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Some approximation results in a non-Archimedean Banach space

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Abstract. Based on the notion of ν-convergence of bounded linear operators defined by A. Mario in [3], we introduce this convergence in a non-Archimedean Banach space and we study its properties. Besides, we introduce the new notion of collectively compact convergence in a non-Archimedean setting.

1. Introduction

The non-Archimedean theory has received major attention during the last years. Its analysis is based on non-Archimedean field valuation, including the p-adic fields which was introduced by K. Hensel, in 1908 [11]. It was not until 1940 that non-Archimedean field analysis was studied, and even since, serveral works have dealt with it form different perspectives. The theory of function analysis was especially due to A. C. M. Van Rooij [19] and A. F. Monna [14]. One of the main objectives of this theory is to study the operator theory. Many works given the properties of operators on non-Archimedean Banach spaces (see [9, 10]). Based on the study of the zeta function of an algebraic variety in finite characteristic by B. Dwork [8], J. P. Serre introduced a theory of completely continuous operators in a Banach space over a non-Archimedean field. Special attention is paid to completely continuous operators and compact operators on this setting. In general, these concepts do not coincide as in the classical Banach space. But, J. P. Serre proved that all completely continuous operators are compact, if the general field is locally compact (see [15]). In [18], M. M. Vishik has constructed the basic notions of the spectral theory of operators in non-Archimedean Banach spaces. Subsequently, many papers have analysed different notions and some aspects of spectrum. Moreover, inspired by the definition used in the study of the non-Archimedean operators theory, A. Ammar, A. Jeribi and N. Lazrag in their work [6], introduced and investigated the notion of generalized convergence between closable linear operators, which represents the convergence between their graphs, in a certain distance.

In the classical Banach space, it is well known that there exist many modes of convergence as pointwise convergence, norm convergence, ν-convergence and collectively compact convergence. The concept of ν-convergence was introduced by M. Ahues [3] and has been subsequently developed by several mathematicians for example S. Snchez-Perales and S. V. Djordjevic [17], A. Ammar and A. Jeribi [4, 5]. This

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convergence make possible to approximate non-compact operators recourse finite rank operators. Moreover, the latter convergence is a pseudo convergence in the sense that it is possible to find T_n ν-convergent to T and T_n ν-convergent to S where $T \neq S$. However, the spectrum of S and the spectrum of T are equal (see [3, Exercice 2.12]). Moreover, S. Snchez-Perales and S. V. Djordjevic have established an analogous statement for the point spectrum (see[17, Theorem 2.1]). In [3], M. Ahues proved that the ν-convergence is equivalent to the concept of collectively compact convergence if we add some conditions. The latter concept of convergence was introduced by S. L. Sobolev, in 1956. More recently, the collectively compact convergence was studied by several mathematicians. We can cite P. M. Anselone [2] and K. E. Atkinson [1]. It is a way to approximate the compact operators resorting to finite rank operators.

The principal aim of this work is to introduced the ν-convergence and the collectively compact convergence in a non-Archimedean Banach space and study some of its properties. This paper is devoted to extend A. Mario's result [3, Exercice 2.12] and S. Snchez-Perales and S. V. Djordjevic's result [17, Theorem 2.1] to non-Archimedean Banach space.

The rest of this paper is organized as follows. In Section 2, some notations, basic concepts and fundamental results about the theory of non-Archimedean are recalled. Moreover, a relationship between pointwise convergence and norm convergence is found. In Section 3, the ν-convergence and the collectively compact convergence are introduced and some properties of these convergence are studied. After that, the relationships between the spectrum of the sequence of bounded linear operators that converges in the ν-convergence (respectively collectively compact convergence) and the spectra of its limit is established.

2. Preliminary and auxiliary results

In this section, we collect some auxiliary results of the theory of non-Archimedean needed in the sequel, in the attempt of making our paper as self-contained as possible. Before beginning, let us recall some basic definitions of valuation, norm and linear operators on a non-Archimedean space.

Definition 2.1. *Let $\mathbb{K}$ be a field. A valuation on $\mathbb{K}$ is a map $|\cdot| : \mathbb{K} \longrightarrow \mathbb{R}$ satisfies*

(i) $|x| \geq 0$ *for any $x \in \mathbb{K}$ with equality only for $x = 0$.*

(ii) $|xy| = |x|\,|y|$ *for any $x, y \in \mathbb{K}$.*

(iii) *For some real number $c \geq 1$ and any $x \in \mathbb{K}$, if $|x| \leq 1$, then $|x + 1| \leq c$.* ◇

Definition 2.2. *Let $\mathbb{K}$ be a field.*

(i) *Let $|\cdot|_1$ and $|\cdot|_2$ be two valuations on the field $\mathbb{K}$. If there exists a positive real numbers λ such that $|\cdot|_2 = |\cdot|_1^{\lambda}$, then $|\cdot|_1$ and $|\cdot|_2$ are equivalent.*

(ii) *A valuation $|\cdot|$ on $\mathbb{K}$ satisfies the triangle inequality, if for any $x, y \in \mathbb{K}$*

$$|x + y| \leq |x| + |y|.$$

(iii) *A valuation $|\cdot|$ on $\mathbb{K}$ satisfies the strong triangle inequality, if for any $x, y \in \mathbb{K}$*

$$|x + y| \leq \max\Big\{|x|, |y|\Big\}.$$ ◇

Lemma 2.3. [10, Propositions 1.6, 1.10 and 1.13] *Let $\mathbb{K}$ be a field.*

(i) *Let $|\cdot|$ be a valuation on $\mathbb{K}$, then it satisfies the triangle inequality if, and only if, one can take $c = 2$ in Definition 2.1.*

(ii) *Every valuation on $\mathbb{K}$ is equivalent to one that satisfies the triangle inequality.*

(iii) *Let $|\cdot|$ be a valuation on $\mathbb{K}$, then it satisfies the strong triangle inequality if, and only if, one can take $c = 1$ in Definition 2.1.* ◇

According to Lemma 2.3 (*ii*), we may and will assume that the valuation $|\cdot|$ on $\mathbb{K}$ satisfies the triangle inequality. Hence, it induces a natural distance function

$$\begin{array}{rcl} d:\mathbb{K}\times\mathbb{K} & \longrightarrow & \mathbb{R}_+ \\ (x,y) & \longmapsto & |x-y|, \end{array}$$

giving it a structure of a metric space $(\mathbb{K}, d)$ (see [10, Proposition 1.17]).

As it is known, there are two kinds of valuation, one is the Archimedean valuation, as in the cases of $\mathbb{C}$ and $\mathbb{R}$, and the other is the non-Archimedean valuation.

Definition 2.4. *Let $\mathbb{K}$ be a field. A valuation on $\mathbb{K}$ is called non-Archimedean, if, it satisfies strong triangle inequality.* ◊

Now, let us assume that $(\mathbb{K}, |\cdot|)$ be a complete non-Archimedean filed.

Definition 2.5. *Let X be a vector space over $\mathbb{K}$. A non-Archimedean norm on X is a map $\|\cdot\| : X \longrightarrow \mathbb{R}_+\backslash\{0\}$ satisfying*
(*i*) $\|x\| = 0$ *if, and only if,* $x = 0$.
(*ii*) $\|\lambda x\| = |\lambda|\,\|x\|$ *for any* $x \in X$ *and any* $\lambda \in \mathbb{K}$.
(*iii*) $\|x+y\| \leq \max\Big\{\|x\|, \|y\|\Big\}$, *for any* $x, y \in X$. ◊

Remark 2.6. (*i*) *The non-Archimedean valuation on $\mathbb{K}$ itself is a non-Archimedean norm* (*see* [10, Example 2.3]).
(*ii*) *Let X be a non-Archimedean normed space. If $x, y \in X$ such that $\|x\| \neq \|y\|$, then we have*

$$\|x+y\| = \max\Big\{\|x\|, \|y\|\Big\} \text{ (see [10, Proposition 2.9]).} \quad ◊$$

Definition 2.7. *Let X be a non-Archimedean normed space and E be a nonempty subset of X.*
(*i*) *The set E is said to be bounded, if the set of real numbers $\Big\{\|x\| : x \in E\Big\}$ is bounded.*
(*ii*) *The set E is said to be absolutely convex, if $\alpha x + \beta y \in E$, for all $x, y \in E$ and $\alpha, \beta \in \overline{\mathrm{B}}_{\mathbb{K}}(0,1) = \{\lambda \in \mathbb{K} : |\lambda| \leq 1\}$, i.e., if E is $\overline{\mathrm{B}}_{\mathbb{K}}(0,1)$-module.*
(*iii*) *The set E is said to be compactoid, if for every $r > 0$ there exists a finite set $F \subset X$ such that*

$$E \subset \overline{\mathrm{B}}_X(0,r) + \overline{CoF},$$

where $\overline{CoF}$ is a intersection for all closed absolutely convex subsets of X that contain F and $\overline{\mathrm{B}}_X(0,r) = \{x \in X : |x| \leq r\}$. ◊

Definition 2.8. *Let X be a non-Archimedean normed space. A sequence $(x_n) \subset X$, converges to $x \in X$, if the sequence of real numbers $\|x_n - x\|$ converges to 0.* ◊

Lemma 2.9. [10, Proposition 2.13] *Let X be a non-Archimedean normed space. If the sequence (x_n) converges in X, then it is bounded.* ◊

Definition 2.10. *A non-Archimedean Banach space X is a non-Archimedean normed space, which is complete with respect to the natural metric induced by the norm*

$$d(x,y) = \|x-y\| \text{ for any } x, y \in X. \quad ◊$$

Let X be a non-Archimedean Banach space and T be an operator acting on X. T is called linear, if $\mathcal{D}(T)$, which designate its domain, is a vector subspace of X, and if $T(\alpha x + \beta y) = \alpha Tx + \beta Ty$, for all $\alpha, \beta \in \mathbb{K}$ and x, $y \in \mathcal{D}(T)$.

Definition 2.11. *Let X be a non-Archimedean Banach space. A linear operator $T : X \longrightarrow X$ is called bounded, if there exists $M \geq 0$ such that*

$$\|Tx\| \leq M\|x\|, \text{ for all } x \in X. \qquad \diamond$$

Denoted by $\mathcal{L}(X)$ the set of all bounded linear operators of X.

For $T \in \mathcal{L}(X)$, $\|T\| = \sup\limits_{x \in X \setminus \{0\}} \dfrac{\|Tx\|}{\|x\|} = \inf\{M \geq 0 : \|Tx\| \leq M\|x\|, \text{ for all } x \in X\}$ is finite.

Lemma 2.12. [10, Theorems 3.5 and 3.6] *Let X be a non-Archimedean Banach space.*
(i) *If $T, S \in \mathcal{L}(X)$ and $\lambda \in \mathbb{K}$, then $T + S$, λT, TS and ST belong to $\mathcal{L}(X)$.*
(ii) *The space $(\mathcal{L}(X), \|\cdot\|)$ of bounded linear operators on X, is a non-Archimedean Banach space.* $\diamond$

Remark 2.13. *Let X be a non-Archimedean Banach space. If $T \in \mathcal{L}(X)$, we defined a descent norm by*

$$\|T\|_0 = \sup_{\substack{x \in X \\ \|x\| \leq 1}} \|Tx\|.$$

(i) *The descent norm $\|\cdot\|_0$ is equivalent to $\|\cdot\|$ but need not be identical with it* (*see* [19, Chapter 3]).
(ii) *If the valuation of $\mathbb{K}$ is dense, then these norms are always equivalent and equal* (*see* [15, Section 2]). $\diamond$

Remark 2.14. *Let X be a non-Archimedean Banach space.*
(i) *In the non-Archimedean theory, the set $\|X\| := \{\|x\| : x \in X\}$ may not be the same as $|\mathbb{K}| := \{|\alpha| : \alpha \in \mathbb{K}\}$. As a consequence, non zero element of X may be fail to have a scalar multiple of norm 1, in fact, the set $\{x \in X : \|x\| = 1\}$ may well very be empty* (*see* [19, Chapter 3]).
(ii) *Assume that $\|X\| \subseteq |\mathbb{K}|$. Let $T \in \mathcal{L}(X)$. Then, the operator norms $\|\cdot\|$ and $\|\cdot\|_0$ are equivalent and equal. Indeed, let $x \in X\backslash\{0\}$. Since $\|X\| \subseteq |\mathbb{K}|$, then there exists $c \in \mathbb{K}\backslash\{0\}$ such that $|c| = \|x\|$. Setting $y = c^{-1}x$. This implies that $y \in X$ and $\|y\| = 1$. Hence,*

$$\|T\| = \sup_{\substack{x \in X \\ \|x\| = 1}} \|Tx\| = \|T\|_0. \qquad \diamond$$

Let X be a non-Archimedean Banach space and let $T \in \mathcal{L}(X)$. The symbols $N(T)$ and $R(T)$ stand respectively for the null and the range space of T, which are defined by

$$N(T) = \Big\{x \in X : Tx = 0\Big\} \text{ and } R(T) = \Big\{Tx : x \in X\Big\}.$$

Moreover, the null and identity operators on X will be denoted respectively by O_X and I_X, which are defined by

$$O_X(x) = 0 \text{ and } I_X(x) = x, \text{ for all } x \in X.$$

Definition 2.15. *Let X be a non-Archimedean Banach space and let $T \in \mathcal{L}(X)$.*

(i) *A sequence (T_n) of bounded linear operators mapping on X is said to be norm convergent, denoted by $T_n \to T$, if $\|T_n - T\| \to 0$ as $n \to \infty$.*

(ii) *A sequence (T_n) of bounded linear operators mapping on X is said to be pointwise convergent to T, denoted by $T_n \xrightarrow{p} T$, if $\|T_n x - Tx\| \to 0$ for every $x \in X$ as $n \to \infty$.* $\diamond$

Now, we recall the concepts of finite rank, completely continuous and compact linear operators in a non-Archimedean Banach space.

Definition 2.16. *Let X be a non-Archimedean Banach space.*
(i) *A bounded linear operator T acting on X is said to be a finite rank, if $R(T)$ is a finite dimensional subspace of X.*
(ii) *A bounded linear operator T acting on X is said to be completely continuous, if there exists a sequence of finite rank linear operators (T_n) such that $\|T - T_n\| \to 0$ as $n \to \infty$.*
(iii) *A linear operator $T : X \longrightarrow X$ is said to be compact, if $T\left(\overline{\mathrm{B}}_X(0, 1)\right)$ is compactoid.* ◊

We denote by $C_c(X)$ (respectively $\mathcal{K}(X)$) the collection of completely continuous (respectively compact) linear operators on X.
In a non-Archimedean Banach space X, we do not have the relationship between $C_c(X)$ and $\mathcal{K}(X)$ as a classical case. But, in [15], J. P. Serre has proved that those concepts coincide, when $\mathbb{K}$ is locally compact.

Lemma 2.17. *Let X be a non-Archimedean Banach space over a locally compact filed $\mathbb{K}$ and let $T \in \mathcal{L}(X)$. Then,*
(i) [15, Proposition 5] *All completely continuous linear operators are compact.*
(ii) [19, chapter 4] *T is compact if, and only if, $T\left(\overline{\mathrm{B}}_X(0, 1)\right)$ has compact closure.* ◊

Now, we study the relationship between the pointwise convergence and the norm convergence in a non-Archimedean Banach space.

Proposition 2.18. *Let X be a non-Archimedean Banach space such that $\|X\| \subseteq |\mathbb{K}|$. Let (T_n) be a sequence of bounded linear operators on X and let $T \in \mathcal{L}(X)$. If $T_n \to T$, then $T_n \xrightarrow{p} T$.* ◊

Proof. If $x = 0$, then $\|T_n(0) - T(0)\| = 0$. Let us assume that $x \in X\backslash\{0\}$. Since $\|X\| \subseteq |\mathbb{K}|$, then there exists $c \in \mathbb{K}\backslash\{0\}$ such that $|c| = \|x\|$. Setting $y = c^{-1}x$. This implies that $y \in X$ and $\|y\| = 1$. Hence, we have

$$\begin{aligned} \|(T_n - T)x\| &= \|(T_n - T)c\, y\| \\ &= |c|\, \|(T_n - T)y\|. \end{aligned} \tag{1}$$

Based on the assumption that $T_n - T \in \mathcal{L}(X)$ for sufficiently larger n, we can conclude from (1) that

$$\begin{aligned} \|(T_n - T)x\| &\leq |c|\, \|T_n - T\|\, \|y\| \\ &\leq |c|\, \|T_n - T\|. \end{aligned} \tag{2}$$

This implies from the fact that $T_n \to T$ and from (2) that

$$\|(T_n - T)x\| \to 0 \text{ as } n \to \infty.$$

This is equivalent to saying that $T_n \xrightarrow{p} T$. □

The example below show that the converse of the above Proposition is false.

Example 2.19. *Let $\mathbb{K}$ be a quadratically closed field and let $\omega = (\omega_j)$ be a sequence of non-zero elements in $\mathbb{K}$, for all $j \in \mathbb{N}$. The non-Archimedean Banach space $\mathbb{E}_\omega$ over $\mathbb{K}$ is defined by*

$$\mathbb{E}_\omega = \left\{x = (x_j) : x_j \in \mathbb{K}, \forall j \in \mathbb{N} \text{ and } \lim_{j\to\infty}\left(|\omega_j|^{\frac{1}{2}}|x_j|\right) = 0\right\},$$

when it is endowed with the following norm

$$\|x\| = \sup_{j\in\mathbb{N}}\left(|\omega_j|^{\frac{1}{2}}|x_j|\right),\ x = (x_j) \subset \mathbb{E}_\omega. \tag{3}$$

Consider the sequence of finite rank linear operators (D_n) defined on $\mathbb{E}_\omega$ by

$$D_n e_j = \begin{cases} e_j & \text{if } j \in \{0, \cdots, n\} \\ 0 & \text{otherwise,} \end{cases}$$

where $\{e_j : j = 0, 1, \cdots\}$ *is the canonical orthogonal basis of* $\mathbb{E}_\omega$ *(see* [10, Proposition 2.43]*) such that* $\|e_j\| = |\omega_j|^{\frac{1}{2}}$*, for all* $j \in \mathbb{N}$*. Since*

$$(D_n - I_{\mathbb{E}_\omega})e_j = \begin{cases} -e_j & \text{if } j \geq n+1 \\ 0 & \text{otherwise,} \end{cases} \tag{4}$$

then for all $x \in \mathbb{E}_\omega$*, we have*

$$\begin{aligned} (D_n - I_{\mathbb{E}_\omega})x &= (D_n - I_{\mathbb{E}_\omega})\left(\sum_{j=0}^{\infty} x_j e_j\right) \\ &= -\sum_{j=n+1}^{\infty} x_j e_j. \end{aligned} \tag{5}$$

Let us assume that $x \in \mathbb{E}_\omega$*. By applying* (5)*, we get that*

$$\begin{aligned} \|(D_n - I_{\mathbb{E}_\omega})x\| &\leq \max_{j \geq n+1} \left\|x_j e_j\right\| \\ &\leq \max_{j \geq n+1} \left\{|x_j|\ \|e_j\|\right\} \\ &\leq \max_{j \geq n+1} \left\{|x_j|\ |\omega_j|^{\frac{1}{2}}\right\}. \end{aligned} \tag{6}$$

Using the fact that $\lim\limits_{j\to\infty}\left(|\omega_j|^{\frac{1}{2}}|x_j|\right) = 0$*, we deduce that* $\max\limits_{j\geq n+1}\{|x_j|\ |\omega_j|^{\frac{1}{2}}\} \to 0$ *as* $n \to \infty$*. Hence, it follows from* (6) *that* $\|D_n x - I_{\mathbb{E}_\omega} x\| \to 0$*, as* $n \to \infty$*. This implies that* $D_n \xrightarrow{p} I_{\mathbb{E}_\omega}$*.*
The use of (4) *makes us conclude that*

$$\begin{aligned} \|D_n - I_{\mathbb{E}_\omega}\| &= \sup_{j\in\mathbb{N}} \frac{\|(D_n - I_{\mathbb{E}_\omega})e_j\|}{\|e_j\|} \\ &= \sup_{j\geq n+1} \frac{\|e_j\|}{\|e_j\|} \\ &= 1. \end{aligned} \tag{7}$$

Hence, it follows from (7) *that*

$$\|D_n - I_{\mathbb{E}_\omega}\| \to 1, \text{ as } n \to \infty.$$

This is equivalent to saying that $D_n \nrightarrow I_{\mathbb{E}_\omega}$*.* ◊

Proposition 2.20. *Let* X *be a non-Archimedean Banach space,* (T_n)*,* (S_n) *be sequences of bounded linear operators on* X *and let* $T \in \mathcal{L}(X)$*. If* $T_n \nrightarrow T$ *and* $S_n \to O_X$*, then* $T_n + S_n \nrightarrow T$*.* ◊

Proof. For $n \in \mathbb{N}$, we have

$$\begin{aligned} \|T_n - T\| &= \|T_n + S_n - T - S_n\| \\ &\leq \max\{\|T_n + S_n - T\|, \|S_n\|\}. \end{aligned} \tag{8}$$

Now, we shall show that

$$\|T_n + S_n - T\| = \max\{\|T_n - T\|, \|S_n\|\}. \tag{9}$$

Suppose that $\|T_n - T\| = \|S_n\|$. Since $S_n \to O_X$, then $T_n \to T$. This contradiction implies that $\|T_n - T\| \neq \|S_n\|$. Hence, it follows from Remark 2.6 (*ii*) that (9) holds. This leads to

$$\|S_n\| \leq \|T_n + S_n - T\|.$$

By referring to (8), we infer that

$$\|T_n - T\| \leq \|T_n + S_n - T\|. \tag{10}$$

Thus, the fact that $T_n \nrightarrow T$ implies from (10) that $T_n + S_n \nrightarrow T$. □

Definition 2.21. *Let X be a non-Archimedean Banach space. If $T \in \mathcal{L}(X)$, then*
(i) *T is said to be one-to-one, if $N(T) = \{0\}$.*
(ii) *T is said to be onto, if $R(T) = X$.*
(iii) *T is said to be invertible, if it is both one-to-one and onto.* ◊

Remark 2.22. *Let X be a non-Archimedean Banach space and let $T \in \mathcal{L}(X)$. If T is invertible, then there exists an unique bounded linear operator denoted $T^{-1} : X \to X$ called the inverse of T such that $T^{-1}T = TT^{-1} = I_X$.* ◊

Lemma 2.23. *Let X be a non-Archimedean Banach space and let $T \in \mathcal{L}(X)$.*
(i) [9, Lemma 2 p 23] *If $\|T\| < 1$, then $(I_X - T)$ is invertible, $(I_X - T)^{-1} = \sum_{k=0}^{\infty} T^k$ and $\|(I_X - T)^{-1}\| \leq 1$.*
(ii) [19, Theorem 3.12] *If $E \subseteq \mathcal{L}(X)$ such that $\Big\{Tx : T \in E\Big\}$ is a bounded set in X, for every $x \in X$. Then, E is a bounded set in $\mathcal{L}(X)$.* ◊

Definition 2.24. *Let X be a non-Archimedean Banach space and let $T \in \mathcal{L}(X)$.*
(i) *The set $\rho(T) = \Big\{\lambda \in \mathbb{K} : \lambda - T \text{ is invertible in } \mathcal{L}(X)\Big\}$ is called the resolvent set of T.*
(ii) *The set $\sigma(T) = \mathbb{K} \backslash \rho(T)$ is called the spectrum of T.*
(iii) *The set $\sigma_p(T) = \Big\{\lambda \in \sigma(T) : N(\lambda - T) \neq \{0\}\Big\}$ is called the point spectrum set of T.* ◊

3. Main results

The aim of this section is to introduce and investigate the ν-convergence and the collectively compact convergence of a sequence of bounded linear operators on a non-Archimedean Banach space.

Definition 3.1. (i) *Let X be a non-Archimedean Banach space and let $T \in \mathcal{L}(X)$. A sequence (T_n) of bounded linear operators mapping on X is said to be ν-convergent to T, denoted by $T_n \xrightarrow{\nu} T$, if*

(i_1) *$(\|T_n\|)$ is bounded,*

(i_2) *$\|(T_n - T)T\| \to 0$ as $n \to \infty$, and*

(i_3) *$\|(T_n - T)T_n\| \to 0$ as $n \to \infty$.*

(ii) *Let X be a non-Archimedean Banach space over a locally compact filed $\mathbb{K}$ and let $T \in \mathcal{L}(X)$. A sequence (T_n) of bounded linear operators mapping on X is said to be convergent to T in the collectively compact convergence, denoted by $T_n \xrightarrow{c.c} T$, if $T_n \xrightarrow{p} T$, and for some positive integer n_0,*

$$\bigcup_{n \geq n_0} \{(T_n - T)x : x \in X, \|x\| \leq 1\},$$

has compact closure of X. ◊

Now, let us study some basic properties of the ν-convergence.

Remark 3.2. (*i*) *The ν-convergence is a way to approximate the non completely continuous linear operators to finite rank linear operators in a non-Archimedean Banach space.*

(*ii*) *If $\mathbb{K}$ is locally compact filed, then by Lemma 2.17* (*i*), *we infer that the latter convergence will approximate the non compact linear operators to finite rank linear operators in a non-Archimedean Banach space, as the classical case.*

(*iii*) *Let X be a non-Archimedean Banach space, (T_n) be a sequence of bounded linear operators on X and let $T, S \in \mathcal{L}(X)$. As the classical Banach space, the ν-convergence is a pseudo-convergence i.e., if $T_n \xrightarrow{\nu} T$ and $T_n \xrightarrow{\nu} S$, then not necessarily $S = T$. In fact, it suffices to consider the following example:* ◊

Example 3.3. *Let $\mathbb{K}^2$ be a non-Archimedean Banach space when it is equipped with the norm defined by*

$$\|(\alpha, \beta)\| = \max\left\{|\alpha|, |\beta|\right\}, \text{ for all } \alpha, \beta \in \mathbb{K}, \text{ (see [10, Example 2.4])}.$$

We consider the following operators defined by

$$T = \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix} \text{ and } T_n = \begin{pmatrix} 0 & \lambda_n \\ 0 & 0 \end{pmatrix},$$

where $T \neq O_{\mathbb{K}^2}$ and (λ_n) is a sequence of $\mathbb{K}$ such that for all $n \in \mathbb{N}$, $0 < |\lambda_n| \leq 1$ and $|\lambda_n| \to 0$ as $n \to \infty$.

Let $\alpha, \beta \in \mathbb{K}$. For all $n \in \mathbb{N}$, we have

$$\begin{aligned} \left\|T_n \begin{pmatrix} \alpha \\ \beta \end{pmatrix}\right\| &= \left\|\begin{pmatrix} \lambda_n \beta \\ 0 \end{pmatrix}\right\| \\ &= \max\left\{|\lambda_n|\,|\beta|, 0\right\} \\ &= |\lambda_n|\,|\beta| \\ &\leq |\lambda_n| \max\left\{|\alpha|, |\beta|\right\} \\ &\leq |\lambda_n| \left\|\begin{pmatrix} \alpha \\ \beta \end{pmatrix}\right\|. \end{aligned} \tag{11}$$

Since $|\lambda_n| \leq 1$, then by applying (11), we obtain

$$\left\|T_n \begin{pmatrix} \alpha \\ \beta \end{pmatrix}\right\| \leq \left\|\begin{pmatrix} \alpha \\ \beta \end{pmatrix}\right\|.$$

Therefore, (T_n) is a sequence of bounded linear operators on $\mathbb{K}^2$. By using the same above reasoning, we find that

$$\left\|T \begin{pmatrix} \alpha \\ \beta \end{pmatrix}\right\| \leq |\lambda| \left\|\begin{pmatrix} \alpha \\ \beta \end{pmatrix}\right\|.$$

This is equivalent to saying that $T \in \mathcal{L}(\mathbb{K}^2)$. Now, we have to prove that $(\|T_n\|)$ is bounded. The fact that $|\lambda_n| \to 0$ as $n \to \infty$ implies from (11) *that*

$$\left\|T_n \begin{pmatrix} \alpha \\ \beta \end{pmatrix}\right\| \to 0 \text{ as } n \to \infty.$$

Hence, it follows from Lemma 2.9 that $\left(T_n \begin{pmatrix} \alpha \\ \beta \end{pmatrix}\right)$ is bounded. This leads to

$$\sup_{n \in \mathbb{N}} \left\|T_n \begin{pmatrix} \alpha \\ \beta \end{pmatrix}\right\| < \infty, \quad \text{for all } \alpha, \beta \in \mathbb{K}.$$

By referring to Lemma 2.23 (*ii*), *we deduce that $(\|T_n\|)$ is bounded. Moreover, we have*

$$\|(T_n - T)T\| = \|(T_n - T)T_n\| = \|(T_n - O_{\mathbb{K}^2})O_{\mathbb{K}^2}\| = \|(T_n - O_{\mathbb{K}^2})T_n\| = 0.$$

This enables us to conclude that $T_n \xrightarrow{\nu} O_{\mathbb{K}^2}$ and $T_n \xrightarrow{\nu} T$. ◊

The aim of the following theorem is to find a relationship between two different limits of a sequence of bounded linear operators which ν-convergent.

Theorem 3.4. *Let X be a non-Archimedean Banach space such that $\|X\| \subseteq |\mathbb{K}|$. Let (T_n) be a sequence of bounded linear operators on X and $T, S \in \mathcal{L}(X)$.*

(i) If $\|(T_n - T)T\| \to 0$ as $n \to \infty$ and $\|(T_n - S)S\| \to 0$ as $n \to \infty$, then we have

$$T_{|R(T)\cap R(S)} = S_{|R(T)\cap R(S)}.$$

(ii) If $T_n \xrightarrow{\nu} T$, $T_n \xrightarrow{\nu} S$, $TT_n = T_nT$ and $ST_n = T_nS$ for sufficiently larger n, then we have

$$T^2 = S^2. \qquad \diamond$$

Proof. (*i*) Let $z \in R(T) \cap R(S)$. Then, there exists $x, y \in X$ such that

$$z = Tx = Sy. \tag{12}$$

For sufficiently larger n, we can write

$$Tz - Sz = Tz - T_nz + T_nz - Sz. \tag{13}$$

It follows from (12) and (13) that

$$\begin{aligned} \|Tz - Sz\| &= \|(T - T_n)Tx + (T_n - S)Sy\| \\ &\leq \max\Big\{\|(T - T_n)Tx\|, \|(T_n - S)Sy\|\Big\}. \end{aligned} \tag{14}$$

Since $T, S \in \mathcal{L}(X)$ and $T_n \in \mathcal{L}(X)$ for sufficiently larger n, then by using Lemma 2.12 (*i*), we get $((T - T_n)T)$ and $((T_n - S)S)$ are sequences of bounded linear operators. The fact that $\|(T_n - T)T\| \to 0$ as $n \to \infty$ and $\|(T_n - S)S\| \to 0$ as $n \to \infty$ implies from Proposition 2.18 that

$$\|(T_n - T)Tx\| \to 0 \text{ as } n \to \infty \text{ and } \|(T_n - S)Sy\| \to 0 \text{ as } n \to \infty.$$

By referring to (14), we deduce that

$$\|Tz - Sz\| = 0, \quad \text{for all } z \in R(T) \cap R(S).$$

This shows that

$$T_{|R(T)\cap R(S)} = S_{|R(T)\cap R(S)}.$$

(*ii*) It is clear that $\mathcal{D}(T^2) = \mathcal{D}(S^2) = X$. Let us assume that $x \in X$. Using the fact that $TT_n = T_nT$ and $ST_n = T_nS$ for sufficiently larger n, we can write

$$\begin{aligned} T^2x - S^2x &= T^2x - T_nTx + TT_nx - ST_nx + T_nSx - S^2x \\ &= T^2x - T_nTx + TT_nx - T_n^2x + T_n^2x - ST_nx + T_nSx - S^2x \\ &= (T - T_n)Tx + (T - T_n)T_nx + (T_n - S)T_nx + (T_n - S)Sx. \end{aligned}$$

This implies that

$$\|T^2x - S^2x\| \leq \max\{\|(T - T_n)Tx\|, \|(T - T_n)T_nx\|, \|(T_n - S)T_nx\|, \|(T_n - S)Sx\|\}. \tag{15}$$

Since $T, S \in \mathcal{L}(X)$ and $T_n \in \mathcal{L}(X)$ for sufficiently larger n, then by using Lemma 2.12 (*i*), we obtain $((T - T_n)T)$, $((T - T_n)T_n)$, $((T_n - S)T_n)$ and $((T_n - S)S)$ are sequences of bounded linear operators. Based on the assumptions $T_n \xrightarrow{\nu} T$ and $T_n \xrightarrow{\nu} S$, we infer from Proposition 2.18 and (15) that

$$\|T^2x - S^2x\| = 0.$$

As a result, $T^2 = S^2$, as desired. $\square$

Remark 3.5. *The following result is a direct consequence of Theorem 3.4 (i): If $T_n \xrightarrow{\nu} T$ and $T_n \xrightarrow{\nu} S$, then $T_{|R(T)\cap R(S)} = S_{|R(T)\cap R(S)}$. Moreover, if $R(T) = R(S) = X$, then $T = S$.* ◊

In the following result, we discuss a relationship between spectrum (respectively point spectrum) of two ν-limits of operators on a non-Archimedean Banach space.

Proposition 3.6. *Let X be a non-Archimedean Banach space such that $\|X\| \subseteq |\mathbb{K}|$. Let (T_n) be a sequence of bounded linear operators on X and $T, S \in \mathcal{L}(X)$ such that $T_n \xrightarrow{\nu} T$ and $T_n \xrightarrow{\nu} S$.*

(i) If T and S have closed ranges, then $\sigma_p(T)\backslash\{0\} = \sigma_p(S)\backslash\{0\}$.

(ii) If $TT_n = T_nT$ and $ST_n = T_nS$ for sufficiently larger n, then $\sigma(T) = \sigma(S)$. ◊

Proof. (*i*) Let us assume that $\lambda \in \sigma_p(T)\backslash\{0\}$. Then,

$$Tx = \lambda x \text{ for some } x \neq 0. \tag{16}$$

It follows from (16) that

$$\begin{aligned} \|T_nx - \lambda x\| &= \|(T_n - T)x\| \\ &= \|(T_n - T)\lambda^{-1}Tx\| \\ &= |\lambda^{-1}|\,\|(T_n - T)Tx\|. \end{aligned} \tag{17}$$

Using the fact that $\|(T_n - T)T\| \to 0$ as $n \to \infty$, we infer from Lemma 2.12 (*i*) and Proposition 2.18 that $\|(T_n - T)Tx\| \to 0$ as $n \to \infty$. Hence, it follows from (17) that

$$\|T_nx - \lambda x\| \to 0 \text{ as } n \to \infty. \tag{18}$$

Since $\|TT_nx - \lambda Tx\| \leq \|T\|\,\|T_nx - \lambda x\|$, then by applying (18), we get that

$$\|TT_nx - \lambda Tx\| \to 0 \text{ as } n \to 0. \tag{19}$$

We have

$$\begin{aligned} \|ST_nx - \lambda Tx\| &= \|ST_nx - T_nT_nx + T_nT_nx - TT_nx + TT_nx - \lambda Tx\| \\ &= \|(S - T_n)T_nx + (T_n - T)T_nx + TT_nx - \lambda Tx\| \\ &\leq \max\Big\{\|(S - T_n)T_nx\|, \|(T - T_n)T_nx\|, \|TT_nx - \lambda Tx\|\Big\}. \end{aligned}$$

The fact that $\|(S - T_n)T_n\| \to 0$ as $n \to \infty$ and $\|(T - T_n)T_n\| \to 0$ as $n \to \infty$ implies from Lemma 2.12 (*i*) and Proposition 2.18 that

$$\|(S - T_n)T_nx\| \to 0 \text{ as } n \to \infty \text{ and } \|(T - T_n)T_nx\| \to 0 \text{ as } n \to \infty.$$

Hence, by refereing to (19), we obtain

$$\max\Big\{\|(S - T_n)T_nx\|, \|(T - T_n)T_nx\|, \|TT_nx - \lambda Tx\|\Big\} \to 0 \text{ as } n \to \infty.$$

Thus, we infer that

$$\|ST_nx - \lambda Tx\| \to 0 \text{ as } n \to \infty.$$

This leads to $\lambda^2 x = \lambda Tx \in \overline{R(S)} = R(S)$ which yields $x \in R(T) \cap R(S)$. The use of Theorem 3.4 (*i*) allows us to conclude that $Sx = Tx$. This implies from (16) that

$$\lambda x - Sx = \lambda x - Tx = 0.$$

This is equivalent to saying that $\lambda \in \sigma_p(S)\backslash\{0\}$. Conversely, a same reasoning as before leads to the result.

(ii) For $\lambda \in \mathbb{K}$ and $T \in \mathcal{L}(X)$, we can write

$$\lambda^2 - T^2 = (\lambda - T)(\lambda + T). \tag{20}$$

Let us assume that $\lambda \in \sigma(T)$. By referring to (20), we infer that $\lambda^2 - T^2$ is not invertible. This implies that $\lambda^2 \in \sigma(T^2)$. It follows from Theorem 3.4 (ii) that $\lambda^2 \in \sigma(S^2)$. Finally, the use of (20), by replacing T by S, gives $\lambda \in \sigma(S)$. Conversely, a same reasoning as before leads to the result. □

Remark 3.7. (i) *The problem of pseudo-convergence was solved due to the equality found between the ν-limit's spectrum (respectively point spectrum), under some conditions.*
(ii) *The results of Proposition 3.6 extend those of* [3, Exercice 2.12] *and* [17, Theorem 2.1] *for linear operators on a classical Banach space to linear operators on a non-Archimedean Banach space.* ◊

We give a simple example to illustrate Proposition 3.6.

Example 3.8. *We consider the following linear operator defined on $\mathbb{K}^2$ by:*

$$T = \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix}.$$

Then, $\sigma_p(T)\backslash\{0\} = \emptyset$ and $\sigma(T) = \{0\}$. Indeed, by referring to Example 3.3, we have $T_n \xrightarrow{\nu} T$ and $T_n \xrightarrow{\nu} O_{\mathbb{K}^2}$. Moreover, we have $\sigma_p(O_{\mathbb{K}^2})\backslash\{0\} = \emptyset$ and $\sigma(O_{\mathbb{K}^2}) = \{0\}$.
Since $\dim R(T)$ *and* $\dim R(O_{\mathbb{K}^2})$ *are finite, then T and $O_{\mathbb{K}^2}$ have closed range. Hence, it follows from (i) of Proposition 3.6 that $\sigma_p(T)\backslash\{0\} = \emptyset$. The fact that $T_nO_{\mathbb{K}^2} = O_{\mathbb{K}^2}T_n$ and $TT_n = T_nT$ for all $n \in \mathbb{N}$ implies from Proposition 3.6 (ii) that $\sigma(T) = \{0\}$.* ◊

Theorem 3.9. *Let X be a non-Archimedean Banach space, (T_n), (S_n) be sequences of bounded linear operators on X and $T, S \in \mathcal{L}(X)$.*

(i) *If $T_n \to T$, then $T_n \xrightarrow{\nu} T$.*

(ii) *If $0 \in \rho(T)$ and $(T_n - T)T \to O_X$, then $T_n \to T$.*

(iii) *If $T_n \xrightarrow{\nu} T$, $S_n \to S$ and $(T_n - T)S \to O_X$, then $T_n + S_n \xrightarrow{\nu} T + S$.*

(iv) *If $(T_n - T)T \to O_X$, $S_n \to S$ and $(T_n + S_n - S - T)(S + T) \to O_X$, then $(T_n - T)S \to O_X$.* ◊

Proof. (i) Since $T \in \mathcal{L}(X)$ and $T_n \in \mathcal{L}(X)$ for sufficiently larger n, then we have

$$\|(T_n - T)T\| \leq \|T_n - T\| \, \|T\| \text{ and } \|(T_n - T)T_n\| \leq \|T_n - T\| \, \|T_n\|. \tag{21}$$

The fact that $T_n \to T$ implies from (21) that

$$\|(T_n - T)T\| \to 0 \text{ as } n \to \infty \text{ and } \|(T_n - T)T_n\| \to 0 \text{ as } n \to \infty.$$

It remains to prove that $(\|T_n\|)$ is bounded. Since $T_n \to T$, then by using Lemma 2.9, we obtain $(T_n - T)$ is bounded. This implies that

$$\sup_{n\in\mathbb{N}} \|T_n - T\| < \infty. \tag{22}$$

For $n \in \mathbb{N}$, we can write

$$\|T_n\| \leq \max\{\|T_n - T\|, \|T\|\}. \tag{23}$$

Now, we discuss two cases.

First case. If $\|T\| < \|T_n - T\|$, for all $n \in \mathbb{N}$, then by using (22) and (23), we infer that

$$\sup_{n\in\mathbb{N}} \|T_n\| < \sup_{n\in\mathbb{N}} \|T_n - T\| < \infty.$$

Second case. If $\|T_n - T\| \leq \|T\|$, for all $n \in \mathbb{N}$, then by referring to (23), we deduce that

$$\sup_{n\in\mathbb{N}} \|T_n\| \leq \|T\| < \infty.$$

This means that $(\|T_n\|)$ is bounded. As a result, $T_n \xrightarrow{\nu} T$, as desired.

(ii) Since $0 \in \rho(T)$ and $\|(T_n - T)T\| \to 0$ as $n \to \infty$, then

$$\|(T_n - T)\| \leq \|(T_n - T)T\| \, \|T^{-1}\| \to 0 \text{ as } n \to \infty.$$

This is equivalent to saying that $T_n \to T$.

(iii) First, we have to prove that $\Big(\|T_n + S_n\|\Big)$ is bounded. We have

$$\begin{aligned} \sup_{n\in\mathbb{N}} \|T_n + S_n\| &= \sup_{n\in\mathbb{N}} \|T_n + S_n - S + S\| \\ &\leq \sup_{n\in\mathbb{N}} \Big[\max\Big\{\|T_n\|, \|S_n - S\|, \|S\|\Big\}\Big]. \end{aligned} \tag{24}$$

Now, we discuss three cases.

First case. If $\max\Big\{\|T_n\|, \|S_n - S\|, \|S\|\Big\} = \|S\|$, for all $n \in \mathbb{N}$, then by using (24), we obtain

$$\sup_{n\in\mathbb{N}} \|T_n + S_n\| < \infty.$$

Second case. If $\max\Big\{\|T_n\|, \|S_n - S\|, \|S\|\Big\} = \|S_n - S\|$, for all $n \in \mathbb{N}$, then by using (24), we get

$$\sup_{n\in\mathbb{N}} \|T_n + S_n\| \leq \sup_{n\in\mathbb{N}} \|S_n - S\|. \tag{25}$$

The fact that $S_n \to S$ implies from (22), by replacing T by S, and (25) that

$$\sup_{n\in\mathbb{N}} \|T_n + S_n\| < \infty.$$

Third case. If $\max\Big\{\|T_n\|, \|S_n - S\|, \|S\|\Big\} = \|T_n\|$, for all $n \in \mathbb{N}$, then by using (24), we obtain $\sup_{n\in\mathbb{N}} \|T_n + S_n\| \leq \sup_{n\in\mathbb{N}} \|T_n\|$. Using the fact that $(\|T_n\|)$ is bounded, we infer that

$$\sup_{n\in\mathbb{N}} \|T_n + S_n\| < \infty.$$

Therefore, we deduce that $\Big(\|T_n + S_n\|\Big)$ is bounded. Second, we have to prove that

$$\|(T_n + S_n - T - S)(S + T)\| \to 0, \text{ as } n \to \infty.$$

Since

$$\begin{aligned} (T_n + S_n - T - S)(S + T) &= \Big[(T_n - T) + (S_n - S)\Big](S + T) \\ &= (T_n - T)T + (S_n - S)(S + T) + (T_n - T)S, \end{aligned}$$

then we obtain

$$\|(T_n + S_n - T - S)(S + T)\| \leq \max\Big\{\|(T_n - T)T\|, \|S_n - S\|\|S + T\|, \|(T_n - T)S\|\Big\}. \tag{26}$$

Using the fact that $(T_n - T)T \to O_X$, $S_n \to S$, $(T_n - T)S \to O_X$ and $S + T$ is a bounded linear operator, then by (26), we obtain

$$\|(T_n + S_n - T - S)(S + T)\| \to 0 \text{ as } n \to \infty.$$

Finally, we have to prove that

$$\|(T_n + S_n - T - S)(S_n + T_n)\| \to 0 \text{ as } n \to \infty. \tag{27}$$

Since

$$\begin{aligned} (T_n + S_n - T - S)(S_n + T_n) &= \Big[(T_n - T) + (S_n - S)\Big](S_n + T_n) \\ &= (T_n - T)S_n + (T_n - T)T_n + (S_n - S)(S_n + T_n), \end{aligned}$$

then we get that

$$\|(T_n + S_n - T - S)(S_n + T_n)\| \leq \max\Big\{\|(T_n - T)T_n\|, \|(T_n - T)S_n\|, \|(S_n - S)\|\,\|(S_n + T_n)\|\Big\}. \tag{28}$$

Using the fact that $S_n \to S$ and $(S_n + T_n)$ is a sequence of bounded linear operators, we infer that

$$\|(S_n - S)\|\,\|(S_n + T_n)\| \to 0 \; as \; n \to \infty. \tag{29}$$

Now, we have

$$\begin{aligned} \|(T_n - T)S_n\| &= \|(T_n - T)(S_n - S + S)\| \\ &= \|(T_n - T)(S_n - S) + (T_n - T)S\| \\ &\leq \max\Big\{\|T_n - T\|\,\|S_n - S\|, \|(T_n - T)S\|\Big\}. \end{aligned} \tag{30}$$

The fact that $\|S_n - S\| \to 0 \; as \; n \to \infty$, $\|(T_n - T)S\| \to 0 \; as \; n \to \infty$ and $(T_n - T)$ is a sequence of bounded linear operators implies from (30) that

$$\|(T_n - T)S_n\| \to 0 \; as \; n \to \infty. \tag{31}$$

According to the hypothesis $\|(T_n - T)T_n\| \to 0 \; as \; n \to \infty$, (28), (29) and (31), we deduce that (27) holds. This shows that $T_n + S_n \xrightarrow{\nu} T + S$.

(iv) Since

$$\begin{aligned} (T_n - T)S &= (T_n - T)(S + T - T) \\ &= (T_n - T)(S + T) - (T_n - T)T \\ &= (T_n + S_n - S - T - S_n + S)(S + T) - (T_n - T)T \\ &= (T_n + S_n - S - T)(S + T) + (S - S_n)(S + T) - (T - T_n)T, \end{aligned}$$

then we obtain

$$\|(T_n - T)S\| \leq \max\Big\{\|(T_n + S_n - T - S)(S + T)\|, \|(T_n - T)T\|, \|S_n - S\|\|T + S\|\Big\}. \tag{32}$$

The fact that $(T_n - T)T \to O_X$, $(T_n + S_n - S - T)(S + T) \to O_X$, $S_n \to S$ and $\|S + T\| < \infty$ implies from (32) that $\|(T_n - T)S\| \to 0$ as $n \to \infty$. □

As an immediate consequence of Theorem 3.9, we have:

Corollary 3.10. *Let X be a non-Archimedean Banach space, (T_n), (S_n) be sequences of bounded linear operators on X and let $T, S \in \mathcal{L}(X)$.*

(i) If $0 \in \rho(T)$ and $T_n \xrightarrow{\nu} T$, then $T_n \to T$.

(ii) We assume that $T_n \xrightarrow{\nu} T$ and $S_n \to S$. Then, $T_n + S_n \xrightarrow{\nu} T + S$ if, and only if, $(T_n - T)S \to O_X$. ◊

Remark 3.11. (*i*) *In a non-Archimedean Banach space, norm convergence implies ν-convergence, but the equivalence can be achieved in the case where* $0 \in \rho(T)$, *as the classical case* (*see* [3, Lemma 2.2]).

(*ii*) *It follows from Corollary 3.10* (*ii*) *that ν-convergence is stable under norm perturbations, as the classical case* (*see* [3, Lemma 2.2]). ◊

Let us study some basic properties of the collectively compact convergence:

Remark 3.12. *The collectively compact convergence make possible to approximate compact operators recourse finite rank operators in non-Archimedean Banach space over a locally compact filed.* ◊

Theorem 3.13. *Let X be a non-Archimedean Banach space over a locally compact filed $\mathbb{K}$, (T_n) be a sequence of bounded linear operators on X and let $T \in \mathcal{K}(X)$. Then:*

(i) $T_n \xrightarrow{c.c} T$ if, and only if, $T_n \xrightarrow{p} T$, and for some positive integer n_0,

$$\bigcup_{n \geq n_0} \{T_n x : x \in X, \|x\| \leq 1\},$$

has compact closure of X.

(ii) We assume that $T_n \xrightarrow{c.c} T$. If $T_n + S_n \xrightarrow{c.c} T$, then $S_n \xrightarrow{c.c} O_X$. ◊

Proof. (*i*) For some $n_0 \in \mathbb{N}$, we show that

$$\bigcup_{n \geq n_0} \{T_n x : x \in X, \|x\| \leq 1\} = \bigcup_{n \geq n_0} \{(T_n - T)x : x \in X, \|x\| \leq 1\} + T\left(\overline{\mathrm{B}}_X(0,1)\right). \tag{33}$$

Let us assume that $y \in \bigcup_{n \geq n_0} \{T_n x : x \in X, \|x\| \leq 1\}$, then there exists $n \geq n_0$ such that $y = T_n x$ for every $x \in X$ satisfies $\|x\| \leq 1$. This implies that

$$y = (T_n - T)x + Tx \in \bigcup_{n \geq n_0} \{(T_n - T)x : x \in X, \|x\| \leq 1\} + T\left(\overline{\mathrm{B}}_X(0,1)\right).$$

Conversely, let $y \in \bigcup_{n \geq n_0} \{(T_n - T)x : x \in X, \|x\| \leq 1\} + T\left(\overline{\mathrm{B}}_X(0,1)\right)$. Then, $y_1 \in \bigcup_{n \geq n_0} \{(T_n - T)x : x \in X, \|x\| \leq 1\}$ and $y_2 \in T\left(\overline{\mathrm{B}}_X(0,1)\right)$ with $y = y_1 + y_2$. This implies that there exists $n \geq n_0$ such that $y = (T_n - T)x + Tx$ for every $x \in X$ satisfies $\|x\| \leq 1$. Hence, $y \in \bigcup_{n \geq n_0} \{T_n x : x \in X, \|x\| \leq 1\}$. Therefore, (33) holds.

At this level, since $\mathbb{K}$ is locally compact and T is compact, then by referring to [19, Exercice 4.S] and (33), we infer that for some positive integer n_0,

$$\bigcup_{n \geq n_0} \{T_n x : x \in X, \|x\| \leq 1\},$$

has compact closure of X if, and only if, for some positive integer n_0,

$$\bigcup_{n \geq n_0} \{(T_n - T)x : x \in X, \|x\| \leq 1\},$$

has compact closure of X.
(ii) For $x \in X$, we have

$$\|S_n x\| \leq \max\{\|S_n x + T_n x - Tx\|, \|T_n x - Tx\|\}. \tag{34}$$

Based on the assumptions that $\|S_n x + T_n x - Tx\| \to 0$ as $n \to \infty$ and $\|T_n x - Tx\| \to 0$ as $n \to \infty$, we infer from (34) that $\|S_n x\| \to 0$ as $n \to \infty$. It remains to prove that for some $n_0 \in \mathbb{N}$ such that

$$\bigcup_{n \geq n_0} \{S_n x : x \in X, \|x\| \leq 1\},$$

has compact closure of X. The fact that $T_n \xrightarrow{c.c} T$, $T_n + S_n \xrightarrow{c.c} T$ and $T \in K(X)$ implies from (i) that for some $n_1, n_2 \in \mathbb{N}$ such that
$\bigcup_{n \geq n_1} \{(S_n + T_n)x : x \in X, \|x\| \leq 1\}$ and $\bigcup_{n \geq n_2} \{T_n x : x \in X, \|x\| \leq 1\}$ have compact closure of X. Let $n_0 = \max\{n_1, n_2\}$ and $y \in \bigcup_{n \geq n_0} \{S_n x : x \in X, \|x\| \leq 1\}$, then there exists $n \geq n_0$ such that $y = S_n x$ for every $x \in X$ satisfies $\|x\| \leq 1$. This implies that $y = (S_n + T_n)x - T_n x$. Hence,

$$y \in \bigcup_{n \geq n_0} \{(S_n + T_n)x : x \in X, \|x\| \leq 1\} - \bigcup_{n \geq n_0} \{T_n x : x \in X, \|x\| \leq 1\}.$$

This shows that

$$\bigcup_{n \geq n_0} \{S_n x : x \in X, \|x\| \leq 1\} \subset \Gamma_n, \tag{35}$$

where $\Gamma_n = \bigcup_{n \geq n_0} \{(S_n + T_n)x : x \in X, \|x\| \leq 1\} - \bigcup_{n \geq n_0} \{T_n x : x \in X, \|x\| \leq 1\}$.

Thus, the fact that $\mathbb{K}$ is locally compact from [19, Exercice 4.S] implies that Γ_n is has compact closure of X. Finally, the use of (35) allows us to conclude that

$$\bigcup_{n \geq n_0} \{S_n x : x \in X, \|x\| \leq 1\},$$

has compact closure of X. □

Theorem 3.14. *Let X be a non-Archimedean Banach space over a locally compact filed $\mathbb{K}$ such that $\|X\| \subseteq |\mathbb{K}|$. Let (T_n), (S_n) be sequences of bounded linear operators on X and let $T \in \mathcal{K}(X)$.*
(i) If $T_n \xrightarrow{c.c} T$, then $T_n \xrightarrow{\nu} T$.
(ii) If $T_n \xrightarrow{c.c} T$ and $S_n \to O_X$, then $T_n + S_n \xrightarrow{\nu} T$. ◇

Proof. (i) First, we have to prove that $(\|T_n\|)$ is bounded. The fact that $\|T_n x - Tx\| \to 0$, for all $x \in X$ as $n \to \infty$ implies from Lemma 2.9 that $(T_n x - Tx)$ is a bounded sequence. This leads to $\sup\limits_{n \in \mathbb{N}} \|(T_n - T)x\| < \infty$, for all $x \in X$. It follows from Lemma 2.23 (ii) that

$$\sup_{n \in \mathbb{N}} \|T_n - T\| < \infty. \tag{36}$$

We have

$$\sup_{n \in \mathbb{N}} \|T_n\| \leq \sup_{n \in \mathbb{N}} \Big[\max\{\|T_n - T\|, \|T\|\} \Big]. \tag{37}$$

We discuss two cases.

First case. If $\|T_n - T\| < \|T\|$, then by using (37), we get $\sup\limits_{n\in\mathbb{N}} \|T_n\| < \|T\| < \infty$.

Second case. If $\|T\| \leq \|T_n - T\|$, then it follows from (36) and (37) that $\sup\limits_{n\in\mathbb{N}} \|T_n\| < \infty$. This means that $(\|T_n\|)$ is bounded. It remains to prove that

$$\|(T_n - T)T\| \to 0 \text{ as } n \to \infty \ \text{ and } \ \|(T_n - T)T_n\| \to 0 \text{ as } n \to \infty.$$

Based on the assumptions that $T_n \xrightarrow{c.c} T$, $T \in K(X)$ and $\mathbb{K}$ is locally compact, we infer from Lemma 2.17 and Theorem 3.13 (i) that $T(\overline{\mathrm{B}}_X)$ and $\bigcup\limits_{n\geq n_0} T_n(\overline{\mathrm{B}}_X)$ have compact closure of X. Hence, $T_n \xrightarrow{p} T$ is equivalent to $T_n \to T$ on the compacts sets $\overline{T(\overline{\mathrm{B}}_X)}$ and $\overline{\bigcup\limits_{n\geq n_0} T_n(\overline{\mathrm{B}}_X)}$. Therefore, according to Remark 2.14 (ii), we obtain

$$\|(T_n - T)T\| = \sup_{\|x\|\leq 1} \|(T_n - T)Tx\| \to 0 \text{ as } n \to \infty.$$

By using the same reasoning as above gives

$$\|(T_n - T)T_n\| \to 0 \text{ as } n \to \infty.$$

This enables us to conclude that $T_n \xrightarrow{\nu} T$.

(ii) Using the fact $T_n \xrightarrow{c.c} T$ and $T \in K(X)$, and applying (i), we have $T_n \xrightarrow{\nu} T$. Since $S_n \to O_X$, then by referring to Corollary 3.10 (ii), we deduce that $T_n + S_n \xrightarrow{\nu} T$.

□

As an immediate consequence of Corollary 3.10 and Theorem 3.14, we have:

Corollary 3.15. *Let X be a non-Archimedean Banach space over a locally compact filed $\mathbb{K}$ such that $\|X\| \subseteq |\mathbb{K}|$. Let (T_n) be a sequence of bounded linear operators on X and let $T \in \mathcal{K}(X)$ such that $0 \in \rho(T)$. If $T_n \xrightarrow{c.c} T$, then $T_n \to T$.* ◊

Remark 3.16. (i) *Let us notice that, according to Proposition 2.20, Corollary 3.10 and Theorem 3.13.*

(i_1) *If $T_n \xrightarrow{\nu} T$ and $S_n \to O_X$, then $T_n + S_n \xrightarrow{\nu} T$.*

(i_2) *If $T_n \nrightarrow T$ and $S_n \to O_X$, then $T_n + S_n \nrightarrow T$.*

(i_3) *If $T_n \xrightarrow{c.c} T$ and $S_n \overset{c.c}{\nrightarrow} O_X$, then $T_n + S_n \overset{c.c}{\nrightarrow} T$.*

(ii) *It follows from (i) that ν-convergence can be available even in the absence of norm convergence and collectively compact convergence.* ◊

The goal of the following results is to discuss the spectrum of a sequence of linear operators in a non-Archimedean Banach space:

Theorem 3.17. *Let X be a non-Archimedean Banach space, (T_n) be a sequence of bounded linear operators on X and let $T \in \mathcal{L}(X)$. If $(T_n - T)T \to O_X$ and $0 \in \rho(T)$, then there exists $n_0 \in \mathbb{N}$, we have*

$$\sigma(T_n) \subset \sigma(T), \text{ for all } n \geq n_0. \quad ◊$$

Proof. Let us assume that $\lambda \in \rho(T)$. Then, for $n \in \mathbb{N}$, we can write

$$\lambda - T_n = (\lambda - T)(I + (\lambda - T)^{-1}(T - T_n)). \tag{38}$$

Using the fact that $(T_n - T)T \to O_X$ and $0 \in \rho(T)$, together with Theorem 3.9 (*ii*), we obtain $T_n \to T$. Then, the sequence of real numbers $(\|T_n - T\|)$ converge to 0. This implies that for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\|T_n - T\| < \varepsilon$, for all $n \geq n_0$. In particular, for $\varepsilon = \|(\lambda - T)^{-1}\|^{-1}$, we have

$$\|T_n - T\| < \|(\lambda - T)^{-1}\|^{-1}, \text{ for all } n \geq n_0.$$

Therefore, for all $n \geq n_0$, we have

$$\begin{aligned} \|(\lambda - T)^{-1}(T - T_n)\| &\leq \|(\lambda - T)^{-1}\| \, \|T - T_n\| \\ &< 1. \end{aligned}$$

By referring to Theorem 2.23 (*i*), we infer that $(I + (\lambda - T)^{-1}(T - T_n))^{-1} \in \mathcal{L}(X)$. Hence, by using (38), we deduce that $(\lambda - T_n)^{-1} \in \mathcal{L}(X)$, for all $n \geq n_0$. Thus, we conclude that $\lambda \in \rho(T_n)$, for all $n \geq n_0$. □

As a direct consequence of Theorem3.17, Corollaries 3.10 and 3.15, we infer the following result:

Corollary 3.18. (*i*) *Let X be a non-Archimedean Banach space, (T_n) be a sequence of bounded linear operators on X and let $T \in \mathcal{L}(X)$. If $T_n \xrightarrow{\nu} T$ and $0 \in \rho(T)$, then there exists $n_0 \in \mathbb{N}$, we have*

$$\sigma(T_n) \subset \sigma(T), \text{ for all } n \geq n_0.$$

(*ii*) *Let X be a non-Archimedean Banach space over a locally compact filed $\mathbb{K}$ such that $\|X\| \subseteq |\mathbb{K}|$. Let (T_n) be a sequence of bounded linear operators on X and let $T \in \mathcal{K}(X)$. If $T_n \xrightarrow{c.c} T$, then there exists $n_0 \in \mathbb{N}$ such that*

$$\sigma(T_n) \subset \sigma(T), \text{ for all } n \geq n_0. \quad \diamond$$

Theorem 3.19. *Let X be a non-Archimedean Banach space, (T_n) be a sequence of bounded linear operators on X and let $T \in \mathcal{L}(X)$. If $T_n \to T$, then $(\lambda - T_n)^{-1} \xrightarrow{\nu} (\lambda - T)^{-1}$, for all $\lambda \in \rho(T)$.* ◊

Proof. Let us assume that $\lambda \in \rho(T)$. Since $T_n \to T$, then by using Theorem 3.17 (*i*), we infer that there exists $n_0 \in \mathbb{N}$ such that $\lambda \in \rho(T_n)$, for all $n \geq n_0$. For all $n \geq n_0$, we can write

$$\begin{aligned} \|(\lambda - T_n)^{-1} - (\lambda - T)^{-1}\| &= \|(\lambda - T_n)^{-1}(\lambda - T - \lambda + T_n)(\lambda - T)^{-1}\| \\ &= \|(\lambda - T_n)^{-1}(T_n - T)(\lambda - T)^{-1}\| \\ &\leq \|(\lambda - T_n)^{-1}\| \, \|(\lambda - T)^{-1}\| \, \|T_n - T\|. \end{aligned} \quad (39)$$

The fact that $\|T_n - T\| \to 0$ as $n \to \infty$, $\|(\lambda - T_n)^{-1}\| < \infty$ and $\|(\lambda - T)^{-1}\| < \infty$, together with (39), we deduce that $\|(\lambda - T_n)^{-1} - (\lambda - T)^{-1}\| \to 0$ as $n \to \infty$. Finally, the use of Theorem 3.9 (*i*) allows us to conclude that $(\lambda - T_n)^{-1} \xrightarrow{\nu} (\lambda - T)^{-1}$. □

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