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A modified generalized viscosity explicit methods for quasi-nonexpansive mappings in Banach spaces

T. M. M. Sow[a]

[a] *Gaston Berger University, Saint Louis, Senegal*

Abstract. The main objective of this paper is to introduce and study an iterative algorithm which is a combination of general iterative method with strongly accretive operators and generalized viscosity explicit methods (GVEM) for finding fixed points of quasi-nonexpansive mappings in Banach spaces. Under suitable conditions, some strong convergence theorems for finding a common element of the set of solutions of fixed points problems involving quasi-nonexpansive mappings and the set of solutions of variational inequality problems are obtained without imposing any compactness assumption. Finally, applications of our results to quadratic optimization problems are given.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle_H$ and norm $\| \cdot \|_H$. An operator $A : H \to H$ is called *monotone* if for all $x, y \in D(A)$, the following inequality holds:

$$\langle Ax - Ay, x - y \rangle_H \geq 0,$$

A is called *k-strongly monotone* if there exists $k \in (0, 1)$ such that for all $x, y \in D(A)$,

$$\langle Ax - Ay, x - y \rangle_H \geq k\|x - y\|^2.$$

An operator $A : H \to H$ is said to be *strongly positive bounded linear* if there exists a constant $k > 0$ such that

$$\langle Ax, x \rangle_H \geq k\|x\|^2, \quad \forall\ x \in H.$$

Remark 1.1. *From the definition of A, we note that strongly positive bounded linear operator A is a $\|A\|$-Lipschitzian and k- strongly monotone operator.*

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Email address: sowthierno89@gmail.com (T. M. M. Sow)

Let X be a real normed space, K be a nonempty subset of X. A map $T : K \to X$ is said to be Lipschitz if there exists an $L \geq 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|, \quad \forall x, y \in K, \tag{1}$$

if $L < 1$, T is called *contraction* and if $L = 1$, T is called nonexpansive.
We denote by $F(T)$ the set of fixed points of the mapping T, that is $F(T) := \{x \in D(T) : x = Tx\}$. We assume that $F(T)$ is nonempty. If T is nonexpansive mapping, it is well known $F(T)$ is closed and convex. A map T is called quasi-nonexpansive if $\|Tx - p\| \leq \|x - p\|$ holds for all x in K and $p \in F(T)$.
The mapping $T : K \to K$ is said to be firmly nonexpansive, if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(x - y) - (Tx - Ty)\|^2, \forall x, y \in K.$$

We note that the following inclusions hold for the classes of the mappings:

firmly nonexpansive $\subset$ nonexpansive $\subset$ quasi-nonexpansive.

We illustrate these by the following example.

Example 1.2. *Let $X = l_\infty$ and $C := \{x \in l_\infty : \|x\|_\infty \leq 1\}$. Define $T : C \to C$ by $Tx = (0, x^2{}_1, x^2{}_2, x^3{}_3, ...)$ for $x = (x_1, x_2, x_3, ...)$ in C. Then, it is clear that T is continuous and map C into C. Moreover, $Tp = p$ if and only if $p = 0$. Futhermore,*

$$\begin{aligned}\|Tx - p\|_\infty &= \|Tx\|_\infty = \|(0, x^2{}_1, x^2{}_2, x^2{}_3, ...)\|_\infty \\ &\leq \|(x_1, x_2, x_3, ...)\|_\infty = \|x\|_\infty \\ &= \|x - p\|_\infty.\end{aligned}$$

Therefore, T is quasi-nonexpansive. However, T is not nonexpansive.

Many problems arising in different areas of mathematics such as optimization, variational analysis, differential equations, mathematical economics, and game theory can be modeled as fixed point equations of the form $x = Tx$, where T is a nonexpansive mapping. Until now there have been many effective algorithms for solving fixed point problems involving nonexpansive mappings (see, e.g., Yao et al. [19], Chidume [4], Marino et al. [10] and the references therein).

Historically, one of the most investigated methods approximating fixed points of nonexpansive mappings dates back to 1953 and is known as Mann's method, in light of Mann [7]. Let C be a nonempty, closed and convex subset of a Banach space X, Mann's scheme is defined by

$$\begin{cases} x_0 \in C, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \end{cases} \tag{2}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$. But Mann's iteration process has only weak convergence, even in Hilbert spaces setting. Therefore, many authors try to modify Mann's iteration to have strong convergence for nonlinear operators.

In the real world, many engineering and science problems can be reformulated as ordinary differential equations. Several numerical methods have been developed for solving ordinary differential equations (ODEs) by numerous authors. Consider the following initial value problem:

$$\begin{cases} x_0 = x(t_0), \\ x'(t) = f(x(t)), \end{cases} \tag{3}$$

where $f : \mathbb{R}^M \to \mathbb{R}^M$ is a continuous function. The implicit midpoint method (IMR) is an implicit method given by the following finite difference scheme [6]:

$$\begin{cases} y_0 = x_0, \\ y_{n+1} = y_n + hf\Big(\dfrac{y_{n+1} + y_n}{2}\Big), \end{cases} \tag{4}$$

where $h > 0$ is a time step. It is known that if $f : \mathbb{R}^M \to \mathbb{R}^M$ is Lipschitz continuous and sufficiently smooth, then the sequence $\{y_n\}$ converges to the exact solution of (3) as $h \to 0$ uniformly over $t \in [t_0,\ t^*]$ for any fixed $t^* > 0$. If we write the function f in the form $f = I - T$, where T is a nonlinear mapping, then equilibrium problem involving differential Equation (3) is the fixed point problem $x = Tx$. Over the last several years, the implicit midpoint rule has become a powerful numerical method for numerically solving time-dependent differential equations (in particular, stiff equations) and differential algebraic equations.

Based on IMR (4), Xu et al. [18] applied the viscosity approximation method introduced by Moudafi [8] to the IMR for a nonexpansive mapping T and proposed the following viscosity implicit midpoint rule (VIMR) in Hilbert spaces H as follows:

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T\Big(\frac{x_{n+1} + x_n}{2}\Big),\ \ n \geq 1, \tag{5}$$

where $\{\alpha_n\}$ is a real control condition in $(0, 1)$. They also proved that the sequence $\{x_n\}$ generated by (5) converges strongly to a point $x^* \in F(T)$.

In numerical analysis, it is clear that the computation by the IMR is not an easy work in practice. Because the IMR need to compute at every time steps, it can be much harder to implement. To overcome this difficulty, for solving (3), we consider the helpful method, the so-called explicit midpoint method (EMR), given by the following finite difference scheme

$$\begin{cases} y_0 = x_0, \\ \bar{y}_{n+1} = y_n + hf(y_n), \\ y_{n+1} = y_n + hf\Big(\dfrac{\bar{y}_{n+1} + y_n}{2}\Big), \end{cases} \tag{6}$$

It is easy to see that the explicit midpoint method calculates the state of a system at the next time from the state of the system at the current time [12].

In 2017, Marino et al. [9] based on EMR (6) established the following so-called general viscosity explicit rule for quasi-nonexpansive mappings T in Hilbert spaces:

$$\begin{cases} \bar{x}_{n+1} = \beta_n x_n + (1-\beta_n)Tx_n,\ \ n \geq 0, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T\big(s_n x_n + (1-s_n)\bar{x}_{n+1}\big), \end{cases} \tag{7}$$

where f is a contraction and $\{\alpha_n\}$, $\{\beta_n\}$, and $\{s_n\}$ are the sequences in $(0, 1)$. They proved, under suitable conditions on the sequence parameters, that the generalized viscosity explicit rule (7) converges strongly to a point $x^* \in F(T)$.

Let C be a nonempty closed convex subset of real Hilbert space H, and $F : C \to H$ be a nonlinear map and T be a map on C, such that $F(T) \neq \emptyset$. The point $u \in F(T)$ is said to be a solution of the variational inequality problem $VI(F, T)$ provided that

$$\langle Fu,\, v - u\rangle \geq 0,\ \forall\, v \in F(T). \tag{8}$$

Variational inequality theory, which was initially introduced by Stampacchia [14] in 1964 is a branch of applicable mathematics with a wide range of applications in industry, physical, regional, social, pure, and applied sciences. This field is dynamic and is experiencing an explosive growth in both theory and applications; as a consequence, research techniques and problems are drawn from various fields. Variational inequalities have been generalized and extended in different directions using the novel and innovative techniques.
However, there were few results established for fixed point problems involving quasi-nonexpansive mappings and variational inequality problems in Banach spaces.

The above results naturally bring us to the following questions.

Question 1: Can results of Marino et al. [9] be extend from Hilbert spaces to Banach spaces?
Question 2: Can we obtain a strong convergence results by using a modified generalized viscosity explicit methods for finding a common element of the set of solutions of fixed points problems involving quasi-nonexpansive mappings and the set of solutions of variational inequality problems ?

The purpose of this paper is to give affirmative answers to these questions mentioned above.
Motivated by Marino et al. [9], we construct an iterative algorithm and prove strong convergence theorems for finding a common element of the set of solutions of fixed points problems involving quasi-nonexpansive and the set of solutions of variational inequality problem in real Banach spaces having a weakly continuous duality maps. No compactness assumption is made. The algorithm and results presented in this paper improve and extend some recents results. Finally, our method of proof is of independent interest.

2. Preliminairies

Let E be a Banach space with norm $\|\cdot\|$ and dual E^*. For any $x\in E$ and $x^*\in E^*$, $\langle x^*,x\rangle$ is used to refer to $x^*(x)$. Let $\varphi:[0,+\infty)\to[0,\infty)$ be a strictly increasing continuous function such that $\varphi(0)=0$ and $\varphi(t)\to+\infty$ as $t\to\infty$. Such a function φ is called gauge. Associed to a gauge a duality map $J_\varphi:E\to 2^{E^*}$ defined by:

$$J_\varphi(x):=\{x^*\in E^*:\langle x,x^*\rangle=||x||\varphi(||x||),||x^*||=\varphi(||x||)\}. \tag{9}$$

If the gauge is defined by $\varphi(t)=t$, then the corresponding duality map is called the *normalized duality map* and is denoted by J. Hence the normalized duality map is given by

$$J(x):=\{x^*\in E^*:\langle x,x^*\rangle=||x||^2=||x^*||^2\},\ \forall\, x\in E.$$

Notice that

$$J_\varphi(x)=\frac{\varphi(||x||)}{||x||}J(x),\ \ x\neq 0.$$

Let E be a real normed space and let $S:=\{x\in E:\|x\|=1\}$. E is said to be *smooth* if

$$\lim_{t\to 0^+}\frac{\|x+ty\|-\|x\|}{t}$$

exists for each $x,y\in S$. E is said to be uniformly smooth if it is smooth and the limit is attained uniformly for each $x,y\in S$.
Let E be a normed space with dimE ≥ 2. The *modulus of smoothness* of E is the function $\rho_E:[0,\infty)\to[0,\infty)$ defined by

$$\rho_E(\tau):=\sup\left\{\frac{\|x+y\|+\|x-y\|}{2}-1:\|x\|=1,\|y\|=\tau\right\};\quad \tau>0.$$

It is known that a normed linear space E is *uniformly smooth* if

$$\lim_{\tau\to 0}\frac{\rho_E(\tau)}{\tau}=0.$$

If there exists a constant $c>0$ and a real number $q>1$ such that $\rho_E(\tau)\leq c\tau^q$, then E is said to be *q-uniformly smooth*. Typical examples of such spaces are the L_p, ℓ_p and W_p^m spaces for $1<p<\infty$ where,

$$L_p\ (or\ l_p)\ or\ W_p^m \text{ is}$$

$$\begin{cases} 2-\text{uniformly smooth and } p-\text{uniformly convex} & \text{if}\quad 2\leq p<\infty;\\ 2-\text{uniformly convex and } p-\text{uniformly smooth} & \text{if}\quad 1<p<2.\end{cases} \tag{10}$$

Let J_q denote the *generalized duality mapping* from E to 2^{E^*} defined by

$$J_q(x) := \left\{ f \in E^* : \langle x, f \rangle = \|x\|^q \text{ and } \|f\| = \|x\|^{q-1} \right\}.$$

J_2 is called the *normalized duality mapping* and is denoted by J. It is known that E is smooth if and only if each duality map J_φ is single-valued, that E is Frechet differentiable if and only if each duality map J_φ is norm-to-norm continuous in E, and that E is uniformly smooth if and only if each duality map J_φ is norm-to-norm uniformly continuous on bounded subsets of E. Following Browder [2], we say that a Banach space has a weakly continuous duality map if there exists a gauge φ such that J_φ is single-valued and is weak-to-weak* sequentially continuous, i.e., if $(x_n) \subset E$, $x_n \xrightarrow{w} x$, then $J_\varphi(x_n) \xrightarrow{w^*} J_\varphi(x)$. It is known that l^p $(1 < p < \infty)$ has a weakly continuous duality map with gauge $\varphi(t) = t^{p-1}$ (see e.g., [3] for more details on duality maps).

Remark 2.1. *Note also that a duality mapping exists in each Banach space. We recall from [1] some of the examples of this mapping in $l_p, L_p, W^{m,p}$-spaces, $1 < p < \infty$.*

(i) $l_p: \ Jx = \|x\|_{l_p}^{2-p} y \in l_q, \ x = (x_1, x_2, \cdots, x_n, \cdots), \ y = (x_1|x_1|^{p-2}, x_2|x_2|^{p-2}, \cdots, x_n|x_n|^{p-2}, \cdots),$

(ii) $L_p: \ Ju = \|u\|_{L_p}^{2-p} |u|^{p-2} u \in L_q,$

(iii) $W^{m,p}: \ Ju = \|u\|_{W^{m,p}}^{2-p} \sum_{|\alpha \leq m|} (-1)^{|\alpha|} D^\alpha \Big(|D^\alpha u|^{p-2} D^\alpha u \Big) \in W^{-m,q},$

where $1 < q < \infty$ is such that $1/p + 1/q = 1$.

Finally recall that a Banach space E satisfies Opial property (see, e.g., [11]) if $\limsup_{n\to+\infty} \|x_n - x\| < \limsup_{n\to+\infty} \|x_n - y\|$ whenever $x_n \xrightarrow{w} x$, $x \neq y$. A Banach space E that has a weakly continuous duality map satisfies Opial's property. Given a gauge φ and E be a smooth real Banach space. A map $A : D(A) \subset E \to E$ is called *accretive* if for each $x, y \in D(A)$,

$$\big\langle Ax - Ay, J_\varphi(x - y) \big\rangle \geq 0.$$

A is called *k- strongly accretive* if there exists $k \in (0, 1)$ such that for each $x, y \in D(A)$,

$$\langle Ax - Ay, J_\varphi(x - y) \rangle \geq k\varphi(||x - y||)||x - y||. \tag{11}$$

In a Hilbert space, the normalized duality map is the identity map. Hence, in Hilbert spaces, *strongly monotonicity* and *strongly accretivity* coincide.

Remark 2.2. *If $\varphi(t) = t^{q-1}$, $q > 1$, inequality (11) becomes*

$$\langle Ax - Ay, J_q(x - y) \rangle \geq k\|x - y\|^q.$$

Lemma 2.3. *[5] Let E be a Banach space satisfying Opial's property, K be a closed convex subset of E, and $T : K \to K$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Then $I - T$ is demiclosed; that is,*

$$\{x_n\} \subset K, \ x_n \rightharpoonup x \in K \ \text{ and } \ (I - T)x_n \to y \ \text{ implies that } (I - T)x = y.$$

Lemma 2.4. *[4] Let E be a real normed space then for any $x, y \in E$, the following inequality hold:*

$$\|x + y\|^p \leq \|x\|^p + \langle y, j_p(x + y) \rangle$$

for all x,y $\in E$, $j_p(x + y) \in J_p(x + y)$. In particular, if $p = 2$, then

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y) \rangle.$$

Theorem 2.5. *[4] Let $q > 1$ be a fixed real number and E be a smooth Banach space. Then the following statements are equivalent:*
(i) E is q-uniformly smooth.
(ii) There is a constant $d_q > 0$ such that for all $x, y \in E$

$$\|x+y\|^q \leq \|x\|^q + q\langle y\ ,\ J_q(x)\rangle + d_q\|y\|^q.$$

(iii) There is a constant $c_1 > 0$ such that

$$\langle x-y\ ,\ J_q(x)-J_q(y)\rangle \ \leq c_1\|x-y\|^q, \quad \forall\ x, y \in E.$$

Lemma 2.6. *[16] Let E be a uniformly convex real Banach space. For arbitrary $r > 0$, let $B(0)_r := \{x \in E : ||x|| \leq r\}$ and $\lambda \in [0,1]$. Then there exists a continuous, strictly increasing and convex function*

$$g : [0,2r] \to \mathbb{R}^+,\ g(0) = 0,$$

such that for all $x, y \in B(0)_r$,

$$\|\lambda x + (1-\lambda)y\|^2 \leq \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - (1-\lambda)\lambda g(\|x-y\|).$$

Lemma 2.7. *[17] Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that $a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\sigma_n + \beta_n,\ n \geq 0$, where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\sigma_n\}$ satisfy the conditions:*
(i) $\alpha_n \subset (0,1),\ \sum_{n=0}^{\infty}\alpha_n = \infty$,
(ii) $\sigma_n \in \mathbb{R},\ \limsup_{n\to\infty}\ \sigma_n \leq 0$ or $\sum_{n=0}^{\infty}|\sigma_n\alpha_n| < \infty$,
(ii) $\beta_n \geq 0$ for all $n \geq 0$ with $\sum_{n=0}^{\infty}|\beta_n| < \infty$. Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.8. *[15] Let H be a real Hilbert space. Let $A : H \to H$ be a k-strongly monotone and L-Lipschitzian operator with $k > 0,\ L > 0$. Assume that $0 < \eta < \dfrac{2k}{L^2}$ and $\tau = \eta\Big(k - \dfrac{L^2\eta}{2}\Big)$. Then for each $t \in \Big(0, min\{1,\ \dfrac{1}{\tau}\}\Big)$, we have*

$$\|(I-t\eta A)x - (I-t\eta A)y\| \leq (1-t\tau)\|x-y\|,\ \forall\, x, y \in H.$$

Lemma 2.9. *Let $q > 1$ be a fixed real number and E be a q-uniformly smooth real Banach space with constant d_q. Let $A : E \to E$ be a k-strongly accretive and L-Lipschitzian operator with $k > 0,\ L > 0$. Assume that $\eta \in \Big(0, min\Big\{1,\ \Big(\dfrac{kq}{d_qL^q}\Big)^{\frac{1}{q-1}}\Big\}\Big)$ and $\tau = \eta\Big(k - \dfrac{d_qL^q\eta^{q-1}}{q}\Big)$. Then for each $t \in \Big(0, min\{1,\ \dfrac{1}{\tau}\}\Big)$, we have*

$$\|(I-t\eta A)x - (I-t\eta A)y\| \leq (1-t\tau)\|x-y\|,\ \forall\, x, y \in E. \tag{12}$$

Proof. Without loss of generality, assume $k < \dfrac{1}{q}$. Then, as $\eta < \Big(\dfrac{kq}{d_qL^q}\Big)^{\frac{1}{q-1}}$, we have $0 < qk - d_qL^q\eta^{q-1}$. Furthermore, from $k < \dfrac{1}{q}$, we have $qk - d_qL^q\eta^{q-1} < 1$ so that $0 < qk - d_qL^q\eta^{q-1} < 1$. By using (ii) of Theorem 2.5 and properties of A, it follows that

$$\begin{aligned}
\|(I-t\eta A)x - (I-t\eta A)y\|^q &\leq \|x-y\|^q + q\langle t\eta Ay - t\eta Ax\ ,\ J_q(x-y)\rangle + d_q\|t\eta Ax - t\eta Ay\|^q \\
&\leq \|x-y\|^q - qt\eta\langle Ax - Ay\ ,\ J_q(x-y)\rangle + d_q(t\eta)^q\|Ax-Ay\|^q \\
&\leq \|x-y\|^q - qtk\eta\|x-y\|^q + d_q(Lt\eta)^q\|x-y\|^q \\
&\leq \Big(1 - qtk\eta + d_qL^qt^q\eta^q\Big)\|x-y\|^q.
\end{aligned}$$

Therefore

$$\|(I-t\eta A)x-(I-t\eta A)y\| \leq \Big(1-qtk\eta+d_qL^qt\eta^q\Big)^{\frac{1}{q}}\|x-y\|. \tag{13}$$

Using definition of τ, inequality (13) and inequality $(1+x)^s \leq 1+sx$, for $x>-1$ and $0<s<1$, we have

$$\begin{aligned}\|(I-t\eta A)x-(I-t\eta A)y\| &\leq \Big(1-tk\eta+\frac{d_qL^qt\eta^q}{q}\Big)\|x-y\|\\ &\leq \Big(1-t\eta(k-\frac{d_qL^q\eta^{q-1}}{q})\Big)\|x-y\|\\ &\leq (1-t\tau)\|x-y\|,\end{aligned}$$

which gives us the required result (12). This completes the proof. □

Remark 2.10. *Lemma 2.9 is one generalization of Lemma 2.8.*

Let C be a nonempty subsets of real Banach space E. A mapping $Q_C : E \to C$ is said to be sunny if

$$Q_C(Q_Cx+t(x-Q_Cx)) = Q_Cx$$

for each $x \in E$ and $t \geq 0$. A mapping $Q_C : E \to C$ is said to be a retraction if $Q_Cx = x$ for each $x \in C$.

Lemma 2.11. *[13] Let C and D be nonempty subsets of a real Banach space E with $D \subset C$ and $Q_D : C \to D$ a retraction from C into D. Then Q_D is sunny and nonexpansive if and only if*

$$\langle z-Q_Dz, j(y-Q_Dz)\rangle \leq 0$$

for all $z \in C$ and $y \in D$.

It is noted that Lemma 2.11 still holds if the normalized duality map is replaced by the general duality map J_φ, where φ is gauge function.

Remark 2.12. *If K is a nonempty closed and convex subset of a Hilbert space H, then the nearest point projection P_K from H to K is the sunny nonexpansive retraction.*

3. Main results

In this section, we present our explicit iterative method for finding fixed points of quasi-nonexpansive mappings in real Banach spaces.

Theorem 3.1. *Let $q>1$ be a fixed real number and E be a q-uniformly smooth and uniformly convex real Banach space having a weakly continuous duality map J_φ. Let $T : E \to E$ be a quasi-nonexpansive mapping such that $F(T) \neq \emptyset$ and $I-T$ is demiclosed at origin. Let $f : E \to E$ be an b-Lipschitzian mapping with a constant $b \geq 0$. Let $A : E \to E$ be an k-strongly accretive and L-Lipschitzian operator. Assume that $\eta \in \Big(0, min\Big\{1, \ \big(\frac{kq}{d_qL^q}\big)^{\frac{1}{q-1}}\Big\}\Big)$ and $0 \leq \gamma b < \tau$ where $\tau = \eta\Big(k-\dfrac{d_qL^q\eta^{q-1}}{q}\Big)$.*
Let $\{x_n\}$ be a sequence defined iteratively from arbitrary $x_0 \in E$ by:

$$\begin{cases}\bar{x}_{n+1} = \beta_nx_n+(1-\beta_n)Tx_n,\\ x_{n+1} = \alpha_n\gamma f(x_n)+(I-\alpha_n\eta A)T\big(s_nx_n+(1-s_n)\bar{x}_{n+1}\big), \ \ n \geq 0,\end{cases} \tag{14}$$

where $\{\beta_n\}$, $\{s_n\}$ and $\{\alpha_n\}$ are sequences in $(0,1)$ satisfying the following conditions:
(i) $\lim\limits_{n\to\infty}\alpha_n = 0$, *(ii)* $\sum\limits_{n=0}^{\infty}\alpha_n = \infty$,
(iii) $\lim\limits_{n\to\infty}\inf(1-s_n)\beta_n(1-\beta_n) > 0$.
Then, the sequence $\{x_n\}$ generated by (14) converges strongly to $x^* \in F(T)$, which is a unique solution of the following variational inequality:

$$\langle \eta Ax^*-\gamma f(x^*), J_\varphi(x^*-p)\rangle \leq 0, \ \ \forall p \in F(T). \tag{15}$$

Proof. We first show that the uniqueness of a solution of the variational inequality (15). Suppose both $x^* \in F(T)$ and $x^{**} \in F(T)$ are solutions to (15), then

$$\langle \eta Ax^* - \gamma f(x^*), J_\varphi(x^* - x^{**})\rangle \leq 0 \tag{16}$$

and

$$\langle \eta Ax^{**} - \gamma f(x^{**}), J_\varphi(x^{**} - x^*)\rangle \leq 0. \tag{17}$$

Adding up (16) and (17) yields

$$\langle \eta Ax^{**} - \eta Ax^* + \gamma f(x^*) - \gamma f(x^{**}), J_\varphi(x^{**} - x^*)\rangle \leq 0. \tag{18}$$

$$\begin{aligned} \frac{d_q L^q \eta^{q-1}}{q} > 0 \quad &\Longleftrightarrow \quad k - \frac{d_q L^q \eta^{q-1}}{q} < k \\ &\Longleftrightarrow \quad \eta\Big(k - \frac{d_q L^q \eta^{q-1}}{q}\Big) < k\eta \\ &\Longleftrightarrow \quad \tau < k\eta. \end{aligned}$$

It follows that

$$0 \leq \gamma b < \tau < k\eta.$$

Noticing that

$$\langle \eta Ax^{**} - \eta Ax^* + \gamma f(x^*) - \gamma f(x^{**}), J_\varphi(x^{**} - x^*)\rangle \geq (k\eta - b\gamma)\varphi(\|x^* - x^{**}\|)\|x^* - x^{**}\|,$$

which implies that $x^* = x^{**}$ and the uniqueness is proved. Below we use x^* to denote the unique solution of (15). Without loss of generality, we can assume $\alpha_n \in \Big(0, min\{1\ , \frac{1}{\tau}\}\Big)$.
For each $n \geq 1$, we put $z_n := s_n x_n + (1 - s_n)\bar{x}_{n+1}$. Let $p \in F(T)$, we have

$$\begin{aligned} \|z_n - p\| &= \|s_n x_n + (1 - s_n)\bar{x}_{n+1} - p\| \\ &\leq s_n\|x_n - p\| + (1 - s_n)\|\bar{x}_{n+1} - p\| \\ &\leq s_n\|x_n - p\| + (1 - s_n)\|\beta_n x_n + (1 - \beta_n)Tx_n - p\| \\ &\leq s_n\|x_n - p\| + (1 - s_n)\Big[\beta_n\|x_n - p\| + (1 - \beta_n)\|Tx_n - p\|\Big]. \end{aligned}$$

Hence,

$$\|z_n - p\| \leq \|x_n - p\|. \tag{19}$$

By Lemma 2.9 and inequality (19), we have

$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_n\gamma f(x_n) + (I - \alpha_n\eta A)Tz_n - p\| \\ &\leq \|\alpha_n\gamma f(x_n) + (I - \eta\alpha_n A)Tz_n - p\| \\ &\leq \alpha_n\gamma\|f(x_n) - f(p)\| + (1 - \tau\alpha_n)\|Tz_n - p\| + \alpha_n\|\gamma f(p) - \eta Ap\| \\ &\leq (1 - \alpha_n(\tau - b\gamma))\|x_n - p\| + \alpha_n\|\gamma f(p) - \eta Ap\| \\ &\leq \max\{\|x_n - p\|, \frac{\|\gamma f(p) - \eta Ap\|}{\tau - b\gamma}\}. \end{aligned}$$

By induction, it is easy to see that

$$\|x_n - p\| \leq \max\{\|x_0 - p\|, \frac{\|\gamma f(p) - \eta Ap\|}{\tau - b\gamma}\}, \quad n \geq 1.$$

Hence $\{x_n\}$ is bounded also are $\{f(x_n)\}$ and $\{Ax_n\}$.
Using Lemma 2.6, convexity of $\|.\|^2$ and (14), we have

$$\begin{aligned}
\|Tz_n - p\|^2 &\leq \|z_n - p\|^2 \\
&= \|s_n x_n + (1-s_n)\bar{x}_{n+1} - p\|^2 \\
&\leq s_n\|x_n - p\|^2 + (1-s_n)\|\bar{x}_{n+1} - p\|^2 \\
&\leq s_n\|x_n - p\|^2 + (1-s_n)\|\beta_n x_n + (1-\beta_n)Tx_n - p\|^2 \\
&\leq s_n\|x_n - p\|^2 + (1-s_n)\Big[\beta_n\|x_n - p\|^2 + (1-\beta_n)\|Tx_n - p\|^2 - \beta_n(1-\beta_n)g(\|x_n - Tx_n\|)\Big] \\
&\leq \|x_n - p\|^2 - (1-s_n)\beta_n(1-\beta_n)g(\|x_n - Tx_n\|).
\end{aligned}$$

Therefore, by Lemmas 2.6 and 2.9, we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\alpha_n\gamma f(x_n) + (I - \alpha_n\eta A)Tz_n - p\|^2 \\
&= \|\alpha_n\gamma\big(f(x_n) - f(p)\big) + (I - \alpha_n\eta A)\big(Tz_n - p\big) + \alpha_n\gamma f(p) - \alpha_n\eta Ap\|^2 \\
&\leq \|\alpha_n\gamma\big(f(x_n) - f(p)\big) + (I - \alpha_n\eta A)\big(Tz_n - p\big)\|^2 + 2\alpha_n\langle\eta Ap - \gamma f(p), J(p - x_{n+1})\rangle \\
&\leq \alpha_n\gamma b\|x_n - p\|^2 + (1 - \tau\alpha_n)\|Tz_n - p\|^2 + 2\alpha_n\langle\eta Ap - \gamma f(p), J(p - x_{n+1})\rangle \\
&\leq \alpha_n\gamma b\|x_n - p\|^2 + (1 - \tau\alpha_n)\Big[\|x_n - p\|^2 - (1-s_n)\beta_n(1-\beta_n)g(\|x_n - Ty_n\|)\Big] \\
&\quad +2\alpha_n\langle\eta Ap - \gamma f(p), J(p - x_{n+1})\rangle \\
&\leq (1 - \alpha_n(\tau - b\gamma))\|x_n - p\|^2 - (1 - \tau\alpha_n)(1-s_n)\beta_n(1-\beta_n)g(\|x_n - Tx_n\|) \\
&\quad +2\alpha_n\langle\eta Ap - \gamma f(p), J(p - x_{n+1})\rangle.
\end{aligned}$$

Therefore,

$$(1-\tau\alpha_n)(1-s_n)\beta_n(1-\beta_n)g(\|x_n - Tx_n\|) \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2\alpha_n\langle\eta Ap - \gamma f(p), J(p - x_{n+1})\rangle. \quad (20)$$

Since $\{x_n\}$ is bounded, then there exists a constant $B > 0$ sucht that

$$\langle\eta Ap - \gamma f(p), J(p - x_{n+1})\rangle \leq B, \ \ \forall n \geq 0.$$

Hence,

$$(1 - \tau\alpha_n)(1 - s_n)\beta_n(1 - \beta_n)g(\|x_n - Tx_n\|) \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2\alpha_n B. \quad (21)$$

Now we prove that $\{x_n\}$ converges strongly to x^*.
We divide the proof into two cases.
Case 1. Assume that the sequence $\{\|x_n - p\|\}$ is monotonically decreasing sequence. Then $\{\|x_n - p\|\}$ is convergent. Clearly, we have

$$\|x_n - p\|^2 - \|x_{n+1} - p\|^2 \to 0.$$

It then implies from (21) that

$$\lim_{n\to\infty}(1 - \tau\alpha_n)(1 - s_n)\beta_n(1 - \beta_n)g(\|x_n - Tx_n\|) = 0. \quad (22)$$

Using the fact that $\lim_{n\to\infty}\inf(1 - s_n)\beta_n(1 - \beta_n) > 0$ and property of g, we have

$$\lim_{n\to\infty}\|x_n - Tx_n\| = 0. \quad (23)$$

Let t_0 be a fixed real number such that $t_0 \in \Big(0, min\{1\ , \dfrac{1}{\tau}\}\Big)$. We observe that $Q_{F(T)}(I + (t_0\gamma f - t_0\eta A))$ is a contraction, where $Q_{F(T)}$ is the sunny nonexpansive retraction from E to $F(T)$. Indeed, for all $x, y \in E$, by Lemma 2.9, we have

$$\begin{aligned}
\|Q_{F(T)}(I+(t_0\gamma f-t_0\eta A))x-Q_{F(T)}(I+(t_0\gamma f-t_0\eta A))y\| &\le \|(I+(t_0\gamma f-t_0\eta A))x-(I+(t_0\gamma f-t_0\eta A))y\|\\
&\le t_0\gamma\|f(x)-f(y)\|+\|(I-t_0\eta A)x-(I-t_0\eta A)y\|\\
&\le (1-t_0(\tau-b\gamma))\|x-y\|.
\end{aligned}$$

Banach's Contraction Mapping Principle guarantees that $Q_{F(T)}(I+(t_0\gamma f-t_0\eta A))$ has a unique fixed point, say $x_1\in E$. That is, $x_1=Q_{F(T)}(I+(t_0\gamma f-t_0\eta A))x_1$. Thus, in view of Lemma 2.11, it is equivalent to the following variational inequality problem

$$\langle \eta Ax_1-\gamma f(x_1), J_\varphi(x_1-p)\rangle\le 0,\quad\forall\ \ p\in F(T).$$

By the uniqueness of the solution of (15), we have $x_1=x^*$.
Next, we prove that $\limsup\limits_{n\to+\infty}\langle \eta Ax^*-\gamma f(x^*), J_\varphi(x^*-x_n)\rangle$. Since E is reflexive and $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that x_{n_k} converges weakly to a in E and

$$\limsup_{n\to+\infty}\langle \eta Ax^*-\gamma f(x^*), J_\varphi(x^*-x_n)\rangle=\lim_{k\to+\infty}\langle \eta Ax^*-\gamma f(x^*), J_\varphi(x^*-x_{n_k})\rangle.$$

From (23) and $I-T$ is demiclosed, we obtain $a\in F(T)$. On other hand, the assumption that the duality mapping J_φ is weakly continuous and the fact that x^* solves variational inequality (15), we then have

$$\begin{aligned}
\limsup_{n\to+\infty}\langle \eta Ax^*-\gamma f(x^*), J_\varphi(x^*-x_n)\rangle &= \lim_{k\to+\infty}\langle \eta Ax^*-\gamma f(x^*), J_\varphi(x^*-x_{n_k})\rangle\\
&= \langle \eta Ax^*-\gamma f(x^*), J_\varphi(x^*-a)\rangle\le 0.
\end{aligned}$$

Finally, we show that $x_n\to x^*$. In fact, since $\Phi(t)=\int_0^t\varphi(\sigma)d\sigma,\ \forall t\ge 0$, and φ is a gauge function, then for $1\ge k\ge 0,\ \Phi(kt)\le k\Phi(t)$. From (14), Lemmas 2.6 and 2.9, we get that

$$\begin{aligned}
\Phi(\|x_{n+1}-x^*\|) &= \Phi(\|\alpha_n\gamma f(x_n)+(I-\eta\alpha_n A)Tz_n-x^*\|)\\
&\le \Phi(\|\alpha_n(\gamma f(x_n)-\gamma f(x^*)+(I-\alpha_n\eta A)(Tz_n-x^*)\|)\\
&\quad +\alpha_n\langle \eta Ax^*-\gamma f(x^*), J_\varphi(x^*-x_{n+1})\rangle\\
&\le \Phi(\alpha_n\gamma\|f(x_n)-f(x^*)\|+\|(I-\alpha_n\eta A)(Tz_n-x^*)\|)\\
&\quad +\alpha_n\langle \eta Ax^*-\gamma f(x^*), J_\varphi(x^*-x_{n+1})\rangle\\
&\le \Phi(\alpha_n b\gamma\|x_n-x^*\|+(1-\alpha_n\tau)\|Tz_n-x^*\|)\\
&\quad +\alpha_n\langle \eta Ax^*-\gamma f(x^*), J_\varphi(x^*-x_{n+1})\rangle\\
&\le \Phi((1-\alpha_n(\tau-b\gamma))\|x_n-x^*\|)+\alpha_n\langle \eta Ax^*-\gamma f(x^*), J_\varphi(x^*-x_{n+1})\rangle\\
&\le (1-\alpha_n(\tau-b\gamma))\Phi(\|x_n-x^*\|)+\alpha_n\langle \eta Ax^*-\gamma f(x^*), J_\varphi(x^*-x_{n+1})\rangle.
\end{aligned}$$

From Lemma 2.7, its follows that $x_n\to x^*$.
Case 2. Assume that the sequence $\{\|x_n-x^*\|\}$ is not monotonically decreasing sequence. Set $B_n=\|x_n-x^*\|$ and $\tau:\mathbb{N}\to\mathbb{N}$ be a mapping for all $n\ge n_0$ (for some n_0 large enough) by $\tau(n)=\max\{k\in\mathbb{N}: k\le n,\ \ B_k\le B_{k+1}\}$.
We have τ is a non-decreasing sequence such that $\tau(n)\to\infty$ as $n\to\infty$ and $B_{\tau(n)}\le B_{\tau(n)+1}$ for $n\ge n_0$. From (21), we have

$$(1-\tau\alpha_{\tau(n)})(1-s_{\tau(n)})\beta_{\tau(n)}(1-\beta_{\tau(n)})g(\|x_{\tau(n)}-Tx_{\tau(n)}\|)\le 2\alpha_{\tau(n)}B\to 0\ \text{ as }\ n\to\infty.$$

Hence,

$$\lim_{n\to\infty}\|x_{\tau(n)}-Tx_{\tau(n)}\|=0. \tag{24}$$

By same argument as in case 1, we can show that $x_{\tau(n)}$ converges weakly in E and $\limsup_{n\to+\infty}\langle \eta Ax^* - \gamma f(x^*), J_\varphi(x^* - x_{\tau(n)})\rangle \leq 0$. We have for all $n \geq n_0$,

$$0 \leq \Phi(\|x_{\tau(n)+1}-x^*\|)-\Phi(\|x_{\tau(n)}-x^*\|) \leq \alpha_{\tau(n)}[-(\tau-b\gamma)\Phi(\|x_{\tau(n)}-x^*\|)+\langle \eta Ax^*-\gamma f(x^*), J_\varphi(x^*-x_{\tau(n)+1})\rangle],$$

which implies that

$$\Phi(\|x_{\tau(n)} - x^*\|) \leq \frac{1}{\tau - b\gamma}\langle \eta Ax^* - \gamma f(x^*), J_\varphi(x^* - x_{\tau(n)+1})\rangle.$$

Then, we have

$$\lim_{n\to\infty} \Phi(\|x_{\tau(n)} - x^*\|) = 0.$$

Therefore,

$$\lim_{n\to\infty} B_{\tau(n)} = \lim_{n\to\infty} B_{\tau(n)+1} = 0.$$

Furthermore, for all $n \geq n_0$, we have $B_{\tau(n)} \leq B_{\tau(n)+1}$ if $n \neq \tau(n)$ (that is, $n > \tau(n)$); because $B_j > B_{j+1}$ for $\tau(n) + 1 \leq j \leq n$. As consequence, we have for all $n \geq n_0$,

$$0 \leq B_n \leq \max\{B_{\tau(n)},\ B_{\tau(n)+1}\} = B_{\tau(n)+1}.$$

Hence, $\lim_{n\to\infty} B_n = 0$, that is $\{x_n\}$ converges strongly to x^*. This completes the proof. □

We now apply Theorem 3.1 for finding fixed points of nonexpansive mappings without demiclosedness assumption.

Theorem 3.2. *Let $q > 1$ be a fixed real number and E be a q-uniformly smooth and uniformly convex real Banach space having a weakly continuous duality map J_φ. Let $T : E \to E$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Let $f : E \to E$ be an b-Lipschitzian mapping with a constant $b \geq 0$. Let $A : E \to E$ be an k-strongly accretive and L-Lipschitzian operator. Assume that $\eta \in \Big(0, min\Big\{1,\ \Big(\frac{kq}{d_qL^q}\Big)^{\frac{1}{q-1}}\Big\}\Big)$ and $0 \leq \gamma b < \tau$ where $\tau = \eta\Big(k - \frac{d_qL^q\eta^{q-1}}{q}\Big)$.*
Let $\{x_n\}$ be a sequence defined iteratively from arbitrary $x_0 \in E$ by:

$$\begin{cases} \bar{x}_{n+1} = \beta_n x_n + (1-\beta_n)Tx_n, \\ x_{n+1} = \alpha_n\gamma f(x_n) + (I - \alpha_n\eta A)T\big(s_nx_n + (1-s_n)\bar{x}_{n+1}\big),\ \ n \geq 0, \end{cases} \tag{25}$$

where $\{\beta_n\}$, $\{s_n\}$ and $\{\alpha_n\}$ are sequences in $(0,1)$ satisfying the following conditions:
(i) $\lim_{n\to\infty} \alpha_n = 0$, $\quad (ii) \sum_{n=0}^{\infty} \alpha_n = \infty$,
(iii) $\lim_{n\to\infty} \inf(1-s_n)\beta_n(1-\beta_n) > 0$.
Then, the sequence $\{x_n\}$ generated by (25) converges strongly to $x^* \in F(T)$, which is a unique solution of the following variational inequality:

$$\langle \eta Ax^* - \gamma f(x^*), J_\varphi(x^* - p)\rangle \leq 0, \quad \forall p \in F(T). \tag{26}$$

Proof. Since every nonexpansive mapping is quasi-nonexpansive mapping, then the proof follows Lemma 2.3 and Theorem 3.1. □

Corollary 3.3. *Assume that $E = l_q$, $1 < q < \infty$. Let $T : E \to E$ be a quasi-nonexpansive mapping such that $F(T) \neq \emptyset$ and $I - T$ is demiclosed at origin. Let $f : E \to E$ be an b-Lipschitzian mapping with a constant $b \geq 0$. Let $A : E \to E$ be an k-strongly accretive and L-Lipschitzian operator. Assume that*

$\eta \in \Big(0, min\Big\{1,\ \big(\frac{kq}{d_q L^q}\big)^{\frac{1}{q-1}}\Big\}\Big)$ *and* $0 \leq \gamma b < \tau$ *where* $\tau = \eta\Big(k - \frac{d_q L^q \eta^{q-1}}{q}\Big)$.
Let $\{x_n\}$ *be a sequence defined iteratively from arbitrary* $x_0 \in E$ *by:*

$$\begin{cases} \bar{x}_{n+1} = \beta_n x_n + (1-\beta_n)Tx_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n \eta A)T\big(s_n x_n + (1-s_n)\bar{x}_{n+1}\big), \ \ n \geq 0, \end{cases} \tag{27}$$

where $\{\beta_n\}$, $\{s_n\}$ *and* $\{\alpha_n\}$ *are sequences in* $(0,1)$ *satisfying the following conditions:*
$(i)\ \lim_{n\to\infty} \alpha_n = 0, \quad (ii) \sum_{n=0}^{\infty} \alpha_n = \infty,$
$(iii)\ \lim_{n\to\infty} \inf(1-s_n)\beta_n(1-\beta_n) > 0.$
Then, the sequence $\{x_n\}$ *generated by* (27) *converges strongly to* $x^* \in F(T)$, *which is a unique solution of the following variational inequality* (15).

Proof. Since $E = l_q,\ 1 < q < \infty$ are uniformly convex and has a weakly continuous duality map . The proof follows from Theorem 3.1. □

Corollary 3.4 (Marino et al. [9]). *Let* H *be a real Hibert space. Let* $T : H \to H$ *be a quasi-nonexpansive mapping such that* $F(T) \neq \emptyset$ *and* $I - T$ *is demiclosed at origin. Let* $f : H \to H$ *be an b-contraction mapping with a constant* $b \in [0,1)$.
Let $\{x_n\}$ *be a sequence defined iteratively from arbitrary* $x_0 \in H$ *by:*

$$\begin{cases} \bar{x}_{n+1} = \beta_n x_n + (1-\beta_n)Tx_n, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T\big(s_n x_n + (1-s_n)\bar{x}_{n+1}\big), \ \ n \geq 0, \end{cases} \tag{28}$$

where $\{\beta_n\}$, $\{s_n\}$ *and* $\{\alpha_n\}$ *are sequences in* $(0,1)$ *satisfying the following conditions:*
$(i)\ \lim_{n\to\infty} \alpha_n = 0, \quad (ii) \sum_{n=0}^{\infty} \alpha_n = \infty,$
$(iii)\ \lim_{n\to\infty} \inf(1-s_n)\beta_n(1-\beta_n) > 0.$
Then, the sequences $\{x_n\}$ *generated by* (28) *converges strongly to* $x^* \in F$, *which is a unique solution of the following variational inequality*

$$\langle x^* - f(x^*), x^* - p\rangle \leq 0, \quad \forall p \in F(T). \tag{29}$$

4. Application

In this section, we apply Theorem 3.1 for quadratic optimization problem.

Problem 4.1. *Let* H *be a real Hilbert space and* T *be a quasi-nonexpansive mappings on* H *such that* $F(T) \neq \emptyset$. *We consider the following constrained convex minimization problem:*

$$\min_{x \in F(T)} \frac{1}{2}\langle Ax, x\rangle, \tag{30}$$

where $A : H \to H$ *be a strongly positive bounded linear operator.*

Remark 4.2. *A necessary condition of optimality for a point* $x^* \in F(T)$ *to be a solution of the minimization problem* (30) *is that* x^* *solves the following variational inequality problem:*

$$\langle Ax^*, x^* - p\rangle \leq 0$$

for all $p \in F(T)$.

Consequently, the following theorem is obtained.

Theorem 4.3. *Let H be a real Hilbert space. Let $A : H \to H$ be strongly bounded linear operator with coefficient $k > 0$. Let $T : H \to H$ be a quasi-nonexpansive mapping such that $F(T) \neq \emptyset$ and $I - T$ is demiclosed at origin. Let $\{x_n\}$ be a sequence defined iteratively from arbitrary $x_0 \in H$ by:*

$$\begin{cases} \bar{x}_{n+1} = \beta_n x_n + (1-\beta_n)Tx_n, \\ x_{n+1} = (I - \alpha_n \eta A)T\big(s_n x_n + (1-s_n)\bar{x}_{n+1}\big), \;\; n \geq 0, \end{cases} \tag{31}$$

where $\{\beta_n\}$, $\{s_n\}$ and $\{\alpha_n\}$ are sequences in $(0,1)$ satisfying the following conditions:

(i) $\lim\limits_{n\to\infty} \alpha_n = 0$, $\quad (ii) \sum\limits_{n=0}^{\infty} \alpha_n = \infty$,

(iii) $\lim\limits_{n\to\infty} \inf(1-s_n)\beta_n(1-\beta_n) > 0$.

Assume that $\eta \in \Big(0, min\Big\{1, \dfrac{2k}{\|A\|^2}\Big\}\Big)$,

Then, the sequence $\{x_n\}$ generated by (31) *converges strongly to a solution of* (30).

Proof. We note that strongly positive bounded linear operator A is a $\|A\|$-Lipschitzian and k- strongly monotone operator. Using Remark 4.2, the proof follows Theorem 3.1 with $f = 0$. □

Remark 4.4. *Our results are applicable for the family of nonexpansive mappings, for example W_n-mapping, a countable family of nonexpansive mappings, and nonexpansive semigroups.*
The proof methods of our result are very different from the ones Marino et al. [9].

Remark 4.5. *Let $\alpha_n = \frac{1}{10n+1}$, $\beta_n = \frac{1}{20n+1} + 0.4$ and $s_n = \frac{1}{30n+1} + 0.3$. It is easy to see that the sequences $\{\alpha_n\}, \{\beta_n\}$, and $\{s_n\}$ satisfy the conditions (i), (ii) and (iii) of Theorem 3.1.*

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