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A Note on Generalized Multiplier Spaces and Applications to αAB-, βAB-, γAB- and NAB-duals

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Abstract. We will start with the set $M(X,Y)$, multiplier space, defined by:

$$M(X,Y) = \{a = (a_k) \in \omega \mid ax \in Y, \text{ for all } x \in X\}$$

where ω denotes the space of all complex-valued sequences and X and Y are sequence spaces. Specially, putting $Y = cs$, where cs is the set of convergent series, the multiplier space becomes the β-dual of X. We will present some generalized results related to X^β and extend some of existing. Finally, we will illustrate these generalizations with some examples and applications.

1. Introduction

The idea of this paper is to give some generalizations and comments on results related to generalized multiplier spaces. Recently, in [Foroutannia D., Roopaei H., The generalized multiplier space and Köthe-Toeplitz and null duals, Math.Commun., 22(2017), 273–285] and [Foroutannia D., Roopaei H., The $\alpha AB-$, $\beta AB-$, $\gamma AB-$ and $NAB-$ duals for Sequence Spaces, Filomat, 31:19(2017), 6219–6231] the authors introduced some generalized multiplier spaces. Here, we will present an alternative approach, using matrix domains. This approach is actually a generalization of the mentioned results on generalized multiplier spaces.

As usual, we will use the standard notation. We write ω for the set of all complex sequences $x = (x_k)_{k=0}^\infty$. Also, we write l_∞, c and c_0 for the sets of all bounded, convergent and null sequences, respectively. By cs and bs we denote the sets of all convergent and bounded series. Beside the above mentioned classical sequence spaces l_∞, c and c_0, in our paper we will use one more classical sequence space $\ell_p = \{x \in \omega : \sum_{k=0}^\infty |x_k|^p < \infty\}$ for $1 \leq p < \infty$.

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Let $A = (a_{nk})_{n,k=0}^{\infty}$ be an infinite matrix of complex entries and $A_n = (a_{nk})_{k=0}^{\infty}$ denote the sequence in the n^{th} row of A. We write

$$A_n x = \sum_{k=0}^{\infty} a_{nk} x_k \text{ and } Ax = (A_n x)_{n=0}^{\infty} \text{ (provided all the series converge).}$$

The set

$$X(A) = \{x \in \omega : Ax \in X\}$$

is called the matrix domain of A in X.

If X and Y are arbitrary subsets of ω, then (X, Y) denotes the class of all matrices that map X into Y, that is the class of all matrices A such hat $X \subset Y(A)$, or $A \in (X, Y)$ if and only if the series $A_n x$ converge for all $x \in X$ and for all n, and $Ax \in Y$ for all $x \in X$. The set $\omega_A = \{x \in \omega : Ax \text{ is defined}\}$ is called the domain of A.

Definition 1.1. *Let X and Y be subsets of ω. The set*

$$M(X, Y) = \{a \in \omega \mid ax = (a_k x_k)_{k=0}^{\infty} \in Y \text{ for all } x \in X\} \tag{1}$$

is called the multiplier space of X and Y.

Remark 1.2. *[8, Example 1.28] (i) $M(c_0, c) = \ell_\infty$, (ii) $M(c, c) = c$, (iii) $M(\ell_\infty, c) = c_0$.*

For further research and additional properties of multiplier spaces of classical sequence spaces the readers are referred to [8].

If the final space Y has the special form, $Y = \ell_1$ or $Y = cs$ or $Y = bs$, the multiplier set $M(X, Y)$ becomes the $\alpha-$ (Köthe-Toeplitz), $\beta-$ or $\gamma-$dual of X, denoted by X^α, X^β, X^γ, respectively. In addition, for $Y = c_0$, the authors in [4] consider the $N-$dual of X, denoted by X^N. Actually, we have:

$$X^\alpha = M(X, \ell_1),\ X^\beta = M(X, cs),\ X^\gamma = M(X, bs),\ X^N = M(X, c_0).$$

$\beta-$duals are very important in the characterizations of matrix transformations between sequence spaces. Actually, we have the following: $A \in (X, Y)$ if and only if $A_n \in X^\beta$ for all n and $Ax \in Y$ for all $x \in X$. But, let us mention, the way of finding their explicit form is not always easy. In this paper we deal with the classical sequence spaces and some special cases obtained from them, whose $\beta-$duals we can find. For further reading the following references could be helpful [1–3, 6, 8, 11].

2. Generalized multiplier spaces and one more generalized approach to them

2.1. "B"-Multiplier Spaces

In [4] the authors introduced the space $M_B(X,Y)$ where B is an infinite matrix with real entries and X and Y are real-valued sequence spaces. In this paper we will modify this definition with complex-valued sequence spaces and matrix B with complex entries, too. Hence, we have the following:

$$M_B(X,Y) = \{a \in \omega \mid \sum_{k=0}^{\infty} b_{nk}a_k \text{ converges for all } n \text{ and } \Big(x_n \sum_{k=0}^{\infty} b_{nk}a_k\Big)_{n=0}^{\infty} \in Y \text{ for all } x \in X\}. \tag{2}$$

Let us note that the condition $\sum_{k=0}^{\infty} b_{nk}a_k$ converges for all n (is actually $B_n a$ converges for all n) is important and necessary for the matrix transformation to be well defined and for the and existence of Ba. The second condition $\left(x_n \sum_{k=0}^{\infty} b_{nk}a_k\right)_{n=0}^{\infty} \in Y$ for all $x \in X$ is actually:

$$\Big(x_n \sum_{k=0}^{\infty} b_{nk}a_k\Big)_{n=0}^{\infty} = \big(x_n B_n a\big)_{n=0}^{\infty} = x \cdot Ba \in Y \text{ for all } x \in X$$

The following result will connect matrix domains and multiplier spaces. This is one of our main results in this paper.

Theorem 2.1. *Let X and Y be sequence spaces in ω and B be an infinite matrix with complex entries. Then:*

$$M_B(X,Y) = M(X,Y)(B). \tag{3}$$

Actually, $M_B(X,Y)$ is matrix domain of B in $M(X,Y)$.

Proof. Applying the definition of matrix domain of matrix in some sequence space, we have that $M(X,Y)(B) = \{a \in \omega \mid Ba \in M(X,Y)\}$. Further, applying the definition of the multiplier space, we get that $M(X,Y)(B) = \{a \in \omega \mid (Ba)x \in Y \text{ for all } x \in X\}$, where $(Ba)x = \left(x_n \sum_{k=0}^{\infty} b_{nk}a_k\right)_{n=0}^{\infty}$, provided that Ba exists, that is, the series $\sum_{k=0}^{\infty} b_{nk}a_k$ converges for all n. In such way, we proved that $M_B(X,Y) = M(X,Y)(B)$. □

Obviously, with this result and such an approach, the results from [4] are the corollaries of the previous theorem. Some of them are listed below.

Corollary 2.2. *(i) $c^{\beta B} = c_0^{\beta B} = \ell_\infty^{\beta B} = \ell_1(B)$ [4, Theorem 4];*
(ii) $\ell_1^{\beta B} = \ell_\infty(B)$, $\ell_p^{\beta B} = \ell_q(B)$, where $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ [4, Theorem 5];
(iii) $M_B(c_0, c) = \ell_\infty(B)$, $M_B(c,c) = c(B)$, $M_B(\ell_\infty, c) = c_0(B)$ [4, Theorem 2].

Proof. It is clear that all results are consequences of the relation proved in Theorem 2.1 and well-known results related to multiplier spaces and special classes of duals. For example, it is clear that (i) follows directly from $X^{\beta B} = M_B(X, cs) = M(X, cs)(B) = X^\beta(B)$ and the well-known result $c^\beta = c_0^\beta = \ell_\infty^\beta = \ell_1$. □

2.2. "AB"-Multiplier Spaces

In [5], the authors "extended" the generalization of multiplier spaces and introduced $M_{A,B}(X,Y)$. For infinite matrices A and B with real entries such that $\sum_{k=1}^{\infty} a_{nk}x_k < \infty$ for all $x \in X$ and all n, they defined the following set:

$$M_{A,B}(X,Y) = \{a \in \omega \mid \sum_{k=0}^{\infty} b_{nk}a_k \text{ converges for all } n \text{ and } \Big(\sum_{k=0}^{\infty} b_{nk}a_k \sum_{k=0}^{\infty} a_{nk}x_k\Big)_{n=0}^{\infty} \in Y \text{ for all } x \in X\} \tag{4}$$

As in the previous subsection, let us suppose that matrices A and B are with complex-valued entries, and X and Y subsets in ω. The condition $\sum_{k=0}^{\infty} a_{nk}x_k < \infty$ for all $x \in X$ and all n in the definition of $M_{A,B}(X,Y)$

becomes $A_n x = \sum_{k=0}^{\infty} a_{nk}x_k$ converges for all $x \in X$ and for all n. This condition is necessary for the existence of Ax. Further, as in the definition of $M_B(X,Y)$, the condition $B_n a = \sum_{k=0}^{\infty} b_{nk}a_k$ converges for all n provides the existence of Ba. Finally, the condition $\left(\sum_{k=0}^{\infty} b_{nk}a_k \sum_{k=0}^{\infty} a_{nk}x_k\right)_{n=0}^{\infty} \in Y$ for all $x \in X$ can be written in the following form: $\left((B_n a)(A_n x)\right)_{n=0}^{\infty} \in Y = (Ba)(Ax) \in Y$ for all $x \in X$.

On the other side, let us consider the following set:

$$M_B(A(X),Y) = \{a \in \omega \mid \sum_{k=0}^{\infty} b_{nk}a_k \text{ converges for all } n \text{ and } \left(z_n \sum_{k=0}^{\infty} b_{nk}a_k\right)_{n=0}^{\infty} \in Y \text{ for all } z \in A(X)\} \quad (5)$$

provided that $\sum_{k=0}^{\infty} a_{nk}x_k$ converges for all $x \in X$ and all n, where $A(X) = \{Ax \mid x \in X\}$. The next result is important for further work and represents our next main result.

Theorem 2.3. *Let X and Y be sequence spaces in ω, A and B be infinite matrices with complex entries and $X \subset \omega_A$. Then:*

$$M_{A,B}(X,Y) = M(A(X),Y)(B). \quad (6)$$

Proof. Applying (2), (4), (5) and Theorem 2.1 we have:

$$M_{A,B}(X,Y) = M_B(A(X),Y) = M(A(X),Y)(B).$$

□

It is clear that if $Y = cs$, we have $X^{\beta AB} = (A(X))^{\beta}(B)$. Similarly, $X^{\alpha AB} = (A(X))^{\alpha}(B)$, $X^{\gamma AB} = (A(X))^{\gamma}(B)$ and $X^{NAB} = (A(X))^{N}(B)$.

Also, if we suppose that the entries of matrix the A satisfy the conditions such that $A \in (X,Z)$, the whole problem about finding generalized multiplier spaces $M_{A,B}(X,Y)$ is transformed into finding multiplier space $M(Z,Y)$ and the matrix domain of B in $M(Z,Y)$, but under one more assumption! The additional condition in the form "A must be invertible matrix" is too general and unnecessary. Actually, here we do not need the inverse of general infinite matrix. We consider matrix mappings between certain sequence spaces, defined by an infinite matrix and an appropriate bounded linear operator associated with that matrix ([8, Theorem 1.23], [11, Theorem 4.2.8]). Hence, in our case, additional condition we need is "operator represented by A must be bijective".

In the following theorem, we recall some of well-known results related to the characterization of matrix transformations between some classical sequence spaces. These results are necessary for what follows.

Theorem 2.4. *[6, 10, 11] The necessary and sufficient conditions for $A \in (X,Y)$ when $X \in \{c, c_0\}$ and $Y \in \{c, c_0, \ell_\infty\}$ can be read from the following table:*

From / *To*	c_0	c
c	**2.**	**4.**
c_0	**3.**	**5.**
ℓ_∞	**1.**	**1.**

where

1.	(1^*)	*where* (1^*) $\sup_n \sum_{k=0}^{\infty} |a_{nk}| < \infty$;
2.	(1^*) *and* (2^*)	*where* (2^*) $\lim_{n\to\infty} a_{nk} = \alpha_k$ *for each* k;
3.	(1^*) *and* (3^*)	*where* (3^*) $\lim_{n\to\infty} a_{nk} = 0$;
4.	(1^*) *and* (2^*) *and* (4^*)	*where* (4^*) $\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} = \alpha$;
5.	(1^*) *and* (3^*) *and* (5^*)	*where* (5^*) $\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} = 0$.

Furthermore, $(c_0, \ell_\infty) = (c_0, \ell_\infty) = (\ell_\infty, \ell_\infty)$.

Using the previous conditions, we can determine the conditions for some generalized multiplier spaces. The next theorem will cover the results from [5, Theorem 2.9] and give some new ones.

Theorem 2.5. *Let A and B be infinite matrices.*
i) If $X \in \{c, c_0, \ell_\infty\}$, A is bijective from X to ℓ_∞ and the entries of the matrix $A = (a_{nk})_{n,k=0}^{\infty}$ satisfy condition (1^*) *from Theorem 2.4, then*

$$M_{A,B}(X, c) = c_0(B).$$

ii) If A is bijective from c_0 to c and the entries of the matrix $A = (a_{nk})_{n,k=0}^{\infty}$ satisfy conditions (1^*) *and* (2^*) *from Theorem 2.4, then*

$$M_{A,B}(c_0, c) = c(B).$$

iii) If A is bijective from c_0 to c_0 and the entries of the matrix $A = (a_{nk})_{n,k=0}^{\infty}$ satisfy conditions (1^*) *and* (3^*) *from Theorem 2.4, then*

$$M_{A,B}(c_0, c) = \ell_\infty(B).$$

Proof. Applying the results from Theorem 2.1, Theorem 2.3 and Theorem 2.4, we obtain the characterizations for (c, ℓ_∞), (c_0, c) and (c_0, c_0) in i), ii) and iii) respectively. Hence, the results are obviously a consequence of the mentioned theorems and well known results from Remark 1.2. □

Obviously, depending on the matrix A and the conditions satisfied by its entries, we obtain a different form of the multiplier space $M_{A,B}(X, Y)$, even when the sequence spaces are the same. The next corollary illustrate this. We will find $M_{A,B}(c, c)$ under different assumptions (conditions) for the entries of matrix A.

Corollary 2.6. *Let $A = (a_{nk})_{n,k=0}^{\infty}$ and B be infinite matrices.*
i) If A is bijective from c to ℓ_∞ and the entries of the matrix $A = (a_{nk})_{n,k=0}^{\infty}$ satisfy condition (1^*) *from Theorem 2.4, then*

$$M_{A,B}(c, c) = c_0(B).$$

ii) If A is bijective from c to c_0 and the entries of the matrix $A = (a_{nk})_{n,k=0}^{\infty}$ satisfy conditions (1^*), (3^*) *and* (5^*) *from Theorem 2.4, then*

$$M_{A,B}(c, c) = \ell_\infty(B).$$

iii) If A is bijective from c to c and the entries of the matrix $A = (a_{nk})_{n,k=0}^{\infty}$ satisfy conditions (1^*), (2^*) *and* (4^*) *from Theorem 2.4, then*

$$M_{A,B}(c, c) = c(B).$$

Proof. i) It is clear that the condition in (1*) means that $A \in (c, \ell_\infty)$. Hence, applying Theorem 2.1 and 2.3 we obtain

$$M_{A,B}(c,c) = M_B(A(c),c) = M_B(\ell_\infty, c) = M(\ell_\infty, c)(B) = c_0(B).$$

ii) As in i), the conditions give the characterization of $A \in (c, c_0)$ and further, the next relation:

$$M_{A,B}(c,c) = M_B(A(c),c) = M_B(c_0, c) = M(c_0, c)(B) = \ell_\infty(B).$$

iii) Finally, since the last conditions give the characterization of the class (c, c), as the consequence we obtain the following:

$$M_{A,B}(c,c) = M_B(A(c),c) = M_B(c,c) = M(c,c)(B) = c(B).$$

□

Corollary 2.7. *Let $A = (a_{nk})_{n,k=0}^{\infty}$ and B be infinite matrices and $\dagger \in \{\alpha, \beta, \gamma, N\}$.*
i) If A is bijective from c_0 into itself, that is, if $A \in (c_0, c_0)$ and is a bijective matrix mapping, then $c_0^{\dagger AB} = \ell_1(B)$.
ii) If A is bijective from c into itself, that is, if $A \in (c, c)$ and is a bijective matrix mapping, then $c^{\dagger AB} = \ell_1(B)$.
iii) If A is bijective from ℓ_∞ into itself, that is, if $A \in (\ell_\infty, \ell_\infty)$ and is a bijective matrix mapping, then $\ell_\infty^{\dagger AB} = \ell_1(B)$.
iv) If A is bijective from ℓ_1 into itself, that is, if $A \in (\ell_1, \ell_1)$ and is a bijective matrix mapping, then $\ell_1^{\dagger AB} = \ell_\infty(B)$.
v) If A is bijective from ℓ_p, $1 < p < \infty$ into itself, that is, if $A \in (\ell_p, \ell_p)$ and is a bijective matrix mapping, then $\ell_p^{\dagger AB} = \ell_q(B)$, $q = \frac{p}{p-1}$.

Remark 2.8. *Let us mention that the previous corollary is a direct consequence of all the results mentioned above, but it is important to remark that the characterization of all classes of matrix transformations considered in the corollary is possible in the form of conditions for entries of appropriate infinite matrix, except in the case (ℓ_p, ℓ_p), $1 < p < \infty$.*

We close this section with a result which will connect the matrix domain of a matrix A and "AB"-multiplier spaces.

Theorem 2.9. *Let A and B be infinite matrices, $X, Y \subset \omega$, $X(A)$ matrix domain of A in X and let A be a bijective mapping from X into itself. Then:*

$$M_{A,B}(X(A), Y) = M(X, Y)(B). \tag{7}$$

Proof. First, it is clear that if $z \in A(X(A))$, then there exist $w \in X(A)$ such that $z = Aw$ and $Aw \in X$. So, $A(X(A)) \subseteq X$. On the other side, let $x \in X$. Since A is bijective, $x = A(A^{-1}x)$ and $A^{-1}x \in X(A)$. Hence, $x \in A(X(A))$ and finally $X = A(X(A))$. Applying Theorem 2.3, we obtain the result:

$$M_{A,B}(X(A), Y) = M_B(A(X(A)), Y) = M(A(X(A)), Y)(B) = M(X, Y)(B).$$

□

The results from [5, Theorem 2.17, Theorem 2.18] are direct consequences of the last theorem. The next corollary illustrates those results related to the βAB−duals of spaces $X(A)$, where X is one of the classical sequence spaces.

Corollary 2.10. *Let A be a bijective mapping from X into itself, B an arbitrary infinite matrix, X be a classical sequence space and $\dagger \in \{\alpha, \beta, \gamma, N\}$. Then:*

$$(X(A))^{\dagger AB} = \begin{cases} \ell_1(B) & \text{for } X = c, c_0, \ell_\infty \\ \ell_\infty(B) & \text{for } X = \ell_1 \\ \ell_q(B) & \text{for } X = \ell_p \ (1 < p < \infty, q = \frac{p}{p-1}). \end{cases} \tag{8}$$

Proof. Applying directly the previous theorem for $Y = \ell_1$, $Y = cs$, $Y = bs$ and $Y = c_0$ and the results related to the α, β, γ, N - duals of the classical sequences spaces respectively, we obtain that (8) holds. □

3. Applications to some difference sequence space

As we have already mentioned at the beginning of this paper, $\beta-$duals are of great importance for the characterization of matrix transformations between sequence space. Hence, in this section, we will give applications related to this kind of duals. All other duals can be considered by applying our results and representations given in the previous sections.

In [4] the authors considered the sequence space $c_0(\Delta)$ and its duals. Further, in [5], they considered different kinds of duals of the space $c_0(A,\Delta)$. Here, we will give the applications just for the generalizations of $\beta-$duals for the spaces $X(\Delta)$ and $X(A,\Delta)$, where $X\in\{c,c_0,\ell_\infty\}$. Applying results related to the $\beta-$duals of matrix domains of triangles [7, 8] and the results and relations considered and obtained in the previous sections, the results from [4, 5] can be covered and obtained in different way.

Let $\Delta=(\delta_{nk})_{n,k=1}^{\infty}$ be the matrix of the first order difference, that is

$$\delta_{nk}=\begin{cases}1 & (k=n)\\ -1 & (k=n-1)\\ 0 & (\text{otherwise })\end{cases}.$$

Its inverse is the matrix $\Sigma=(\sigma_{nk})_{n,k=1}^{\infty}$ with entries:

$$\sigma_{nk}=\begin{cases}1 & (1\le k\le n)\\ 0 & (k>n)\end{cases}.$$

In [7] the authors determined the $\beta-$duals of spaces $X(\Delta)$, where $X\in\{c,c_0,\ell_\infty\}$. The following result will be helpful for our work.

Proposition 3.1. *[7, Proposition 3.7] Let R denote the transpose of the matrix Σ and $\mathbf{n}=(n)_{n=1}^{\infty}$. Then we have:*
(a) $a\in(c_0(\Delta))^\beta$ if and only if

$$Ra\in\ell_1 \text{ and } Ra\in\mathbf{n}^{-1}*\ell_\infty; \tag{9}$$

(b) $a\in(\ell_\infty(\Delta))^\beta$ if and only if

$$Ra\in\ell_1 \text{ and } Ra\in\mathbf{n}^{-1}*c_0; \tag{10}$$

furthermore $(c(\Delta))^\beta=(\ell_\infty(\Delta))^\beta$.

Using the notation from the previous proposition, we obtain the next results which connect some matrix domains of triangles and generalized $\beta B-$ duals, for example.

Corollary 3.2. *i) $a\in(c_0(\Delta))^{\beta B}$ if and only if*

$$R(Ba)\in\ell_1 \text{ and } R(Ba)\in\mathbf{n}^{-1}*\ell_\infty; \tag{11}$$

ii) $a\in(\ell_\infty(\Delta))^{\beta B}$ if and only if

$$R(Ba)\in\ell_1 \text{ and } R(Ba)\in\mathbf{n}^{-1}*c_0; \tag{12}$$

furthermore $(c(\Delta))^{\beta B}=(\ell_\infty(\Delta))^{\beta B}$.

Proof. The results are direct consequences of Theorem 3.1 and Theorem 2.3.
Actually, $a\in(X(\Delta))^{\beta B}$ is equivalent to $a\in(X(\Delta))^\beta(B)$, that is $Ba\in(X(\Delta))^\beta$. Further,

$$(c(\Delta))^{\beta B}=(c(\Delta))^\beta(B)=(\ell_\infty(\Delta))^\beta(B)=(\ell_\infty(\Delta))^{\beta B}.$$

□

Let us assume that A is a bijective mapping from X into itself where X is a sequence space. By $X(A,\Delta)$ we will denote matrix domain of matrix ΔA in X. It is worthwile to mention that Δ is a triangle, which allows us to use $(\Delta A)x = \Delta(Ax)$. (*A matrix* $T = (t_{nk})_{n,k=0}^{\infty}$ *is said to be a triangle if* $t_{nk} = 0$ *for all* $k > n$ *and* $t_{nn} \neq 0$ $(n = 0, 1 \ldots)$). This is not true in general for arbitrary infinite matrices, that is, we cannot use $(AB)x = A(Bx)$ [11, 1.4.].

Theorem 3.3. *Let A be a bijective mapping from X into itself and $X \subset \omega$. Then we have:*

$$M_{A,B}(X(A,\Delta), Y) = M_B(X(\Delta), Y).$$

Proof. Since $X(A,\Delta) = (X(\Delta))(A)$, applying Theorem 2.9 we obtain:

$$\begin{aligned} M_{A,B}(X(A,\Delta), Y) &= M_B(A(X(A,\Delta)), Y) = M(A(X(\Delta)(A)), Y)(B) = \\ &= M(X(\Delta), Y)(B) = M_B(X(\Delta), Y). \end{aligned} \tag{13}$$

□

The previous theorem explains why the results from [4, Theorem 10] and [5, Theorem 3.12], for example, are the same. We give the explanations and equivalences in the next corollary.

Corollary 3.4. *Let A be a bijective mapping from X into itself and $X \subset \omega$. Then*

$$(X(A,\Delta))^{\dagger AB} = (X(\Delta))^{\dagger B},$$

where $\dagger \in \{\alpha, \beta, \gamma, N\}$.

Proof. The statement is a direct consequence of the previous theorem and the relation:

$$(X(A,\Delta))^{\dagger AB} = M_{A,B}(X(A,\Delta), Y)$$

where

$$Y = \begin{cases} \ell_1 & \text{for } \dagger = \alpha \\ cs & \text{for } \dagger = \beta \\ bs & \text{for } \dagger = \gamma \\ c_0 & \text{for } \dagger = N. \end{cases} \tag{14}$$

□

At the end of this paper, let us comment some special cases, notations and meanings. For that purpose, we will start with the question: What does the $\beta\Delta-$ dual mean? Intuitively, in one's mind it occurs when one of the matrices A or B is Δ and the other one is equal to I (with all necessary assumptions about the bijective mapping associated with A). But, also one can conclude that the $\beta\Delta-$ dual is a special case of the βB dual, for $B = \Delta$. All these interpretations are intuitively correct but give the readers different results.

Remark 3.5. *In order to illustrate everything mentioned above, let us consider $X = bv$, the set of all sequences of bounded variation [9, 11]. This space can be represented as the matrix domain of the triangle Δ in ℓ_1, that is, $bv = \ell_1(\Delta)$. It is well-known that $bv^\beta = cs$ [11]. Also, cs can be represented as matrix domain of Σ in c, i.e. $cs = c(\Sigma)$. The matrices Δ and Σ are triangle, so invertible [11]. It is clear, as a natural consequence of everything mentioned in the previous sections, that:*

$$(bv)^{\beta AB} = M_{A,B}(bv, cs) = M_B(A(bv), cs) = M(A(bv), cs)(B) = (A(bv))^\beta(B);$$

$$(bv)^{\beta I\Delta} = M_{I,\Delta}(bv, cs) = (I(bv))^\beta(\Delta) = bv^\beta(\Delta) = cs(\Delta) = c;$$

$$(bv)^{\beta\Delta I} = M_{\Delta,I}(bv, cs) = (\Delta(bv))^\beta(I) = \ell_1^\beta = \ell_\infty;$$

$$(bv)^{\beta\Delta} = M_\Delta(bv, cs) = M(bv, cs)(\Delta) = bv^\beta(\Delta) = cs(\Delta) = c(\Sigma)(\Delta) = c.$$

Hence, $M_{I,\Delta}(X, Y) \neq M_{\Delta,I}(X, Y)$ and $M_{I,B}(X, Y) = M_B(X, Y)$.

Corollary 3.6. *Let A and B be infinite matrices, $X, Y \subset \omega$. Then:*
i) $M_{I,A}(X, Y) = M_A(X, Y)$;
ii) In general, $M_{I,A}(X, Y) \neq M_{A,I}(X, Y)$, *that is* $M_{A,I}(X, Y) \neq M_A(X, Y)$.

Remark 3.7. *The previous results concern bijective matrix mappings and bounded linear operators. We are not considering the problem here of checking if a matrix map is bijective, which can be problematic, for example in cases when the matrix is not a triangle. We remark that bounded linear operators between sequence spaces are not always given by matrix transformations ([11]), for instance, if the initial space is* ℓ_∞ . *If* X *and* Y *are BK spaces and* $A \in (X, Y)$ *defines a bijective operator* $L_A \in \mathcal{B}(X, Y)$, *then* L_A^{-1} *exists but may not be given by matrix (for example in the cases, where the space* Y *does not have AK). Hence, all our results are generalizations in the theoretical sense, but checking if the matrix mappings are bijective may not always be possible.*

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