Namely take $G = \mathbb{Z}$, the additive group of integer numbers and define the weight function $\omega$ on $\mathbb{Z}$ as the following

\[
\omega(n) = (1 + |n|) \quad (n \in \mathbb{Z}).
\]

Since $1/\omega \in \ell^{1}(\mathbb{Z})$, it follows that $f \ast g$ exists as a function for all $f, g \in \ell^{p}(\mathbb{Z},\omega)$ [3, Proposition 2.7]. Whereas $G$ is a $\sigma$-compact non-compact group.

Since $L^{p}(G,\omega) \times L^{p}(G,\omega)$ is a Banach space, the main results given in [3] are immediately obtained, with observing the proof of Theorem 1.1.

**Corollary 2.2.** [3, Theorem 2.2] Let $G$ be a locally compact group, $\omega$ be a symmetric and submultiplicative weight function on $G$ and $2 < p < \infty$. If $f \ast g$ exists for all $f, g \in L^{p}(G,\omega)$, then $\omega^{-1}(F)$ is contained in a compact subset of $G$, for all compact subsets $F$ of $[1, \infty)$,

**Corollary 2.3.** [3, Corollary 2.3] Let $G$ be a locally compact group, $\omega$ be a symmetric and submultiplicative weight function on $G$ and $2 < p < \infty$. If $f \ast g$ exists for all $f, g \in L^{p}(G,\omega)$, then $\omega^{-1}([1, m])$ is contained in a compact subset of $G$, for all $m \in \mathbb{N}$.

Recall from [3] that for a submultiplicative weight function $\omega$, the weight function $\omega^{\ast} = \omega \omega$ is a symmetric and submultiplicative weight function on $G$. Now by using [3, Lemma 2.1] together with Corollary 2.3, the following result is provided.

**Corollary 2.4.** [3, Theorem 2.5] Let $G$ be a locally compact group, $\omega$ be a submultiplicative weight function on $G$ and $2 < p < \infty$. If $f \ast g$ exists for all $f, g \in L^{p}(G,\omega)$, then $G$ is $\sigma$-compact.
Note 2.5. Note that in the proof of [3, Theorem 2.5], it was used [3, Corollary 2.3] that the continuity of \( \omega \) is assumed. But the theorem has been based for an arbitrary submultiplicative weight function. The reason is that every submultiplicative weight function \( \omega \) is equivalent to a continuous weight function \( \alpha \) [7, Theorem 2.7]; i.e. there are the positive constants \( C_1 \) and \( C_2 \) such that
\[
C_1 \leq \frac{\omega(x)}{\alpha(x)} \leq C_2,
\]
locally almost everywhere on \( G \). It follows that \( L^p(G, \omega) = L^p(G, \alpha) \) and so the result is concluded by replacing \( \alpha \) instead of \( \omega \); i.e.
\[
G = \bigcup_{m=1}^{\infty} (\alpha^*)^{-1}([1, m]).
\]
Although, one can get the result without considering this point. Indeed, by [3, Theorem 2.2] for every \( m \in \mathbb{N} \) \((\alpha^*)^{-1}([1, m])\) is contained in a compact subset of \( G \) and since \( G = \bigcup_{m=1}^{\infty} (\alpha^*)^{-1}([1, m]) \) it follows that \( G \) is \( \sigma \)-compact.

Remarks 2.6. Let \( G \) be a locally compact group, \( \omega \) be a weight function on \( G \) and \( 0 < p < \infty \). Then

1. For \( 0 < p < 1 \), with the arguments given in [15], we showed that \( f * g \) exists as a function for all \( f, g \in L^p(G) \) if and only if \( G \) is discrete [2]. Also recently we considered the complete metric space \( L^p(G, \omega) \) and showed that if \( \omega \) is of moderate growth then \( L^p(G, \omega) \) is closed under convolution if and only if \( G \) is discrete and \( \omega \) is quasi submultiplicative, that is with some constant \( C > 0 \),
\[
\omega(xy) \leq C\omega(x)\omega(y),
\]
for all \( x, y \in G \). Moreover, in the class of submultiplicative weight functions, we proved that \( f * g \) exists as a function for all \( f, g \in L^p(G, \omega) \) if and only if \( G \) is discrete [3].

2. As a known result, \( f * g \) exists as a function where \( f, g \in L^1(G) \). In the weighted case, \( L^1(G, \omega) \) is closed under convolution just when \( \omega \) is equivalent to a submultiplicative weight function [11, Theorem 3.1]. It follows that for such a weight function, \( f * g \) exists as a function for all \( f, g \in L^1(G, \omega) \). Note that submultiplicativity of \( \omega \) is not a necessary condition for the existence of convolution of every two functions belonging to \( L^1(G, \omega) \). For example for every bounded below weight function \( \omega \), \( f * g \) exists as a function for all \( f, g \in L^1(G, \omega) \).

3. For \( 2 < p < \infty \), \( f * g \) exists as a function for all \( f, g \in L^p(G, \omega) \) if and only if \( G \) is compact. This subject first was considered by Richert [12]. In fact it was shown that if \( G \) is not compact, then for every compact, symmetric neighborhood \( K \) of the identity element of \( G \), there exist functions \( f, g \in L^p(G) \) such that \( f * g(x) = \infty \), for each \( x \in K \); see also [1].

4. If \( \omega \) is submultiplicative, then the existence of \( f * g \) as a function for all \( f, g \in L^p(G, \omega) \), where \( 2 < p < \infty \), implies that \( G \) is \( \sigma \)-compact [3, Theorem 2.5]. Note that \( \sigma \)-compactness of \( G \) is not in general a necessary condition whenever \( \omega \) is not submultiplicative or \( 1 < p \leq 2 \); see [3, Remark 2.6] and also [3, Proposition 2.7].

We end this work with the following example which describes Remark 2.6 for discrete groups.

Examples 2.7. Let \( G \) be a discrete group, \( \omega \) be a submultiplicative weight function on \( G \) and \( 0 < p < \infty \). Then

1. For \( 0 < p \leq 2 \), \( f * g \) exists as a function for all \( f, g \in L^p(G, \omega) \), by [4, Theorem 2.5], [11, Theorem 3.1] and [3, Remark 2.6(a)].

2. If \( 2 < p < \infty \), then [3, Proposition 2.9] implies that \( f * g \) as a function for all \( f, g \in L^p(G, \omega) \) if and only if \( L^p(G, \omega) \subseteq \ell^2(G) \), where \( \widehat{\omega}(x) = \omega(x^{-1}) \), for all \( x \in G \). (Note that there is a misprint in the reference [3, Proposition 2.9] and \( \ell^2(G) \) has been printed in stead of \( \ell^1(G) \).)
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