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Characterization of some mappings on rings and algebras

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Abstract. The main purpose of this paper is to present some functional equations characterizing (σ, τ)-derivations and $*$-derivations, where σ or τ is an automorphism. For instance, we prove the following theorem. Let $\mathcal{R}$ be an $(n-1)!$-torsion free semiprime ring and σ or τ be an automorphism of $\mathcal{R}$. Suppose that there exist additive mappings $\mathfrak{Q}, \mathfrak{L} : \mathcal{R} \to \mathcal{R}$ satisfying the relations

$$2\mathfrak{Q}(x^n) = \mathfrak{Q}(x^{n-1})\sigma(x) + \tau(x^{n-1})\mathfrak{L}(x) + \mathfrak{L}(x)\sigma(x^{n-1}) + \tau(x)\mathfrak{Q}(x^{n-1})$$

$$2\mathfrak{L}(x^n) = \mathfrak{L}(x^{n-1})\sigma(x) + \tau(x^{n-1})\mathfrak{Q}(x) + \mathfrak{Q}(x)\sigma(x^{n-1}) + \tau(x)\mathfrak{L}(x^{n-1})$$

for all $x \in \mathcal{R}$ and some integer $n > 1$. Then both $\mathfrak{L}$ and $\mathfrak{Q}$ are (σ, τ)-derivations and $\mathfrak{Q} = \mathfrak{L}$. In addition, we achieve a result concerning $*$-derivations as follows. Let $\mathcal{H}$ be a real or complex Hilbert space, $dim\mathcal{H} > 1$, and let $\mathfrak{A}$ be a standard operator algebra on $\mathcal{H}$. Let $n > 1$ be an integer and let $d : \mathfrak{A} \to B(\mathcal{H})$ be an additive mapping satisfying

$$d(A^n) = \sum_{j=1}^{n} A^{n-j} d(A) {A^*}^{j-1}$$

for all $A \in \mathfrak{A}$. Then there exists a unique linear operator $T \in B(\mathcal{H})$ such that $d(A) = AT - TA^*$ holds for all $A \in \mathfrak{A}$.

1. Introduction

Throughout this paper, $\mathcal{R}$ denotes an associative ring with the center $Z(\mathcal{R})$. The set of all invertible elements of a unital ring is denoted by $Inv(\mathcal{R})$. We first recall the basic notions that play a fundamental role in what follows. A ring $\mathcal{R}$ is said to be prime if for $a, b \in \mathcal{R}$, $a\mathcal{R}b = \{0\}$ implies that $a = 0$ or $b = 0$, and $\mathcal{R}$ is semiprime if for $a \in \mathcal{R}$, $a\mathcal{R}a = \{0\}$ implies that $a = 0$. A ring $\mathcal{R}$ is said to be n-torsion free, where $n > 1$ is an integer, if $nx = 0$ with $x \in \mathcal{R}$, implies $x = 0$. An additive mapping $d : \mathcal{R} \to \mathcal{R}$ is called a derivation if $d(xy) = d(x)y + xd(y)$ for all $x, y \in \mathcal{R}$ and is called a Jordan derivation if $d(x^2) = d(x)x + xd(x)$ is fulfilled for all $x \in \mathcal{R}$. Evidently, every derivation is a Jordan derivation, but the converse is in general not true (for instance,

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see [1]). A classical result of Herstein [9] asserts that any Jordan derivation on a 2-torsion free prime ring is a derivation. A brief proof of Herstein's result can be found in [4]. Cusack [8] generalized Herstein's result to 2-torsion free semiprime rings (see also [2] for an alternative proof). An additive mapping $d : \mathcal{R} \to \mathcal{R}$, where $\mathcal{R}$ is an arbitrary ring, is called a Jordan triple derivation if $d(xyx) = d(x)yx + xd(y)x + xyd(x)$ holds for all $x, y \in \mathcal{R}$.

M. Brešar [3] proved the following result regarding Relation (1), which is known as a Jordan triple derivation.

Theorem A. *Let $\mathcal{R}$ be a 2-torsion free semiprime ring and let $d : \mathcal{R} \to \mathcal{R}$ be an additive mapping satisfying the relation*

$$d(xyx) = d(x)yx + xd(y)x + xyd(x) \tag{1}$$

for all $x, y \in \mathcal{R}$. Then d is a derivation.

One can easily prove that any Jordan derivation d on an arbitrary 2-torsion free ring $\mathcal{R}$ satisfies Relation (1). It means that Theorem A generalizes Cusack's generalization of Herstein's result which we have mentioned above. Motivated by the above result, Vukman [19] showed that if $\mathcal{R}$ is a 2-torsion free semiprime ring and $d : \mathcal{R} \to \mathcal{R}$ is an additive mapping such that either $d(xyx) = d(xy)x + xyd(x)$ or $d(xyx) = d(x)yx + xd(yx)$ holds for all $x, y \in \mathcal{R}$, then d is a derivation. For more results and other techniques in this regard, see [16–18], and their references.

Let σ and τ be two endomorphisms of $\mathcal{R}$. An additive mapping $\mathfrak{L} : \mathcal{R} \to \mathcal{R}$ is said to be a (σ, τ)-derivation (resp. Jordan (σ, τ)-derivation) on $\mathcal{R}$ if $\mathfrak{L}(xy) = \mathfrak{L}(x)\sigma(y) + \tau(x)\mathfrak{L}(y)$ (resp. $\mathfrak{L}(x^2) = \mathfrak{L}(x)\sigma(x) + \tau(x)\mathfrak{L}(x)$) holds for all $x, y \in \mathcal{R}$. Moreover, an additive mapping $\mathfrak{Q} : \mathcal{R} \to \mathcal{R}$ is called a generalized (σ, τ)-derivation (resp. generalized Jordan (σ, τ)-derivation) on $\mathcal{R}$ if there exists a (σ, τ)-derivation (resp. Jordan (σ, τ)-derivation) $\mathfrak{L} : \mathcal{R} \to \mathcal{R}$ such that $\mathfrak{Q}(xy) = \mathfrak{Q}(x)\sigma(y) + \tau(x)\mathfrak{L}(y)$ (resp. $\mathfrak{Q}(x^2) = \mathfrak{Q}(x)\sigma(x) + \tau(x)\mathfrak{L}(x)$) for all $x, y \in \mathcal{R}$. Clearly, every generalized (σ, τ)-derivation is a generalized Jordan (σ, τ)-derivation, but the converse is in general not true (see [1, Example 3.1]). An additive mapping $\mathfrak{L} : \mathcal{R} \to \mathcal{R}$ is called a Jordan triple (σ, τ)-derivation if $\mathfrak{L}(xyx) = \mathfrak{L}(x)\sigma(yx) + \tau(x)\mathfrak{L}(y)\sigma(x) + \tau(xy)\mathfrak{L}(x)$ holds for all $x, y \in \mathcal{R}$. Indeed, every (σ, τ)-derivation is a Jordan triple (σ, τ)-derivation. Brešar and Vukman [5] proved that every Jordan (σ, τ)-derivation on a prime ring of characteristic different from two is a (σ, τ)-derivation, where σ and τ are automorphisms of $\mathcal{R}$. For these kind of results, we refer the reader to [5, 12, 13], where further references can be found. In 2007, Charles Lanski [12] achieved a very interesting generalization of the above result to 2-torsion free semiprime rings whenever one of σ or τ is assumed to be an automorphism. In view of the above results, it is our objective in this article to characterize (σ, τ)-derivations on semiprime rings using certain functional equations, where σ or τ is an automorphism. Some of our results in the current paper have been motivated by a work of Širovnik [14]. One of the main result of this article is as follows. Let $\mathcal{R}$ be an $(n-1)!$-torsion free semiprime ring and σ or τ be an automorphism of $\mathcal{R}$. Suppose that there exist additive mappings $\mathfrak{Q}, \mathfrak{L} : \mathcal{R} \to \mathcal{R}$ satisfying the relations

$$2\mathfrak{Q}(x^n) = \mathfrak{Q}(x^{n-1})\sigma(x) + \tau(x^{n-1})\mathfrak{L}(x) + \mathfrak{L}(x)\sigma(x^{n-1}) + \tau(x)\mathfrak{Q}(x^{n-1})$$

$$2\mathfrak{L}(x^n) = \mathfrak{L}(x^{n-1})\sigma(x) + \tau(x^{n-1})\mathfrak{Q}(x) + \mathfrak{Q}(x)\sigma(x^{n-1}) + \tau(x)\mathfrak{L}(x^{n-1})$$

for all $x \in \mathcal{R}$ and some integer $n > 1$. Then both $\mathfrak{L}$ and $\mathfrak{Q}$ are (σ, τ)-derivations and $\mathfrak{L} = \mathfrak{Q}$. We also give a characterization of (σ, τ)-derivations and generalized (σ, τ)-derivations as follows. Let $\mathcal{R}$ be an $(n+1)!$-torsion free semiprime ring and σ or τ be an automorphism of $\mathcal{R}$. Suppose that there exist additive mappings $\mathfrak{Q}, \mathfrak{L} : \mathcal{R} \to \mathcal{R}$ satisfying the relation

$$2\mathfrak{Q}(x^{n+1}) = \mathfrak{Q}(x)\sigma(x^n) + \tau(x)\mathfrak{L}(x^n) + \mathfrak{Q}(x^n)\sigma(x) + \tau(x^n)\mathfrak{L}(x)$$

for all $x \in \mathcal{R}$ and some integer $n > 1$. Then $\mathfrak{L}$ is a (σ, τ)-derivation and $\mathfrak{Q}$ is a generalized (σ, τ)-derivation. Another objective of this research is to characterize Jordan $*$-derivations. First, let us introduce some basic notions that play a fundamental role in this regard. An additive map $x \mapsto x^*$ of $\mathcal{R}$ into itself is called an involution if (i) $(xy)^* = y^*x^*$ and (ii) $(x^*)^* = x$ hold for all $x, y \in \mathcal{R}$. A ring equipped with an involution is known as a ring with involution or a $*$-ring. An element x in a ring with involution $*$ is said to be hermitian if $x^* = x$ and skew-hermitian if $x^* = -x$. Let $\mathcal{R}$ be a $*$-ring. An additive mapping $d : \mathcal{R} \to \mathcal{R}$ is called a $*$-derivation in case $d(xy) = d(x)y^* + xd(y)$ holds for all pairs $x, y \in \mathcal{R}$ and is called a Jordan $*$-derivation if $d(x^2) = d(x)x^* + xd(x)$ is fulfilled for all $x \in \mathcal{R}$. It is easy to prove that there are no nonzero $*$-derivations on noncommutative prime $*$-rings (see [6] for the details). Note that the mapping $x \mapsto xa - ax^*$, where $a \in \mathcal{R}$ is a fixed element, is a Jordan $*$-derivation; such Jordan $*$-derivations are said to be inner. The study of Jordan $*$-derivations has been motivated by the problem of the representability of quadratic forms by bilinear forms (for the results concerning this problem, we refer to [10, 11, 21]). In this work, we present a characterization of Jordan $*$-derivations as follows. Let $n > 1$ be an integer, $\mathcal{R}$ be an $n!$-torsion free $*$-ring and let $d : \mathcal{R} \to \mathcal{R}$ be an additive mapping satisfying

$$d(a^n) = \sum_{j=1}^{n} a^{n-j} d(a) {a^*}^{j-1}$$

for all $a \in \mathcal{R}$. Then d is a Jordan $*$-derivation on $\mathcal{R}$. As a consequence of this result, we show that if $\mathcal{A}$ is a commutative complex $*$-algebra such that $\mathcal{A}a = \{0\}$ or $a\mathcal{A} = \{0\}$ (where $a \in \mathcal{A}$) implies $a = 0$ and if $d : \mathcal{A} \to \mathcal{A}$ is an additive mapping satisfying

$$d(a^n) = \sum_{j=1}^{n} a^{n-j} d(a) {a^*}^{j-1}$$

for all $a \in \mathcal{A}$ and some integer $n > 1$, then d is a $*$-derivation on $\mathcal{A}$. Another result in this regard is as follows. Let $\mathcal{H}$ be a real or complex Hilbert space, $dim\mathcal{H} > 1$, and let $\mathfrak{A}$ be a standard operator algebra on $\mathcal{H}$. Let $n > 1$ be an integer and let $d : \mathfrak{A} \to B(\mathcal{H})$ be an additive mapping satisfying

$$d(A^n) = \sum_{j=1}^{n} A^{n-j} d(A) {A^*}^{j-1}$$

for all $A \in \mathfrak{A}$. Then there exists a unique linear operator $T \in B(\mathcal{H})$ such that $d(A) = AT - TA^*$ holds for all $A \in \mathfrak{A}$. Moreover, we present the following characterization of some linear mappings. Let $\mathcal{A}$ be a unital semiprime Banach algebra and let $f, d : \mathcal{A} \to \mathcal{A}$ be additive mappings satisfying

$$d(x) = -x^2 d(x^{-1}) \text{ and } f(x) = -x^2 f(x^{-1}) + x^3 (d(x^{-1}))^2$$

for all $x \in Inv(\mathcal{A})$, where $Inv(\mathcal{A})$ denotes the set of all invertible elements of $\mathcal{A}$. Then $f = \frac{1}{2}d^2 + D$, where D is a derivation mapping $\mathcal{A}$ into $Z(\mathcal{A})$. In addition, some other related results are presented.

2. Main Results

Throughout this section, without further mention, the rings are unital with the unity e and $\sigma, \tau : \mathcal{R} \to \mathcal{R}$ are assumed to be endomorphisms with $\sigma(e) = e = \tau(e)$. Our first result reads as follows.

Theorem 2.1. *Let $\mathcal{R}$ be an $(n-1)!$-torsion free semiprime ring and let $\mathfrak{L} : \mathcal{R} \to \mathcal{R}$ be an additive mapping satisfying*

$$2\mathfrak{L}(x^n) = \mathfrak{L}(x^{n-1})\sigma(x) + \tau(x^{n-1})\mathfrak{L}(x) + \mathfrak{L}(x)\sigma(x^{n-1}) + \tau(x)\mathfrak{L}(x^{n-1}),$$

for all $x \in \mathcal{R}$ and some integer $n > 1$, where σ or τ is an automorphism of $\mathcal{R}$. Then $\mathfrak{L}$ is a (σ, τ)-derivation.

Proof. We have the relation

$$2\mathfrak{L}(x^n) = \mathfrak{L}(x^{n-1})\sigma(x) + \tau(x^{n-1})\mathfrak{L}(x) + \mathfrak{L}(x)\sigma(x^{n-1}) + \tau(x)\mathfrak{L}(x^{n-1}) \tag{2}$$

for all $x \in \mathcal{R}$. Replacing x by e in Relation (2) and using the assumption that $\mathcal{R}$ is an $(n-1)!$-torsion free ring, it is obtained that $\mathfrak{L}(e) = 0$. Replacing x by $x + \lambda e$ for $\lambda > 0$, we obtain that

$$\begin{aligned} 2\sum_{i=0}^{n}\binom{n}{i}\mathfrak{L}(x^{n-i}(\lambda e)^i) &= \left(\sum_{i=0}^{n-1}\binom{n-1}{i}\mathfrak{L}(x^{n-1-i}(\lambda e)^i)\right)\sigma(x+\lambda e) \\ &+ \left(\sum_{i=0}^{n-1}\binom{n-1}{i}\tau(x^{n-1-i}(\lambda e)^i)\right)\mathfrak{L}(x+\lambda e) \\ &+ \mathfrak{L}(x+\lambda e)\left(\sum_{i=0}^{n-1}\binom{n-1}{i}\sigma(x^{n-1-i}(\lambda e)^i)\right) \\ &+ \tau(x+\lambda e)\left(\sum_{i=0}^{n-1}\binom{n-1}{i}\mathfrak{L}(x^{n-1-i}(\lambda e)^i)\right). \end{aligned}$$

Collecting all expression with coefficients λ^{n-2} from the above relations and using $\mathfrak{L}(e) = 0$, we get that

$$\begin{aligned} 2\binom{n}{n-2}\mathfrak{L}(x^2) &= \binom{n-1}{n-2}\mathfrak{L}(x)\sigma(x) + \binom{n-1}{n-3}\mathfrak{L}(x^2) \\ &+ \binom{n-1}{n-2}\tau(x)\mathfrak{L}(x) + \binom{n-1}{n-2}\mathfrak{L}(x)\sigma(x) \\ &+ \binom{n-1}{n-3}\mathfrak{L}(x^2) + \binom{n-1}{n-2}\tau(x)\mathfrak{L}(x). \end{aligned}$$

The above equation reduces into

$$(n-1)\mathfrak{L}(x^2) = 2n(n-1)\mathfrak{L}(x)\sigma(x) + 2n(n-1)\tau(x)\mathfrak{L}(x) + (n-1)(n-2)\mathfrak{L}(x^2)$$

for all $x \in \mathcal{R}$. Since $\mathcal{R}$ is an $(n-1)!$-torsion free ring, therefore we get that $\mathfrak{L}(x^2) = \mathfrak{L}(x)\sigma(x) + \tau(x)\mathfrak{L}(x)$, for all $x \in \mathcal{R}$. Hence, $\mathfrak{L}$ is a Jordan (σ,τ)-derivation and it follows from [12, Theorem 2] that $\mathfrak{L}$ is a (σ,τ)-derivation. □

Below, we present another characterization of (σ,τ)-derivations.

Theorem 2.2. *Let $\mathcal{R}$ be an $(n-1)!$-torsion free semiprime ring and σ or τ be an automorphism of $\mathcal{R}$. Suppose that there exist additive mappings $\mathfrak{Q}, \mathfrak{L} : \mathcal{R} \to \mathcal{R}$ satisfying the relations*

$$2\mathfrak{Q}(x^n) = \mathfrak{Q}(x^{n-1})\sigma(x) + \tau(x^{n-1})\mathfrak{L}(x) + \mathfrak{L}(x)\sigma(x^{n-1}) + \tau(x)\mathfrak{Q}(x^{n-1})$$

$$2\mathfrak{L}(x^n) = \mathfrak{L}(x^{n-1})\sigma(x) + \tau(x^{n-1})\mathfrak{Q}(x) + \mathfrak{Q}(x)\sigma(x^{n-1}) + \tau(x)\mathfrak{L}(x^{n-1})$$

for all $x \in \mathcal{R}$ and some integer $n > 1$. Then both $\mathfrak{L}$ and $\mathfrak{Q}$ are (σ,τ)-derivations and $\mathfrak{L} = \mathfrak{Q}$.

Proof. We have

$$2\mathfrak{Q}(x^n) = \mathfrak{Q}(x^{n-1})\sigma(x) + \tau(x^{n-1})\mathfrak{L}(x) + \mathfrak{L}(x)\sigma(x^{n-1}) + \tau(x)\mathfrak{Q}(x^{n-1}) \tag{3}$$

$$2\mathfrak{L}(x^n) = \mathfrak{L}(x^{n-1})\sigma(x) + \tau(x^{n-1})\mathfrak{Q}(x) + \mathfrak{Q}(x)\sigma(x^{n-1}) + \tau(x)\mathfrak{L}(x^{n-1}) \tag{4}$$

for all $x \in \mathcal{R}$. Subtracting the above two equations, we get

$$2\mathfrak{A}(x^n) = \mathfrak{A}(x^{n-1})\sigma(x) - \tau(x^{n-1})\mathfrak{A}(x) - \mathfrak{A}(x)\sigma(x^{n-1}) - \tau(x)\mathfrak{A}(x^{n-1}) \tag{5}$$

for all $x \in \mathcal{R}$, where $\mathfrak{A} = \mathfrak{Q} - \mathfrak{L}$. Putting e for x in Equation (5) and using the assumption that $\mathcal{R}$ is an $(n-1)!$-torsion free ring, we get

$$\mathfrak{A}(e) = 0 \tag{6}$$

Now, replacing x by $x + \lambda e$ in Relation (5), we find that

$$\begin{aligned} 2\sum_{i=0}^{n}\binom{n+1}{i}\mathfrak{A}(x^{n-i}(\lambda e)^i) &= \left(\sum_{i=0}^{n-1}\binom{n-1}{i}\mathfrak{A}(x^{n-1-i}(\lambda e)^i)\right)\sigma(x+\lambda e) \\ &- \left(\sum_{i=0}^{n-1}\binom{n-1}{i}\tau(x^{n-1-i}(\lambda e)^i)\right)\mathfrak{A}(x+\lambda e) \\ &- \mathfrak{A}(x+\lambda e)\left(\sum_{i=0}^{n-1}\binom{n-1}{i}\sigma(x^{n-1-i}(\lambda e)^i)\right) \\ &- \tau(x+\lambda e)\left(\sum_{i=0}^{n-1}\binom{n-1}{i}\mathfrak{A}(x^{n-1-i}(\lambda e)^i)\right). \end{aligned}$$

Collecting all expressions with coefficients λ^{n-1} and using $\mathfrak{A}(e) = 0$, we get that

$$2n\mathfrak{A}(x) = 2\mathfrak{A}(x) - 2(n-1)\mathfrak{A}(x)$$

for all $x \in \mathcal{R}$. Since $\mathcal{R}$ is $(n-1)!$-torsion free, it follows from the above relation that $\mathfrak{A}(x) = 0$ for all $x \in \mathcal{R}$. Therefore, we have $\mathfrak{Q} = \mathfrak{L}$ and consequently, Equations (3) and (4) reduce into one relation, which is

$$2\mathfrak{L}(x^n) = \mathfrak{L}(x^{n-1})\sigma(x) + \tau(x^{n-1})\mathfrak{L}(x) + \mathfrak{L}(x)\sigma(x^{n-1}) + \tau(x)\mathfrak{L}(x^{n-1}).$$

Using Theorem 2.1, we conclude that $\mathfrak{L}$ is a (σ, τ)-derivation. □

In the following, we present a functional equation which is satisfied by both Jordan (σ, τ)-derivations and Jordan generalized (σ, τ)-derivations.

Theorem 2.3. *Let $\mathcal{R}$ be an $(n+1)!$-torsion free semiprime ring and σ or τ be an automorphism of $\mathcal{R}$. Suppose that there exist additive mappings $\mathfrak{Q}, \mathfrak{L} : \mathcal{R} \to \mathcal{R}$ satisfying the relation*

$$2\mathfrak{Q}(x^{n+1}) = \mathfrak{Q}(x)\sigma(x^n) + \tau(x)\mathfrak{L}(x^n) + \mathfrak{Q}(x^n)\sigma(x) + \tau(x^n)\mathfrak{L}(x)$$

for all $x \in \mathcal{R}$ and some integer $n \geq 1$. Then $\mathfrak{L}$ is a (σ, τ)-derivation and $\mathfrak{Q}$ is a generalized (σ, τ)-derivation.

Proof. We have the relation

$$2\mathfrak{Q}(x^{n+1}) = \mathfrak{Q}(x)\sigma(x^n) + \tau(x)\mathfrak{L}(x^n) + \mathfrak{Q}(x^n)\sigma(x) + \tau(x^n)\mathfrak{L}(x) \qquad (7)$$

for all $x \in \mathcal{R}$. Replacing x by e in Equation (7), we get $2\mathfrak{Q}(e) = 2\mathfrak{Q}(e) + 2\mathfrak{L}(e)$ which implies that $2\mathfrak{L}(e) = 0$. Since $\mathcal{R}$ is $(n+1)!$-torsion free, we get $\mathfrak{L}(e) = 0$. Now, replacing x by $x + \lambda e$, where $\lambda > 0$, we get

$$\begin{aligned} 2\sum_{i=0}^{n+1}\binom{n+1}{i}\mathfrak{Q}(x^{n+1-i}(\lambda e)^i) &= \mathfrak{Q}(x+\lambda e)\left(\sum_{i=0}^{n}\binom{n}{i}\sigma(x^{n-i}(\lambda e)^i)\right) \\ &+ \tau(x+\lambda e)\left(\sum_{i=0}^{n}\binom{n}{i}\mathfrak{L}(x^{n-i}(\lambda e)^i)\right) \\ &+ \left(\sum_{i=0}^{n}\binom{n}{i}\mathfrak{Q}(x^{n-i}(\lambda e)^i)\right)\sigma(x+\lambda e) \\ &+ \left(\sum_{i=0}^{n}\binom{n}{i}\tau(x^{n-i}(\lambda e)^i)\right)\mathfrak{L}(x+\lambda e) \end{aligned}$$

Collecting all expressions with coefficient λ^n and all expressions with coefficient λ^{n-1} from the above relations and using $\mathfrak{L}(e) = 0$, we obtain that

$$\begin{aligned} 2\binom{n+1}{n}\mathfrak{Q}(x) &= \mathfrak{Q}(x) + \binom{n}{n-1}\mathfrak{Q}(e)\sigma(x) \\ &+ \binom{n}{n-1}\mathfrak{L}(x) + \mathfrak{Q}(e)\sigma(x) \\ &+ \binom{n}{n-1}\mathfrak{Q}(x) + \mathfrak{L}(x) \end{aligned}$$

and

$$\begin{aligned} 2\begin{pmatrix} n+1 \\ n-1 \end{pmatrix}\mathfrak{Q}(x^2) &= \begin{pmatrix} n \\ n-1 \end{pmatrix}\mathfrak{Q}(x)\sigma(x) + \begin{pmatrix} n \\ n-2 \end{pmatrix}\mathfrak{Q}(e)\sigma(x^2) \\ &+ \begin{pmatrix} n \\ n-1 \end{pmatrix}\tau(x)\mathfrak{L}(x) + \begin{pmatrix} n \\ n-2 \end{pmatrix}\mathfrak{L}(x^2) \\ &+ \begin{pmatrix} n \\ n-1 \end{pmatrix}\mathfrak{Q}(x)\sigma(x) + \begin{pmatrix} n \\ n-2 \end{pmatrix}\mathfrak{Q}(x^2) \\ &+ \begin{pmatrix} n \\ n-1 \end{pmatrix}\tau(x)\mathfrak{L}(x), \end{aligned}$$

respectively. The above equations reduce into

$$(n+1)\mathfrak{Q}(x) = (n+1)\mathfrak{Q}(e)\sigma(x) + (n+1)\mathfrak{L}(x) \tag{8}$$

and

$$\begin{aligned} 2\frac{n(n+1)}{2!}\mathfrak{Q}(x^2) &= n\mathfrak{Q}(x)\sigma(x) + \frac{n(n-1)}{2!}\mathfrak{Q}(e)\sigma(x^2) \\ &+n\tau(x)\mathfrak{L}(x) + \frac{n(n-1)}{2!}\mathfrak{L}(x^2) \\ &+n\mathfrak{Q}(x)\sigma(x) + \frac{n(n-1)}{2!}\mathfrak{Q}(x^2) \\ &+n\tau(x)\mathfrak{L}(x), \end{aligned} \tag{9}$$

respectively. Since $\mathcal{R}$ is $(n+1)!$-torsion free, Equation (8) reduces into

$$\mathfrak{Q}(x) = \mathfrak{Q}(e)\sigma(x) + \mathfrak{L}(x) \tag{10}$$

Now, multiplying Equation (9) on both side by 2, we find that

$$\begin{aligned} 2n(n+1)\mathfrak{Q}(x^2) &= 4n\mathfrak{Q}(x)\sigma(x) + 4n\tau(x)\mathfrak{L}(x) + n(n-1)\mathfrak{Q}(x^2) \\ &+n(n-1)\mathfrak{Q}(e)\sigma(x^2) + n(n-1)\mathfrak{L}(x^2) \end{aligned} \tag{11}$$

Using Equation (10) and the assumption that $\mathcal{R}$ is $(n+1)!$-torsion free, we obtain that

$$\mathfrak{L}(x^2) = \mathfrak{L}(x)\sigma(x) + \tau(x)\mathfrak{L}(x),$$

for all $x \in \mathcal{R}$. Hence, $\mathfrak{L}$ is a Jordan (σ,τ)-derivation of $\mathcal{R}$. It follows from [12, Theorem 2] that $\mathfrak{L}$ is a (σ,τ)-derivation. Reusing Equation (10), we find that

$$\mathfrak{Q}(x^2) = \mathfrak{Q}(e)\sigma(x^2) + \mathfrak{L}(x)\sigma(x) + \tau(x)\mathfrak{L}(x) = \mathfrak{Q}(x)\sigma(x) + \tau(x)\mathfrak{L}(x)$$

for all $x \in \mathcal{R}$, which means that $\mathfrak{Q}$ is a generalized Jordan (σ,τ)-derivation of $\mathcal{R}$. Using [12, Theorem 3], we infer that $\mathfrak{Q}$ is a generalized (σ,τ)-derivation. This proves our theorem. □

Theorem 2.4. *Let $\mathcal{R}$ be a k-torsion free semiprime ring, where $k \in \{2, m, n\}$ and σ or τ be an automorphism of $\mathcal{R}$. Suppose that $d : \mathcal{R} \to \mathcal{R}$ is an additive mapping satisfying*

$$d(x^{m+n}) = d(x^m)\sigma(x^n) + \tau(x^m)d(x^n)$$

for all $x \in \mathcal{R}$ and some integers $m, n \geq 1$. Then d is a (σ,τ)-derivation.

Proof. We have the relation

$$d(x^{m+n}) = d(x^m)\sigma(x^n) + \tau(x^m)d(x^n) \tag{12}$$

for all $x \in \mathcal{R}$. Putting e instead of x in the above relation, we obtain

$$d(e) = 0. \tag{13}$$

Setting $x + \lambda e$ for x in the above expression, we find that

$$\begin{aligned}\sum_{i=0}^{m+n}\begin{pmatrix} m+n \\ i \end{pmatrix} d(x^{m+n-i}(\lambda e)^i) &= \sum_{i=0}^{m}\begin{pmatrix} m \\ i \end{pmatrix} d(x^{m-i}(\lambda e)^i)\times \\ &\sum_{i=0}^{n}\begin{pmatrix} n \\ i \end{pmatrix} \sigma(x^{n-i}(\lambda e)^i) \\ &+\sum_{i=0}^{m}\begin{pmatrix} m \\ i \end{pmatrix} \tau(x^{m-i}(\lambda e)^i)\times \\ &\sum_{i=0}^{n}\begin{pmatrix} n \\ i \end{pmatrix} d(x^{n-i}(\lambda e)^i)\end{aligned}$$

Collecting all expressions with coefficient λ^{m+n-2} and using the Relation (13), we find that

$$\begin{aligned}\begin{pmatrix} m+n \\ m+n-2 \end{pmatrix} d(x^2) &= \begin{pmatrix} m \\ m-2 \end{pmatrix}\begin{pmatrix} n \\ n \end{pmatrix} d(x^2) + \begin{pmatrix} m \\ m-1 \end{pmatrix}\begin{pmatrix} n \\ n-1 \end{pmatrix} d(x)\sigma(x) \\ &+\begin{pmatrix} m \\ m-1 \end{pmatrix}\begin{pmatrix} n \\ n-1 \end{pmatrix} \tau(x)d(x) + \begin{pmatrix} m \\ m \end{pmatrix}\begin{pmatrix} n \\ n-2 \end{pmatrix} d(x^2).\end{aligned}$$

The above expression reduces into

$$\begin{aligned}(m+n)(m+n-1)d(x^2) &= m(m-1)d(x^2) + n(n-1)d(x^2) \\ &+2mn\,(d(x)\sigma(x) + \tau(x)d(x))\end{aligned}$$

Since $\mathcal{R}$ is k-torsion free, it follows from the above relation that

$$d(x^2) = d(x)\sigma(x) + \tau(x)d(x)$$

for all $x \in \mathcal{R}$. In other words, d is a Jordan (σ, τ)-derivation and [12, Theorem 2] implies that d is a (σ, τ)-derivation. □

In the following, we are going to give a characterization of $*$-derivations. Before it, we recall the definition of a $*$-derivation. Let $\mathcal{R}$ be a $*$-ring. An additive mapping $d : \mathcal{R} \to \mathcal{R}$ is called a $*$-derivation (resp. Jordan $*$-derivation) if $d(xy) = d(x)y^* + xd(y)$ (resp. $d(x^2) = d(x)x^* + xd(x)$) for all $x, y \in \mathcal{R}$. For instance, let $\mathcal{R}$ be a $*$-ring and let a be an arbitrary fixed element of $\mathcal{R}$. Then an additive mapping $d : \mathcal{R} \to \mathcal{R}$ defined by $d(x) = xa - ax^*$, where $x \in \mathcal{R}$, is a Jordan $*$-derivation on $\mathcal{R}$. It is clear that, if $\mathcal{R}$ is a commutative $*$-ring, then the additive mapping d is an $*$-derivation on $\mathcal{R}$. For more material about $*$-derivations and Jordan $*$-derivations, see, e.g. [6, 7, 15, 17, 21] and the references therein.

In the next theorem, we present a characterization of Jordan $*$-derivations.

Theorem 2.5. *Let $n > 1$ be an integer, $\mathcal{R}$ be an $n!$-torsion free $*$-ring and let $d : \mathcal{R} \to \mathcal{R}$ be an additive mapping satisfying*

$$d(a^n) = \sum_{j=1}^{n} a^{n-j} d(a) {a^*}^{j-1} \tag{14}$$

for all $a \in \mathcal{R}$. Then d is a Jordan $*$-derivation on $\mathcal{R}$.

Proof. Let c be a hermitian element of $Z(\mathcal{R})$ such that $d(c) = 0$. Replacing a by $a + c$ in Equation (14), we have

$$
\begin{aligned}
d\left(\sum_{i=0}^{n}\binom{n}{i}a^{n-i}c^{i}\right) &= \sum_{j=1}^{n}\left(\sum_{k_1=0}^{n-j}\binom{n-j}{k_1}a^{n-j-k_1}c^{k_1}d(a)\sum_{k_2=0}^{j-1}\binom{j-1}{k_2}a^{*j-1-k_2}c^{k_2}\right)\\
&= \sum_{k_1=0}^{n-1}\binom{n-1}{k_1}a^{n-1-k_1}c^{k_1}d(a)+\\
&\sum_{k_1=0}^{n-2}\binom{n-2}{k_1}a^{n-2-k_1}c^{k_1}d(a)\sum_{k_2=0}^{1}\binom{1}{k_2}a^{*1-k_2}c^{k_2}+\\
&\ldots+\sum_{k_1=0}^{1}\binom{1}{k_1}a^{1-k_1}c^{k_1}d(a)\sum_{k_2=0}^{n-2}\binom{n-2}{k_2}a^{*n-2-k_2}c^{k_2}+\\
&d(a)\sum_{k_2=0}^{n-1}\binom{n-1}{k_2}a^{*n-1-k_2}c^{k_2}.
\end{aligned}
$$

Rearranging the above relations with respect to involving equal number of factors of c, we arrive at

$$\sum_{i=1}^{n-1}\gamma_i(a,a^*,c)=0, \tag{15}$$

where,

$$\gamma_i(a,a^*,c)=\binom{n}{i}d(a^{n-i}c^i)-\sum_{k=1}^{n-i}\binom{n}{i}a^{n-i-k}c^i d(a)a^{*k-1}. \tag{16}$$

Replacing c by c, $2c$, $3c$,..., $(n-1)c$ in Equation (15), we obtain a system of $n-1$ homogeneous equations as follows:

$$
\begin{cases}
\sum_{i=1}^{n-1}\gamma_i(a,a^*,c)=0\\
\sum_{i=1}^{n-1}\gamma_i(a,a^*,2c)=0\\
\sum_{i=1}^{n-1}\gamma_i(a,a^*,3c)=0\\
\quad\vdots\\
\sum_{i=1}^{n-1}\gamma_i(a,a^*,(n-1)c)=0
\end{cases}
$$

It is observed that the coefficient matrix of the above system is:

$$
X=\begin{bmatrix}
\binom{n}{1} & \binom{n}{2} & \binom{n}{3} & \cdots & \binom{n}{n-1}\\
2\binom{n}{1} & 2^2\binom{n}{2} & 2^3\binom{n}{3} & \cdots & 2^{n-1}\binom{n}{n-1}\\
\vdots & \vdots & \vdots & & \vdots\\
(n-1)\binom{n}{1} & (n-1)^2\binom{n}{2} & (n-1)^3\binom{n}{3} & \cdots & (n-1)^{n-1}\binom{n}{n-1}
\end{bmatrix}
$$

In view of the determinant of a square Vandermonde matrix, one can easily get that

$$det\ X = \left(\prod_{k=1}^{n-1}\binom{n}{k}\right)(n-1)! \prod_{1\le i<j\le n-1}(i-j).$$

It is evident that the determinant of X is nonzero, i.e $det\ X \neq 0$, and it implies that the system has only a trivial solution. In particular, $\gamma_{n-2}(a, a^*, c) = 0$. It means that

$$\binom{n}{n-2}d(a^2c^{n-2}) - \sum_{k=1}^{2}\binom{n}{n-2}a^{2-k}c^{n-2}d(a){a^*}^{k-1} = 0. \tag{17}$$

Since $\mathcal{R}$ is an n!-torsion free ring, it follows from Equation (17) that

$$d(a^2c^{n-2}) = ac^{n-2}d(a) + c^{n-2}d(a)a^*. \tag{18}$$

Using Equation (14) along with the fact that $e^* = e$, we get that $d(e) = 0$. Thus, we can put e instead of c in Equation (18) to obtain that $d(a^2) = d(a)a^* + ad(a)$ and it means that d is a Jordan $*$-derivation on $\mathcal{R}$. □

There are two consequences of the above theorem as follows:

Corollary 2.6. *Let $\mathcal{A}$ be a commutative complex $*$-algebra such that $\mathcal{A}a = \{0\}$ or $a\mathcal{A} = \{0\}$ (where $a \in \mathcal{A}$) implies $a = 0$. Let $n > 1$ be an integer and let $d : \mathcal{A} \to \mathcal{A}$ be an additive mapping satisfying*

$$d(a^n) = \sum_{j=1}^{n} a^{n-j}d(a){a^*}^{j-1} \tag{19}$$

for all $a \in \mathcal{A}$. Then d is a $*$-derivation on $\mathcal{A}$.

Proof. It follows from Theorem 2.5 that $d(a^2) = d(a)a^* + ad(a)$ for all $a \in \mathcal{A}$. According to [7, Theorem 2.1] there exists a uniqe double centralizer (T, S) such that $d(a) = T(a^*) - S(a)$ for all $a \in \mathcal{A}$. Indeed, T is a left centralizer and S is a right centralizer on $\mathcal{A}$. So, we have

$$\begin{aligned} d(ab) &= T((ab)^*) - S(ab) \\ &= T(b^*a^*) - S(ab) \\ &= T(a^*b^*) - S(ab) \\ &= T(a^*)b^* - aS(b), \end{aligned}$$

which means that

$$d(ab) = T(a^*)b^* - aS(b) \ for\ all\ a, b \in \mathcal{A}. \tag{20}$$

It follows from the proof of [7, Theorem 2.1] that $aT(b) = S(a)b$ for all $a, b \in \mathcal{A}$. Hence, we have

$$\begin{aligned} d(a)b^* + ad(b) &= (T(a^*) - S(a))b^* + a(T(b^*) - S(b)) \\ &= T(a^*)b^* - S(a)b^* + aT(b^*) - aS(b) \\ &= T(a^*)b^* - aT(b^*) + aT(b^*) - aS(b) \\ &= T(a^*)b^* - aS(b), \end{aligned}$$

which means that

$$d(a)b^* + ad(b) = T(a^*)b^* - aS(b) \ for\ all\ a, b \in \mathcal{A}. \tag{21}$$

Comparin (20) and (21), we see that

$$d(ab) = d(a)b^* + ad(b) \ for\ all\ a, b \in \mathcal{A}.$$

This completes the proof of our corollary. □

Let $\mathcal{H}$ be a real or complex Hilbert space. By $B(\mathcal{H})$ we mean the algebra of all bounded linear operators on $\mathcal{H}$. We denote by $\mathcal{F}(\mathcal{H})$ the subalgebra of bounded finite rank operators. We shall call a subalgebra $\mathfrak{A}$ of $B(\mathcal{H})$ standard, provided $\mathfrak{A}$ contains $\mathcal{F}(\mathcal{H})$. It is easy to see that $\mathcal{F}(\mathcal{H})$ is a prime ring; that is, $A, B \in \mathcal{F}(\mathcal{H})$ and $A\mathcal{F}(\mathcal{H})B = \{0\}$ imply $A = 0$ or $B = 0$.

Corollary 2.7. *Let $\mathcal{H}$ be a real or complex Hilbert space, $\dim\mathcal{H} > 1$, and let $\mathfrak{A}$ be a standard operator algebra on $\mathcal{H}$. Let $n > 1$ be an integer and let $d : \mathfrak{A} \to B(\mathcal{H})$ be an additive mapping satisfying*

$$d(A^n) = \sum_{j=1}^{n} A^{n-j} d(A) A^{*\,j-1} \tag{22}$$

for all $A \in \mathfrak{A}$. Then there exists a unique linear operator $T \in B(\mathcal{H})$ such that $d(A) = AT - TA^*$ holds for all $A \in \mathfrak{A}$.

Proof. This is an immediate conclusion of Theorem 2.5 and the main theorem of [15]. □

We finish the article with the following two theorems. In what follows, the set of all invertible elements of an algebra $\mathcal{A}$ is denoted by $Inv(\mathcal{A})$.

Theorem 2.8. *Let $\mathcal{A}$ be a unital Banach algebra and let m_0, n_0, p_0 be positive integers satisfying $2n_0 - m_0p_0 = 2$. Let $d : \mathcal{A} \to \mathcal{A}$ be a mapping satisfying $d(x) = x^{m_0} d(x^{-1})$ and further $d(x) \in Z(\mathcal{A})$ for any $x \in Inv(\mathcal{A})$. If $f : \mathcal{A} \to \mathcal{A}$ is an linear mapping such that $f(x) = x^2 f(x^{-1}) + x^{n_0}(d(x^{-1}))^{p_0}$ for any $x \in Inv(\mathcal{A})$, then $f(a) = af(e)$ for all $a \in \mathcal{A}$.*

Proof. First, notice that

$$f(x^{-1}) = x^{-2} f(x) + x^{-n_0}(d(x))^{p_0} \ \textit{for all } x \in Inv(\mathcal{A}).$$

According to the above assumptions, for every invertible element $x \in \mathcal{A}$, we have the following expressions:

$$\begin{aligned} f(x) &= x^2\Big(x^{-2} f(x) + x^{-n_0}(d(x))^{p_0}\Big) + x^{n_0}(d(x^{-1}))^{p_0} \\ &= f(x) + x^{2-n_0}(d(x))^{p_0} + x^{n_0}(d(x^{-1}))^{p_0} \\ &= f(x) + x^{2-n_0}(d(x))^{p_0} + x^{n_0}\left(x^{-m_0} d(x)\right)^{p_0} \\ &= f(x) + x^{2-n_0}(d(x))^{p_0} + x^{n_0-p_0m_0}\left(d(x)\right)^{p_0} \qquad (d(x) \in Z(\mathcal{A})) \\ &= f(x) + \Big(x^{2-n_0} + x^{n_0-p_0m_0}\Big)(d(x))^{p_0}, \end{aligned}$$

which means that

$$\Big(x^{2-n_0} + x^{n_0-p_0m_0}\Big)(d(x))^{p_0} = 0. \tag{23}$$

It follows from $2n_0 - m_0p_0 = 2$ that $n_0 - m_0p_0 = 2 - n_0$ and consequently, Equation (23) reduces to

$$2x^{2-n_0}\left(d(x)\right)^{p_0} = 0.$$

Since x is an invertible element of $\mathcal{A}$, one can obtain that $(d(x))^{p_0} = 0$ for all $x \in Inv(\mathcal{A})$. Hence, we infer that $f(x) = x^2 f(x^{-1})$ for all $x \in Inv(\mathcal{A})$. According to the proof of [20, Theorem 5], we get that $f(a) = af(e)$ for all $a \in \mathcal{A}$. Thereby, we achieve our goal. □

Theorem 2.9. *Let $\mathcal{A}$ be a unital semiprime Banach algebra and let $f, d : \mathcal{A} \to \mathcal{A}$ be additive mappings satisfying*

$$d(x) = -x^2 d(x^{-1}) \ \textit{and} \ f(x) = -x^2 f(x^{-1}) + x^3(d(x^{-1}))^2$$

for all $x \in Inv(\mathcal{A})$. Then $f = \frac{1}{2}d^2 + D$, where D is a derivation mapping $\mathcal{A}$ into $Z(\mathcal{A})$.

Proof. It follows from the above-mentioned equations that $d(e) = f(e) = 0$. Let x be an arbitrary element of $\mathcal{A}$. We can choose a positive integer n such that both a and $e - a$ are invertible elements of $\mathcal{A}$, where $a = ne + x$. In the following, we use the well-known Hua identity $a^2 = a - \left(a^{-1} + (e-a)^{-1}\right)^{-1}$. So, we have $\left(a^{-1} + (e-a)^{-1}\right)^{-1} = a - a^2 = a(e-a)$. According to the aforementioned assumption, we have

$$f(x) = -x^2 f(x^{-1}) + x^3(d(x^{-1}))^2 \tag{24}$$

for all $x \in Inv(\mathcal{A})$. Applying Equation (24), we have

$$\begin{aligned}
f(a^2) &= f(a) - f\Big((a^{-1} + (e-a)^{-1})^{-1}\Big)\\
&= f(a) + (a^{-1} + (e-a)^{-1})^{-2} f(a^{-1} + (e-a)^{-1}) - (a^{-1} + (e-a)^{-1})^{-3}\Big(d(a^{-1} + (e-a)^{-1})\Big)^2\\
&= f(a) + a^2(e-a)^2 f(a^{-1}) + a^2(e-a)^2 f((e-a)^{-1}) - a^3(e-a)^3\Big(d(a^{-1}) + d((e-a)^{-1})\Big)^2\\
&= f(a) + a^2(e-a)^2\Big(-a^{-2}f(a) + a^{-3}(d(a))^2\Big) + a^2(e-a)^2\Big(-(e-a)^{-2}f(e-a) + (e-a)^{-3}(d(e-a))^2\Big)\\
&- a^3(e-a)^3\Big(-a^{-2}d(a) - (e-a)^{-2}d(e-a)\Big)^2\\
&= f(a) - (e-a)^2 f(a) + a^{-1}(e-a)^2(d(a))^2 + a^2 f(a) + a^2(e-a)^{-1}(d(a))^2\\
&- a^3(e-a)^3\Big(-a^{-2}d(a) + (e-a)^{-2}d(a)\Big)^2\\
&= f(a) - f(a) + 2af(a) - a^2 f(a) + (a^{-1} - 2e + a)(d(a))^2 + a^2 f(a) + a^2(e-a)^{-1}(f(a))^2\\
&- a^3(e-a)^3\Big((e-a)^{-2}(2a-e)a^{-2}\Big)^2(d(a))^2\\
&= 2af(a) + (a^{-1} - 2e + a)(d(a))^2 + a^2(e-a)^{-1}(d(a))^2 - a^3(e-a)^3(e-a)^{-4}(2a-e)^2a^{-4}(d(a))^2\\
&= 2af(a) + \Big(a^{-1} - 2e + a + a(e-a)^{-1}a - a^{-1}(e-a)^{-1}(2a-e)^2\Big)(d(a))^2\\
&= 2af(a) + \Big(a^{-1} - 2e + a + (e-a)^{-1} - e)a - a^{-1}(e-a)^{-1}(2a-e)^2\Big)(d(a))^2\\
&= 2af(a) + \Big(a^{-1} - 2e + (e-a)^{-1}(-3a - a^{-1} + 4e)\Big)(d(a))^2\\
&= 2af(a) + \Big(a^{-1} - 2e + (e-a)^{-1}a^{-1}(e-a)(3a-e)\Big)(d(a))^2\\
&= 2af(a) + (d(a))^2,
\end{aligned}$$

and hence

$$f(a^2) = 2af(a) + (d(a))^2. \tag{25}$$

Putting $a = ne + x$ in the previous equation and using the fact that $f(e) = d(e) = 0$, we obtain that

$$f(x^2) = 2xf(x) + (d(x))^2, \tag{26}$$

for all $x \in \mathcal{A}$. According to the proof of [20, Theorem 5], d is a Jordan left derivation and it follows from [20, Theorem 2] that d is a derivation which maps $\mathcal{A}$ into $Z(\mathcal{A})$. Note that

$$(d(x))^2 = \frac{1}{2}d^2(x^2) - xd^2(x) \quad (x \in \mathcal{A}). \tag{27}$$

Using Equations (26) and (27), we get that $D(x^2) = 2xD(x)$ for all $x \in \mathcal{A}$, where $D = f - \frac{1}{2}d^2$. It means that D is a Jordan left derivation on $\mathcal{A}$ and reusing [20, Theorem 2] implies that D is a derivation which maps $\mathcal{A}$

into $Z(\mathcal{A})$. Therefore, we can infer that $f(\mathcal{A}) \subseteq Z(\mathcal{A})$. In this case, we see that

$$\begin{aligned} f(xy) &= D(xy) + \frac{1}{2}d^2(xy) \\ &= D(x)y + xD(y) + \frac{1}{2}d^2(x)y + d(x)d(y) + \frac{1}{2}xd^2(y) \\ &= f(x)y + xf(y) + d(x)d(y) \end{aligned}$$

for all $x, y \in \mathcal{A}$. □

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