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A new iterative technique for solving fixed point problem involving quasi-nonexpansive and firmly nonexpansive mappings

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Abstract. In this paper, we introduce a modified Halpern algorithm to approximate a common fixed points of quasi-nonexpansive and firmly nonexpansive mappings in real Hilbert spaces. We start by showing that $Fix(T_1 \circ T_2) = Fix(T_1) \cap Fix(T_2)$ without commuting assumption and establish strong convergence theorems for the proposed iterative process. Our strong convergence theorems extend and improve some known corresponding results in the contemporary literature for a wider class of nonexpansive type mappings in Hilbert spaces. Finally, applications of our theorems to equilibrium problems and monotone inclusion problems are given.

1. Introduction

Let H be a real Hilbert space and K be a nonempty, closed and convex subset of H. Let $T : H \to H$ be a nonlinear mapping, a point $x \in H$ is called a fixed point of T if $Tx = x$. We denote the set of all fixed points of T by $Fix(T)$. Let $D(T) \subset H$, then T is said to be

(1) a contraction if there exists $b \in [0, 1)$ such that:

$$\|Tx - Ty\| \leq b\|x - y\| \;\; x, y \in D(T).$$

If $b = 1$, T is called nonexpansive;
(2) quasi-nonexpansive if $Fix(T) \neq \emptyset$ and

$$\|Tx - p\| \leq \|x - p\|, \;\; x \in D(T), \;\; p \in Fix(T);$$

(4) firmly nonexpansive if for all $x, y \in D(T)$, we have

$$\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y\rangle.$$

Remark 1.1. *Easily, we obtain the following conclusions:*

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1 Every firmly nonexpansive mapping is nonexpansive.

2 Every nonexpansive mapping with a a fixed point is quasi-nonexpansive.

Finding the fixed points of nonlinear operators is an important topic in mathematics, due to the fact that many nonlinear problems can be reformulated as fixed point equations of nonlinear mappings. One of efficient methods to solve fixed point problem involving nonlinear mappings is the iterative method. Constructed iteration approaches to find fixed points of nonlinear mappings have received vast investigation, (see, e.g., Yao et al.[17], Chidume [2], Marino et al. [7], Moudafi [10], Halpern [4], Sow et al. [11] and the references therein).

For nonexpansive mappings with fixed points, Mann iterative method [6] is a ordinary tool to study them. However, only weak convergence is guaranteed in infinite dimensional spaces. Thus a natural question rises: could we obtain a strong convergence result by using the well-known Krasnoselskii-Mann method for nonexpansive mappings? In this connection, in 1975, Genel and Lindenstrass [3] gave a counterexample. Hence the modification is necessary in order to guarantee the strong convergence of Mann's iterative method. In order to get the strong convergence, in 1967, Halpern [4] constructed the following iteration scheme for computing a fixed point of a nonexpansive mapping T. For fixed $u \in K$ and an initial guess $x_0 \in K$, let the sequence $\{x_n\}$ be generated iteratively by

$$x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n, \; n \geq 0, \tag{1}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$. Algorithm (1) was referred to as the Halpern algorithm. Halpern pointed out that the control conditions:
$(C1)$ $\lim\limits_{n\to\infty} \alpha_n = 0$,
$(C2)$ $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$ are necessary for the strong convergence of the iteration (1) to a fixed point of T. At the same time, he also put forth the following open problem.

Problem 1.2. *Are the control conditions $(C1)$ and $(C2)$ sufficient for the convergence of the Halpern iteration (1) to a fixed point of T?*

Many researchers carefully considered this problem, for instance, [5, 8, 15]. However, in 2005, Suzuki [4] gave the a counterexample which shows that $(C1)$ and $(C2)$ are not sufficient for the strong convergence.

In 2017, Yao et al. [16], motivated by the fact that firmly nonexpansive mappings play an important role in nonlinear analysis, proved the following theorem.

Theorem 1.3. *[16] Let K be a nonempty closed convex subset of a Hilbert space H. Let $T : K \to K$ be a firmly nonexpansive mapping with $Fix(T) \neq \emptyset$. Assume $\{\alpha_n\}$ satisfies the following conditions:*
$(C1)$ $\lim\limits_{n\to\infty} \alpha_n = 0$,
$(C2)$ $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$.
Then, the sequence $\{x_n\}$ generated by (1) converges strongly to $x^* = P_{Fix(T)}u$.

Remark 1.4. *Note that Suzuki's conclusion can not be used to the class of firmly type nonexpansive mappings. Theorem 1.3 gives a positive answer to the Halpern open problem for the class of firmly nonexpansive mappings.*

Inspired by the results in the literature, we consider the following fixed point problem :

$$\textit{find } x^* \in K \;\; \textit{such that } x^* \in Fix(T_1 \circ T_2), \tag{2}$$

where T_1 and T_2 are quasi-nonexpansive and firmly nonexpansive mappings respectively.

Almost results existing for solving fixed points problem with a finite family of nonexpansive mappings commuting assumptions are needeed on the operators to get strong convergence (see, e.g., [2]) and authors assume that $Fix(T_1) \cap Fix(T_2)$ is nonempty.

Above discussion suggests the following questions.

Question 1.5. *Is it always true that* $Fix(T_1) \cap Fix(T_2) = Fix(T_1 \circ T_2)$ *without commuting assumptions?*

Question 1.6. *Could we construct a modified Halpern algorithm such that it converges strongly to a solution of problem* (2) *in Hilbert spaces without compactness assumption?*

The purpose of this paper is to give affirmative answers to these questions mentioned above. Applications are also considered.

2. Preliminaries

Let us recall the following definitions and results which will be used in the sequel.

Definition 2.1. *Let K be a nonempty, closed convex subset of a real Hilbert space H and $T : K \to K$ be a single-valued mapping. $I - T$ is said to be demiclosed at* 0 *if for any sequence $\{x_n\} \subset D(T)$ such that $\{x_n\}$ converges weakly to p and $\|x_n - Tx_n\|$ converges to zero, then $p \in Fix(T)$.*

Lemma 2.2 ([1]). *Let H be a real Hilbert space, K be a closed convex subset of H, and $T : K \to K$ be a nonexpansive mapping. Then $I - T$ is demiclosed.*

Lemma 2.3 ([2]). *Let H be a real Hilbert space. Then for any $x, y \in H$, the following inequalities hold:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle.$$

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - (1 - \lambda)\lambda\|x - y\|^2, \ \ \lambda \in (0, 1).$$

Lemma 2.4 ([14]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that $a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n\sigma_n$ for all $n \geq 0$, where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\sigma_n\}$ is a sequence in $\mathbb{R}$ such that* (a) $\sum_{n=0}^{\infty} \alpha_n = \infty$, (b) $\limsup_{n\to\infty} \sigma_n \leq 0$ *or* $\sum_{n=0}^{\infty} |\sigma_n\alpha_n| < \infty$. *Then* $\lim_{n\to\infty} a_n = 0$.

Lemma 2.5. *[9] Let t_n be a sequence of real numbers that does not decrease at infinity in a sense that there exists a subsequence t_{n_i} of t_n such that t_{n_i} such that $t_{n_i} \leq t_{n_{i+1}}$ for all $i \geq 0$. For sufficiently large numbers $n \in \mathbb{N}$, an integer sequence $\{\tau(n)\}$ is defined as follows:*

$$\tau(n) = \max\{k \leq n : \ t_k \leq t_{k+1}\}.$$

Then, $\tau(n) \to \infty$ as $n \to \infty$ and

$$\max\{t_{\tau(n)}, \ t_n\} \leq t_{\tau(n)+1}.$$

3. Main Results

We start by the following result.

Lemma 3.1. *Let H be a real Hilbert space and let K be a nonempty closed convex subset of H. Let $T_1 : K \to K$ be a quasi-nonexpansive mapping and $T_2 : K \to K$ be a firmly nonexpansive mapping. Then, $Fix(T_1) \cap Fix(T_2) = Fix(T_1 \circ T_2)$ and $T_1 \circ T_2$ is a quasi-nonexpansive mapping on K.*

Proof. We split the proof into two steps.
Step 1: First, we show that $Fix(T_1) \cap Fix(T_2) = Fix(T_1 \circ T_2)$. We note that $Fix(T_1) \cap Fix(T_1) \subset Fix(T_1 \circ T_2)$. Thus, we only need to show that $Fix(T_1 \circ T_2) \subseteq Fix(T_1) \cap Fix(T_2)$. Let $p \in Fix(T_1) \cap Fix(T_2)$ and $q \in Fix(T_1 \circ T_2)$. By using properties of T_1 and T_2, we have

$$\begin{aligned} \|q - p\|^2 &= \|T_1 \circ T_2 q - T_1 p\|^2 \\ &\leq \|T_2 q - p\|^2. \end{aligned} \tag{3}$$

Using the fact that T_2 is firmly nonexpansive, we have

$$\begin{aligned} \|T_2 q - p\|^2 &\leq \langle T_2 q - p, q - p \rangle \\ &= \frac{1}{2}(\|T_2 q - p\|^2 + \|q - p\|^2 - \|T_2 q - q\|^2), \end{aligned}$$

which yields

$$\|T_2 q - p\|^2 \leq \|q - p\|^2 - \|T_2 q - q\|^2. \tag{4}$$

Using (3) implies that (4) becomes

$$\begin{aligned} \|T_2 q - p\|^2 &\leq \|q - p\|^2 - \|T_2 q - q\|^2 \\ &\leq \|T_2 q - p\|^2 - \|T_2 q - q\|^2. \end{aligned}$$

Clearly, $\|T_2 q - q\| = 0$ which implies that

$$q = T_2 q.$$

Keeping in mind that $T_1 \circ T_2 q = q$, we have

$$q = T_1 \circ T_2 q = T_1 q.$$

Thus, $q \in Fix(T_1) \cap Fix(T_2)$. Hence, $Fix(T_1) \cap Fix(T_2) = Fix(T_1 \circ T_2)$.
Step 2: We show $T_1 \circ T_2$ is a quasi-nonexpansive mapping on K. Let $x \in K$ and $p \in Fix(T_1 \circ T_2)$. Then, $p \in Fix(T_1) \cap Fix(T_2)$ by step 1. We observe that,

$$\begin{aligned} \|T_1 \circ T_2 x - p\| &= \|T_1 \circ T_2 x - T_1 p\| \\ &\leq \|T_2 x - p\| \\ &\leq \|x - p\|. \end{aligned}$$

This completes the proof. □

We now prove the following theorem.

Theorem 3.2. *Let K be a nonempty, closed convex subset of real Hilbert space H. Let $T_1 : K \to K$ be a quasi-nonexpansive mapping and $T_2 : K \to K$ be a firmly nonexpansive mapping such that $Fix(T_1 \circ T_2) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $(0,1)$. Let $\{x_n\}$ be a sequence defined as follows:*

$$\begin{cases} x_0 \in K, \\ y_n = \beta_n x_n + (1 - \beta_n) T_1 \circ T_2 x_n, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n) T_1 \circ T_2 y_n, \end{cases} \tag{5}$$

where $u \in K$ is fixed. Suppose the following conditions hold:
(i) $\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=0}^{\infty} \alpha_n = \infty$,
(ii) $\liminf_{n\to\infty} \beta_n(1 - \beta_n) > 0$. *Assume that $I - T_1 \circ T_2$ is demiclosed at origin. Then, the sequence $\{x_n\}$ generated by (5) converges strongly to $x^* \in Fix(T_1) \cap Fix(T_2)$, where $x^* = P_{Fix(T_1) \cap Fix(T_2)} u$.*

Proof. Pick $p \in Fix(T_1 \circ T_2)$. By using (5) and Lemmas 3.1 and 2.3, we have

$$\begin{aligned}
\|y_n - p\|^2 &= \Big\|\beta_n x_n + (1-\beta_n)T_1 \circ T_2 x_n - p\Big\|^2 \\
&= \Big\|\beta_n (x_n - p) + (1-\beta_n)(T_1 \circ T_2 x_n - p)\Big\|^2 \\
&= \beta_n\|x_n - p\|^2 + (1-\beta_n)\|T_1 \circ T_2 x_n - p\|^2 - \beta_n(1-\beta_n)\|T_1 \circ T_2 x_n - x_n\|^2.
\end{aligned}$$

Using the fact that $T_1 \circ T_2$ is quasi-nonexpansive, we obtain

$$\begin{aligned}
\|y_n - p\|^2 &\le \beta_n\|x_n - p\|^2 + (1-\beta_n)\|x_n - p\|^2 - \beta_n(1-\beta_n)\|T_1 \circ T_2 x_n - x_n\|^2 \\
&\le \|x_n - p\|^2 - \beta_n(1-\beta_n)\|T_1 \circ T_2 x_n - x_n\|^2. \qquad (6)
\end{aligned}$$

Since $\beta_n \in]0,1[$, we have,

$$\|y_n - p\Big\| \le \|x_n - p\Big\|. \qquad (7)$$

From (5) and (7), we have

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n u + (1-\alpha_n)T_1 \circ T_2 y_n - p\| \\
&\le \alpha_n\|u - p\| + (1-\alpha_n)\|y_n - p\| \\
&\le (1-\alpha_n)\|x_n - p\| + \alpha_n\|u - p\| \\
&\le \max\{\|x_n - p\|, \|u - p\|\}.
\end{aligned}$$

By induction, it is easy to see that

$$\|x_n - p\| \le \max\{\|x_0 - p\|, \|u - p\|\}, \quad n \ge 1.$$

Hence, $\{x_n\}$ is bounded and $\{y_n\}$.
By using (6) and convexity of $\|.\|^2$, we obtain

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\le \|\alpha_n u + (1-\alpha_n)T_1 \circ T_2 y_n - p\|^2 \\
&\le \alpha_n\|u - p\|^2 + (1-\alpha_n)\|T_1 \circ T_2 y_n - p\|^2 \\
&\le \alpha_n\|u - p\|^2 + (1-\alpha_n)\|y_n - p\|^2 \\
&\le \alpha_n\|u - p\|^2 + (1-\alpha_n)(\|x_n - p\|^2 - \beta_n(1-\beta_n)\|T_1 \circ T_2 x_n - x_n\|^2).
\end{aligned}$$

Thus,

$$(1-\alpha_n)(1-\beta_n)\beta_n\|T_1 \circ T_2 x_n - x_n\|^2 \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n\|u - p\|^2.$$

Hence,

$$(1-\alpha_n)(1-\beta_n)\beta_n\|T_1 \circ T_2 x_n - x_n\|^2 \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n\|u - p\|^2. \qquad (8)$$

Now we prove that $\{x_n\}$ converges strongly to x^*. We divide the proof into two cases.
Case I. Assume that the sequence $\{\|x_n - x^*\|\}$ is monotonically decreasing. Then $\{\|x_n - x^*\|\}$ is convergent. Clearly, we have

$$\lim_{n\to\infty}\Big[\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2\Big] = 0.$$

It then implies from (8) that

$$\lim_{n\to\infty}(1-\beta_n)\beta_n\|T_1 \circ T_2 x_n - x_n\|^2 = 0. \qquad (9)$$

Since $\beta_n \in]0,1[$ and $\lim_{n\to\infty}\inf \beta_n(1-\beta_n) > 0$, we have

$$\lim_{n\to\infty}\Big\|x_n - T_1 \circ T_2 x_n\Big\| = 0. \qquad (10)$$

Next, we prove that $\limsup_{n\to+\infty}\langle x^*-u, x^*-x_n\rangle \leq 0$. Since H is reflexive and $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that x_{n_k} converges weakly to a in K and

$$\limsup_{n\to+\infty}\langle x^*-u, x^*-x_n\rangle = \lim_{k\to+\infty}\langle x^*-u, x^*-x_{n_k}\rangle.$$

From (10) and $I - T_1\circ T_2$ is demiclosed, we obtain $a \in Fix(T_1\circ T_2)$. Using Lemma 3.1, we have $a \in Fix(T_2)\cap Fix(T_1)$. On other hand, using property of x^* ($x^* = P_{Fix(T_1)\cap Fix(T_2)}u$), we then have

$$\begin{aligned}\limsup_{n\to+\infty}\langle x^*-u, x^*-x_n\rangle &= \lim_{k\to+\infty}\langle x^*-u, x^*-x_{n_k}\rangle\\ &= \langle x^*-u, x^*-a\rangle \leq 0.\end{aligned}$$

Finally, we show that $x_n \to x^*$. Applying Lemma 2.3, we get

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &= \|\alpha_n u + (1-\alpha_n)T_1\circ T_2 y_n - x^*\|^2\\ &\leq \|(1-\alpha_n)(T_2\circ T_1 y_n - x^*)\|^2 + 2\alpha_n\langle x^*-u, x^*-x_{n+1}\rangle\\ &\leq (1-\alpha_n)\|y_n-x^*\|^2 + 2\alpha_n\langle x^*-u, x^*-x_{n+1}\rangle\\ &\leq (1-\alpha_n)\|x_n-x^*\|^2 + 2\alpha_n\langle x^*-u, x^*-x_{n+1}\rangle.\end{aligned}$$

From Lemma 2.4, its follows that $x_n \to x^*$.

Case II. Assume that the sequence $\{\|x_n - x^*\|\}$ is not monotonically decreasing. Set $\Gamma_n = \|x_n - x^*\|$ and $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping for all $n \geq n_0$ (for some n_0 large enough) by $\tau(n) = \max\{k \in \mathbb{N} : k \leq n,\ \Gamma_k \leq \Gamma_{k+1}\}$. We have τ is a non-decreasing such that $\tau(n) \to \infty$ as $n \to \infty$ and $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ for $n \geq n_0$. From (8), we have

$$(1-\alpha_{\tau(n)})(1-\theta_{\tau(n)})\theta_{\tau(n)}\|x_{\tau(n)} - T_1\circ T_2 x_{\tau(n)}\|^2 \leq \alpha_{\tau(n)}\|u-p\|^2 \to 0 \text{ as } n\to\infty.$$

Since $\beta_n \in]0,1[$ and $\lim_{n\to\infty}\inf(1-\theta_{\tau(n)})\theta_{\tau(n)} > 0$, we can deduce

$$\lim_{n\to\infty}\|x_{\tau(n)} - T_1\circ T_2 x_{\tau(n)}\| = 0. \tag{11}$$

By a similar argument as in Case I, we can show that $x_{\tau(n)}$ and $y_{\tau(n)}$ are bounded in K and $\limsup_{\tau(n)\to+\infty}\langle x^* - u, x^* - x_{\tau(n)}\rangle \leq 0$. We have for all $n \geq n_0$,

$$0 \leq \|x_{\tau(n)+1} - x^*\|^2 - \|x_{\tau(n)} - x^*\|^2 \leq \alpha_{\tau(n)}[-\|x_{\tau(n)} - x^*\|^2 + 2\langle x^*-u, x^*-x_{\tau(n)+1}\rangle],$$

which implies that

$$\|x_{\tau(n)} - x^*\|^2 \leq 2\langle x^*-u, x^*-x_{\tau(n)+1}\rangle.$$

Then, we have

$$\lim_{n\to\infty}\|x_{\tau(n)} - x^*\|^2 = 0.$$

Therefore,

$$\lim_{n\to\infty}\Gamma_{\tau(n)} = \lim_{n\to\infty}\Gamma_{\tau(n)+1} = 0.$$

Thus, by Lemma 2.5, we conclude that

$$0 \leq \Gamma_n \leq \max\{\Gamma_{\tau(n)},\ \Gamma_{\tau(n)+1}\} = \Gamma_{\tau(n)+1}.$$

Hence, $\lim_{n\to\infty}\Gamma_n = 0$, that is $\{x_n\}$ converges strongly to x^*. This completes the proof. □

We now apply Theorem 3.2 when T_1 is a nonexpansive mapping. In this case demiclosedness assumption ($I - T_1\circ T_2$ is demiclosed at origin) is not necessary.

Theorem 3.3. *Let K be a nonempty, closed convex subset of real Hilbert space H. Let $T_1 : K \to K$ be a nonexpansive mapping and $T_2 : K \to K$ be a firmly nonexpansive mapping such that $Fix(T_1 \circ T_2) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $(0, 1)$. Let $\{x_n\}$ be a sequence defined as follows:*

$$\begin{cases} x_0 \in K, \\ \\ y_n = \beta_n x_n + (1 - \beta_n)T_1 \circ T_2 x_n, \\ \\ x_{n+1} = \alpha_n u + (1 - \alpha_n)T_1 \circ T_2 y_n, \end{cases} \tag{12}$$

where $u \in K$ is fixed. Suppose the following conditions hold:
(i) $\lim\limits_{n\to\infty} \alpha_n = 0, \quad \sum\limits_{n=0}^{\infty} \alpha_n = \infty,$
(ii) $\lim\limits_{n\to\infty} \inf \beta_n(1-\beta_n) > 0$. *Then, the sequence $\{x_n\}$ generated by (12) converges strongly to $x^* \in Fix(T_1) \cap Fix(T_2)$, where $x^* = P_{Fix(T_1)\cap Fix(T_2)}u$.*

Proof. We have $T_1 \circ T_2$ is nonexpansive mapping, then, the proof follows Lemma 2.2, Remark 1.1 and Theorem 3.2. □

Remark 3.4. *Our results are applicable for finding a common fixed point of two firmly nonexpansive mappings without demiclosedness assumption.*

4. Applications

In this section, we apply our main results for finding a common solution of fixed points problems involving quasi-nonexpansive mapping and equilibrium problem.
Let H be a real Hilbert space and let K be a nonempty closed convex subset of H. Let $g : K \times K \to \mathbb{R}$ be a bifunction where $\mathbb{R}$ is the set of real numbers. The equilibrium problem corresponding to g is to find $x^* \in K$ such that

$$g(x^*, y) \geq 0, \forall\, y \in K. \tag{13}$$

The set of solutions of (13) is denoted by $EP(g)$. Numerous problems in physics, optimization, and economics are reduced to find the solution of an equilibrium problem (e.g., see [13]). For solving the equilibrium problem we assume that the bifunction g satisfies the following conditions:
$(A1)$ $g(x, x) = 0$ for all $x \in K$;
$(A2)$ g is monotone, i.e., $g(x, y) + g(y, x) \leq 0$ for all $x, y \in K$;
$(A3)$ for each $x, y, z \in K$,

$$\lim_{t\to 0} g(tz + (1 - t)x, y) \leq g(x, y);$$

$(A4)$ for each $x \in K$, $y \to g(x, y)$ is convex and lower semicontinuous.
For solving (13), many authors introduce the following lemma.

Lemma 4.1. *[13] Assume that $g : K \times K \to \mathbb{R}$ satisfies $(A1)$-$(A4)$. For $r > 0$ and $x \in H$, define a mapping $T_r^g : H \to K$ as follows*

$$T_r^g(x) = \{z \in K,\ g(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in K\},$$

for all $x \in H$. Then, the following hold:
1.T_r^g is single-valued;
2.T_r^g is firmly nonexpansive, i.e., $\|T_r^g(x) - T_r^g(y)\|^2 \leq \langle T_r^g x - T_r^g y, x - y\rangle$ for any $x, y \in H$;
3.$Fix(T_r^g) = EP(g)$;
4.$EP(g)$ is closed and convex.

Now, we introduce the following fixed point problem:

Problem 4.2.

$$\textit{find } x \in K \ \textit{ such that } x = Tx, \tag{14}$$

where $T : K \to K$ be a qasi-nonexpansive mapping. Therefore, by Theorem 3.2, the following result is obtained.

Theorem 4.3. *Let K be a nonempty, closed and convex subset of a real Hilbert space H. Let g be a bifunction from $K \times K \to \mathbb{R}$ satisfies (A1)-(A4) and, let $T : K \to K$ be a quasi-nonexpansive mapping such that $Fix(T \circ T_r^g) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $(0,1)$. Let $\{x_n\}$ be a sequence defined as follows:*

$$\begin{cases} x_0 \in K, \\ \\ y_n = \beta_n x_n + (1-\beta_n) T \circ T_r^g x_n, \\ \\ x_{n+1} = \alpha_n u + (1-\alpha_n) T \circ T_r^g y_n, \end{cases} \tag{15}$$

where $u \in K$ is fixed. Suppose the following conditions hold:

(i) $\lim\limits_{n\to\infty} \alpha_n = 0, \quad \sum\limits_{n=0}^{\infty} \alpha_n = \infty,$

(ii) $\lim\limits_{n\to\infty} \inf \beta_n(1-\beta_n) > 0$. *Assume that $I - T \circ T_r^g$ is demiclosed at origin. Then, the sequence $\{x_n\}$ generated by (15) converges strongly to a common solution of problem (13) and problem (14).*

Finally, we apply our main results for finding a common solution of fixed points problems involving quasi-nonexpansive mapping and monotone inclusion problem.
We consider the following inclusion problem.

Problem 4.4.

$$\textit{find } x \in H \ \textit{ such that } 0 \in Ax, \tag{16}$$

where A be a maximal monotone operator. Given a maximal monotone operator $A : H \to 2^H$ and $\lambda > 0$, its associed resolvent of order λ, defined by

$$J_\lambda^A := (I + \lambda A)^{-1},$$

where I denotes the identity operator, is a firmly nonexpansive mapping from H to H with full domain and the set of fixed points of J_λ^A coincides with the solutions set of problem 4.4.

Theorem 4.5. *Let H be a real Hilbert space H. Let A be a maximal monotone operator on H and, let $T : H \to H$ be a quasi-nonexpansive mapping such that $Fix(T \circ J_\lambda^A) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $(0,1)$. Let $\{x_n\}$ be a sequence defined as follows:*

$$\begin{cases} x_0 \in H, \\ \\ y_n = \beta_n x_n + (1-\beta_n) T \circ J_\lambda^A x_n, \\ \\ x_{n+1} = \alpha_n u + (1-\alpha_n) T \circ J_\lambda^A y_n, \end{cases} \tag{17}$$

where $u \in H$ is fixed. Suppose the following conditions hold:

(i) $\lim\limits_{n\to\infty} \alpha_n = 0, \quad \sum\limits_{n=0}^{\infty} \alpha_n = \infty,$

(ii) $\lim\limits_{n\to\infty} \inf \beta_n(1-\beta_n) > 0$. *Assume that $I - T \circ J_\lambda^A$ is demiclosed at origin. Then, the sequence $\{x_n\}$ generated by (17) converges strongly to a common solution of Problem 4.2 and Problem 4.4.*

References

[1] F. E. Browder, *Convergenge theorem for sequence of nonlinear operator in Banach spaces*, Math.Z.100(1967) 201-225. Vol. EVIII, part 2, 1976.

[2] C. E. Chidume, *Geometric Properties of Banach spaces and Nonlinear Iterations*, Springer Verlag Series: Lecture Notes in Mathematics, Vol. 1965,(2009), ISBN 978-1-84882-189.

[3] A. Genel and J. Lindenstrass, *An example concerning fixed points, Israel J. Math.,* 22 (1975), No. 1, 81-86.

[4] B. Halpern, Fixed points of nonexpansive maps, Bull. Amer. Math. Soc., 3 (1967), 957-961.

[5] P.-L. Lions, Approximation de points fixes de contractions, R. Acad. Sci. Paris Sr. A-B, 284 (1977), 1357-1359. 1

[6] W.R. Mann, Mean value methods in iteration, Proc. Amer. Math. Soc., 4 (1953) 506-510.

[7] G. Marino, H.K. Xu, *Weak and strong convergence theorems for strict pseudo-contractions in Hilbert spaces* , J. Math. Math. Appl., 329(2007), 336-346.

[8] P.-E. Maing, Approximation methods for common fixed points of nonexpansive mappings in Hilbert spaces, J. Math. Anal. Appl., 325 (2007), 469-479.

[9] P. E. Mainge, *Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization,* Set-Valued Analysis, 16, 899-912 (2008).

[10] A. Moudafi, Viscosity approximation methods for fixed-point problems, J. Math. Anal. Appl. 241 (2000), 46-55.

[11] T.M.M. Sow, N. Djitt, and C.E. Chidume, *A path convergence theorem and construction of fixed points for nonexpansive mappings in certain Banach spaces*, Carpathian J.Math.,32(2016),No.2,217-226,2016.

[12] T. Suzuki, Reichs problem concerning halperns convergence, Arch. Math. 92 (2009) 602-613.

[13] H.K. Xu, *A variable Krasnoselskii-Mann algorithm and the multiple set split feasiblity problem, Inverse Problem*, 26 (2006), 2021-2034.

[14] H.K. Xu, *Iterative algorithms for nonlinear operators,* J. London Math. Soc. **66** (2002), no. 2, 240 - 256.

[15] Y.-S. Song, X.-K. Chai, Halpern iteration for firmly type nonexpansive mappings, Nonlinear Anal., 71 (2009), 4500-4506.

[16] Y. Yao, M. Postolache, N. Shahzad, *Strong convergence of Halpern method for firmly type nonexpansive mappings,* J. Nonlinear Sci. Appl., 10 (2017), 5932-5938.

[17] Y. Yao, H. Zhou, Y. C. Liou, *Strong convergence of modified Krasnoselskii-Mann iterative algorithm for nonexpansive mappings,* J. Math. Anal. Appl. Comput. 29 (2009) 383-389.