Functional Analysis, Approximation and Computation
12 (1) (2020), 15–22

Published by Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/faac

Polynomials associated by Humbert polynomials

Snežana S. Djordjević[a], Gospava B. Djordjević[a]

[a]*University of Niš, Faculty of Technology in Leskovac, 16000 Leskovac, Serbia*

Abstract. In this note we define the polynomials $w_{n,m}^{(r,s)}(x)$ where $r+s>1$, $m\geq 2$, which are related with the generalized Humbert polynomials $u_{n,m}^{(r)}(x)$. Here we find many recurrence relations and explicit representations for $w_{n,m}^{(r,s)}(x)$. Also, we present some special classes of the polynomials $u_{n,m}^{(r)}(x)$.

1. Introduction

In the paper [9] the polynomials $u_{n,m}^{(r)}(x)$ are introduced by

$$F(x,t)=(1-p(x)t-q(x)t^m)^{-r}=\sum_{n=0}^{\infty}u_{n+1,m}^{(r)}(x)t^n. \tag{1}$$

Namely, the polynomials $u_{n,m}^{(r)}(x)$ are the generalized Humbert polynomials $P_n(m,x,y,p,c)$ which are defined by ([6])

$$\sum_{n=0}^{\infty}P_n(m,x,y,p,c)t^n=(c-mxt+yt^m)^p. \tag{2}$$

Clearly, depending on the choice of the functions $p(x)$ and $q(x)$, and also on the choice of the parameters m and r, the polynomials $u_{n,m}^{(r)}(x)$ present the wide class of the known polynomials, which we consider at the end of this manuscript.

First we give some important properties of the polynomials $u_{n,m}^{(r)}(x)$ ([9]).

Differentiating both sides of (1) to t and comparing the coefficients on t^n, we find the following recurrence relation

$$nu_{n+1,m}^{(r)}(x)=p(x)(r+n-1)u_{n,m}^{(r)}+q(x)(mr+n-m)u_{n+1-m,m}^{(r)}(x). \tag{3}$$

2010 *Mathematics Subject Classification*. Primary11B83 (mandatory); Secondary 11B37, 11B39. (optionally)
Keywords. Generating function; Explicit formula; Recurrence relation; Convolution; Differential equation.
Received: 18 December 2018; Accepted: 3 October 2019
Communicated by Dijana Mosić
Email addresses: snezanadjordjevicle@gmail.com (Snežana S. Djordjević), gospava48@gmail.com (Gospava B. Djordjević)

Using (1) again, we can obtain the explicit formula

$$u_{n+1,m}^{(r)}(x) = \sum_{n=0}^{\infty} \frac{(r)_{n-(m-1)k}}{k!(n-mk)!}(p(x))^{n-mk}(q(x))^k, \tag{4}$$

where

$$u_{n+1,m}^{(r)}(x) = \frac{(r)_n}{n!}(p(x))^n, \quad n = 0, 1, \ldots, m-1,$$

and $(r)_n = r(r+1)\cdots(r+n-1),\ r \neq 0, -1, \ldots, 1-n$.

Next, using the relations ([8])

$$(r)_{n-k} = \frac{(-1)^k (r)_n}{(1-r-n)_k}, \quad (n-k)! = \frac{(-1)^k n!}{(-n)_k},$$

we have

$$\frac{(r)_{n-(m-1)k}}{(n-mk)!} = \frac{(-1)^{mk}(r)_n(-n)_{mk}}{n!(1-r-n)_{(m-1)k}}.$$

Hence, the representation (4) becomes

$$u_{n+1,m}^{(r)}(x) = \frac{(r)_n(-p(x))^n}{n!} \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}(q(x))^k}{(1-r-n)_{(m-1)k}k!(-(p(x))^{mk}}. \tag{5}$$

Further, we introduce the polynomials $v_{n,m}^{(s)}(x)$ $(s \geq 1)$ by

$$V(x,t) = \left(\frac{2-p(x)t}{1-p(x)t-q(x)t^m}\right)^s = \sum_{n=0}^{\infty} v_{n,m}^{(s)}(x)t^n. \tag{6}$$

From (1) and (6), we find the following explicit formula

$$v_{n,m}^{(s)}(x) = 2^s \sum_{j=0}^{s} (-1)^j \binom{s}{j} \left(\frac{p(x)}{2}\right)^j u_{n+1-j,m}^{(s)}(x), \tag{7}$$

or

$$v_{n,m}^{(s)}(x) = \sum_{j=0}^{s} (-1)^j \binom{s}{j} 2^{s-j} \sum_{k=0}^{[(n-j)/m]} \frac{(s)_{n-j-(m-1)k}}{k!(n-j-mk)!}(p(x))^{n-1-mk}(q(x))^k. \tag{8}$$

The polynomials $v_{n,m}^{(s)}(x)$ are the s-th convolutions of the polynomials $v_{n,m}^{(1)}(x) = v_{n,m}(x)$.

2. Mixed convolutions

In this section we introduce the polynomials $w_{n,m}^{(r,s)}(x)$, $r+1 \geq 1$, by

$$F_m(x,t) = \frac{(2-p(x)t)^s}{(1-p(x)t-q(x)t^m)^{r+s}} = \sum_{n=0}^{\infty} w_{n,m}^{(r,s)}(x)t^n. \tag{9}$$

Theorem 2.1. *The polynomials $w_{n,m}^{(r,s)}(x)$ have the following explicit representation*

$$w_{n,m}^{(r,s)}(x) = \sum_{i=0}^{s} 2^{s-i}\binom{s}{i}(-1)^i (p(x))^i u_{n+1-i,m}^{(r+s)}(x). \tag{10}$$

Proof. Using (1), and from (9), we find

$$\begin{aligned} F_m(x,t) &= (2-p(x)t)^s \sum_{n=0}^{\infty} u_{n+1,m}^{(r+s)}(x)t^n \\ &= \sum_{n=0}^{\infty}\sum_{i=0}^{s}(-1)^i\binom{s}{i}2^{s-i}(p(x))^i u_{n+1,m}^{(r+s)}(x)t^{n+i} \\ &= \sum_{n=0}^{\infty}\sum_{i=0}^{s}(-1)^i\binom{s}{i}2^{s-i}(p(x))^i u_{n+1-i,m}^{(r+s)}(x)t^{n}. \end{aligned}$$

So, from the last equalities, we conclude that the formula (10) is correct. □

Theorem 2.2. *The polynomials $w_{n,m}^{(r,s)}(x)$ satisfy the following explicit formula*

$$w_{n,m}^{(r,s)}(x) = \sum_{i=0}^{s-j}(-1)^i\binom{s-j}{i}2^{s-j-i}(p(x))^i w_{n-i,m}^{(r+s-j,j)}(x). \tag{11}$$

Proof. It holds

$$\frac{(2-p(x)t)^s}{(1-p(x)t-q(x)t^m)^{r+s}} = \frac{(2-p(x)t)^{s-j}}{(1-p(x)t-q(x)t^m)^{r+s-j}} \cdot \left(\frac{2-p(x)t}{1-p(x)t-q(x)t^m}\right)^j,$$

so

$$\begin{aligned} F_m(x,t) &= (2-p(x)t)^{s-j}\sum_{n=0}^{\infty} w_{n,m}^{(r+s-j,j)}(x)t^n \\ &= \sum_{n=0}^{\infty}\sum_{i=0}^{s-j}\binom{s-j}{i}2^{s-j-i}(-1)^i(p(x))^i w_{n-i,m}^{(r+s-j,j)}(x)t^n. \end{aligned}$$

From the last equalities we conclude that (11) holds. □

Remark 2.3. *If $r=0$, then we get $w_{n,m}^{(0,s)} = v_{n,m}^{(s)}(x)$; and if $s=0$ then we have $w_{n,m}^{(r,0)}(x) = u_{n+1,m}^{(r)}(x)$.*

Theorem 2.4. *For the polynomials $w_{n,m}^{(r,s)}(x)$ $(r+s>1)$ it holds*

$$w_{n,m}^{(r+s,r+s)}(x) = \sum_{k=0}^{n} w_{n-k,m}^{(s,r)}(x)w_{k,m}^{(r,s)}(x). \tag{12}$$

Proof. It is easy to prove the relation (12) starting from the generating function (9). □

Remark 2.5. *For $r = s$ the relation (12) yields*

$$w_{n,m}^{(2r,2r)}(x) = \sum_{k=0}^{n} w_{n-k,m}^{(r,r)}(x) w_{k,m}^{(r,r)}(x).$$

3. Some special cases

In this section we consider the special case of the polynomials $u_{n,m}^{(r)}(x)$ - the generalized Humbert polynomials ([9]).

1. For $p(x) = 2x + 1$, $q(x) = 1$, from (1), we have

$$F(x,t) = (1 - (2x+1)t - t^m)^{-r} = \sum_{n=0}^{\infty} u_{n+1,m}^{(r)}(x) t^n. \tag{13}$$

Differentiating (13) to x, one-by-one, s-times, we get

$$D_{x^s}\{u_{n+1,m}^{(r)}(x)\} = (r)_s \cdot 2^s u_{n+1-s,m}^{(r+s)}(x).$$

2. If $p(x) = 2x$ and $q(x) = -1$, then (1) becomes ([7])

$$F(x,t) = (1 - 2xt + t^m)^{-r} = \sum_{n=0}^{\infty} p_{n,m}^r(x) t^n, \tag{14}$$

where $p_{n,m}^r(x)$ are the special case of Humbert polynomials:

$$p_{n,m}^r(x) = \left(\frac{2}{m}\right)^r P_n\left(m, x, \frac{m}{2}, -r, \frac{m}{2}\right),$$

or

$$p_{n,m}^r(x) = \Pi_{n,m}^r\left(\frac{2x}{m}\right),$$

where $\Pi_{n,m}(x)$ are the generalized Humbert polynomials.

Many properties of the polynomials $p_{n,m}^r(x)$ are given in [5].

Now we give one of its interesting properties.

The polynomial $p_{n,m}^r(x)$ is a particular solution of the following differential equation

$$y^{(m)}(x) + \sum_{s=0}^{m} a_s x^s y^{(s)}(x) = 0 \tag{15}$$

with coefficients

$$a_s = \frac{2^m}{s!m} \Delta^s f_0 \quad (s = 0, 1, \ldots, m), \tag{16}$$

where

$$f(t) = f_t = (n-t)\left(\frac{n - t + m(r+t)}{m}\right)_{m-1}.$$

Example 3.1. *For $m = 2$, from (15) and (16), we get the following differential equation*

$$(1 - x^2)y''(x) - (2r + 1)xy'(x) + n(n + 2r)y(x) = 0, \tag{17}$$

which corresponds to polynomials $G_n^r(x)$ - Gegenbauer polynomials ([7]).

Furthermore, for $r = \dfrac{1}{2}$ in (17), we have the next differential equation

$$(1 - x^2)y''(x) - 2xy'(x) + n(n + 1)y(x) = 0,$$

which corresponds to Legendre polynomials.

3. If $p(x, y) = 2(x + y)$ and $q(x, y) = -(2xy + 1)$, then we have the polynomials $G_n^r(x, y)$ - the generalized Gegenbauer polynomials with two variables x and y:

$$F = (1 - 2(x + y)t + (2xy + 1)t^m)^{-r} = \sum_{n=0}^{\infty} G_n^r(x, y)t^n. \tag{18}$$

Thus, from (18), we have the following explicit formula

$$G_n^r(x, y) = \sum_{k=0}^{[n/m]} \frac{(-1)^k (r)_{n-(m-1)k}}{k!(n - mk)!}(2x + 2y)^{n-mk}(2xy + 1)^k. \tag{19}$$

Easily, for $y = 0$ we get $G_n^r(x, 0) = G_n^r(x)$ ([7]).

Next, we are going to prove that the polynomials $G_n^r(x, y)$ have the hypergeometric representation. Namely, the following statement holds.

Theorem 3.2. *We have*

$$G_n^r(x, y) = \frac{2^n (r)_n}{n!}(x + y)^n {}_mF_{m-1}[a; b; z], \tag{20}$$

where

$$a = -\frac{n}{m}, \frac{1 - n}{m}, \ldots, \frac{m - 1 - n}{m}; \tag{21}$$

$$b = \frac{1 - r - n}{m - 1}, \frac{2 - r - n}{m - 1}, \ldots, \frac{m - 1 - r - n}{m - 1}; \tag{22}$$

$$z = \frac{m^m (2xy + 1)}{(m - 1)^{m-1}(2x + 2y)^m}. \tag{23}$$

Proof. Using the known relations, as well as the relations (21) - (23), it is easy to prove the relation (20). □

Remark 3.3. *For $m = 2$, from (19) and (21) - (23), we obtain*

$$G_n^r(x, y) = \frac{2^n (r)_n}{n!} {}_2F_1\left[-\frac{n}{2}, \frac{1 - n}{2}; 1 - r - n; \frac{2xy + 1}{(x + y)^2}\right].$$

4. If $p(x) = x$ and $q(x) = -1$, then we have ([4])

$$F(x,t) = (1 - xt + t^m)^{-r} = \sum_{n=0}^{\infty} V_{n,m}^{r-1}(x)t^n,$$

where $V_{n,m}^{r-1}(x)$ are the convolutions of the generalized Chebyshev polynomials.

5. If $p(x) = 1 + x + x^2$ and $q(x) = -\lambda x^2$, for $m = 2$, then we get the polynomials $f_n^{(\lambda,r)}(x)$ - Dilcher polynomials ([1]):

$$(1 - (1 + x + x^2)t + \lambda x^2 t^2)^{-r} = \sum_{n=0}^{\infty} f_n^{(\lambda,r)}(x)t^n, \ \lambda > 0, \ r > 1/2. \tag{24}$$

Easily, these polynomials are related by Gegenbauer polynomials $G_n^r(x)$ as follows

$$f_n^{(\lambda,r)}(x) = x^n \lambda^{n/2} G_n^r\left(\frac{1 + x + x^2}{2x\sqrt{\lambda}}\right).$$

6. If $p(x) = x$, $q(x) = 1$ and $r = 1$, then

$$F(x,t) = (1 - xt - t^m)^{-1} = \sum_{n=0}^{\infty} f_{n+1,m}(x)t^n, \tag{25}$$

where $f_{n,m}(x)$ are the generalized Fibonacci polynomials.

So, differentiating (25) to x, one-by-one r-times, we get

$$\frac{\partial^r F(x,t)}{\partial x^r} = \frac{r!\, t^{r+1}}{(1 - xt - t^m)^{r+1}} = \sum_{n=0}^{\infty} D_{x^r}\{f_{n+1,m}(x)\}t^{n-1-k}. \tag{26}$$

Thus, we see that, by (1) and (26),

$$D_{x^r}\{f_{n+1,m}(x)\} = r!\, u_{n-r,m}^{(r+1)}(x).$$

7. For $p(x) = x$ and $q(x) = -2$, by (1) it follows

$$F(x,t) = (1 - xt + 2t^m)^{-r} = \sum_{n=0}^{\infty} a_{n,m}^{(r-1)}(x)t^n,$$

where $a_{0,m}^{(r-1)}(x) = 0$, $a_{n,m}^{(r-1)}(x) = \dfrac{(r)_n x^n}{n!}$, $n = 0, 1, \ldots, m-1$.

The polynomials $a_{n,m}^{(r-1)}(x)$ are the generalized Fermat polynomials.

Also, the polynomials $a_{n,m}^{(r-1)}(x)$ are the particular solution of the homogenous differential equation of the m-th order

$$y^{(m)}(x) + \sum_{s=0}^{m} a_s x^s y^{(s)}(x) = 0, \tag{27}$$

where a_s $(s = 0, 1, \ldots, m)$ can be computed as

$$a_s = \frac{1}{2ms!}\Delta^s f_0, \tag{28}$$

and

$$f(t) = f_t = (n-t)\binom{\frac{n-t+m(r+t)}{m}}{m-1}. \tag{29}$$

Using (28) and (29), we find a_0, a_1, a_m:

$$\begin{aligned}
a_0 &= \frac{1}{2m} n \binom{\frac{n+mr}{m}}{m-1},\\
a_1 &= \frac{1}{2m}(n-1)\binom{\frac{n-1+m(r+1)}{m}}{m-1} - \frac{1}{2m} n \binom{\frac{n+mr}{m}}{m-1},\\
a_m &= -\frac{1}{2m}\left(\frac{m-1}{m}\right)^{m-1}.
\end{aligned}$$

For $m = 2$, the differential equation (27) becomes

$$\left(1-\frac{1}{8}x^2\right)y''(x) - \frac{1+2r}{8}xy'(x) + \frac{1}{8}n(n+2r)y(x) = 0, \tag{30}$$

and, for $r = 1$ the differential equation (30) becomes the next equation

$$\left(1-\frac{1}{8}x^2\right)y''(x) - \frac{3}{8}xy'(x) + \frac{1}{8}n(n+2)y(x) = 0,$$

which corresponds to Fermat polynomials.

8. For $p(x) = x + p$, $q(x) = -q$ and $r = 1$, (p and q are arbitrary real parameters ($q \neq 0$)), we have

$$f(x,t) = (1-(x+p)t+qt^m)^{-1} = \sum_{n=0}^{\infty} u^{(1)}_{n+1,m}(x)t^n. \tag{31}$$

Differentiating (31), one-by-one r-times, with respect to x, we get the following relation

$$D_{x^r}\{u_{n+1,m}(x)\} = r!\, u^{(r+1)}_{n+1-r,m}(x),$$

where

$$u_{0,m}(p;q;x) = 0, \;\; u_{n,m}(p;q;x) = (x+p)^{n-1}, \;\; n = 1, 2, \ldots, m-1.$$

9. If $p(x) = 1$, $q(x) = 2x$ and $r = 1$, from (1) we obtain

$$F(x,t) = (1-t-2xt^m)^{-1} = \sum_{n=1}^{\infty} J_{n,m}(x)t^{n-1}, \tag{32}$$

where $J_{n,m}(x)$ are the generalized Jacobsthal polynomials ([3]).

Next, using the known method, from (32), we find the following relation

$$D_{x^r}\{J_{n,m}(x)\} = (2r)!!\, u^{(r+1)}_{n-mr,m}(x).$$

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