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The star order on C^*-modular operators

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Abstract. By Moore-Penrose properties and block matrix forms of C^*-modular operators we prove that $T \leqslant_* S$ is equivalent to $T \leqslant^* S$ that define ordering relation, when T and S have closed ranges, we give an explicit formula for Moore-Penrose product of $S^\dagger$ and T, in the case it is idempotent.

1. Introduction

Let $M_{m,n}(\mathbb{C})$ be the algebra of all $m \times n$ complex matrices, and let $B(\mathcal{H})$ be the algebra of all bounded linear operators on an infinite-dimensional complex Hilbert space $\mathcal{H}$.

One of such orders is the star partial order, which was defined by Drazin [4] for complex matrices, and Dolinar [3] stated the equivalent definition of the star partial order on $B(\mathcal{H})$, by using orthogonal projections.

Drazin [4] introduced two binary relations in the set of complex matrices by combining each of the conditions

$$T^*T = T^*S \quad \text{and} \quad TT^* = ST^*, \tag{1}$$

and

$$T^\dagger T = T^\dagger S = S^\dagger T \quad \text{and} \quad TT^\dagger = TS^\dagger = ST^\dagger, \tag{2}$$

The star partial ordering defined by (1) is due to Drazin [4]. Hartwig [7] inspired from Drazin [4] and introduced the plus partial order (or minus partial order).

The star order is investigated by some authors, that we refer to the [1, 2, 6, 7].

In this paper, we introduce star order and Moore-Penrose order in Hilbert C^*-modules. Let $\mathcal{X}, \mathcal{Y}$ be Hilbert $\mathcal{A}$-modules and $T, S \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ have closed ranges. We denote the star order by

$$T \leqslant_* S \quad \text{whenever} \quad T^*T = T^*S \quad \text{and} \quad TT^* = ST^*, \tag{3}$$

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and Moore-Penrose order by

$$T \leqslant^* S \quad \text{whenever } T^\dagger \text{ exists such that } T^\dagger T = T^\dagger S \text{ and } TT^\dagger = ST^\dagger. \tag{4}$$

By Moore-Penrose properties and block matrix forms of C^*-modular operators we show that $T \leqslant_* S$ is equivalent to $T \leqslant^* S$ that define ordering relation, when T and S have closed ranges, and we give an explicit formula for Moore-Penrose product of $S^\dagger$ and T, in the case it is idempotent. We obtain some results that one of two binary relation holds, such as $T^*T = T^*S$ and $ST^\dagger = TT^\dagger$ that is equivalent with $T \leqslant^* S$.

Inner product C^*-modules are generalizations of inner product spaces by allowing inner products to take values in some C^*-algebras instead of the field of complex numbers. More precisely, an inner-product module over a C^*-algebra $\mathfrak{A}$ is a right $\mathfrak{A}$-module equipped with an $\mathfrak{A}$-valued inner product $\langle\cdot,\cdot\rangle : \mathcal{X}\times\mathcal{X} \to \mathfrak{A}$. If $\mathcal{X}$ is complete with respect to the induced norm defined by $\|x\| = \|\langle x, x\rangle\|^{\frac{1}{2}}$ $(x \in \mathcal{X})$, then $\mathcal{X}$ is called a *Hilbert $\mathfrak{A}$-module*. Some fundamental properties of inner product spaces are no longer valid in inner product C^*-modules in their complete generality. Consequently, when we are studying inner product C^*-modules, it is always of interest under which conditions as well as which more general, situations might appear. The book [9] is used as a standard reference source.

Throughout the rest of this paper, $\mathfrak{A}$ denotes a C^*-algebra and $\mathcal{X},\mathcal{Y}$ denote Hilbert $\mathfrak{A}$-modules. Let $\mathcal{L}(\mathcal{X},\mathcal{Y})$ be the set of operators $T : \mathcal{X} \to \mathcal{Y}$ for which there is an operator $T^* : \mathcal{Y} \to \mathcal{X}$ such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for any $x \in \mathcal{X}$ and $y \in \mathcal{Y}$. It is known that any element $T \in \mathcal{L}(\mathcal{X},\mathcal{Y})$ must be bounded and $\mathfrak{A}$-linear. In general, a bounded operator between Hilbert C^*-modules may be not adjointable. We call $\mathcal{L}(\mathcal{X},\mathcal{Y})$ the set of all adjointable operators from $\mathcal{X}$ to $\mathcal{Y}$. In the case when $\mathcal{X} = \mathcal{Y}$, $\mathcal{L}(\mathcal{X},\mathcal{X})$, abbreviated to $\mathcal{L}(\mathcal{X})$, is a C^*-algebra. For any operator T between linear spaces, the range and the null space of T are denoted by $\mathcal{R}(T)$ and $\mathcal{N}(T)$, respectively.

A closed submodule M of $\mathcal{X}$ is said to be *orthogonally complemented* if $\mathcal{X} = M \oplus M^\perp$, where $M^\perp = \Big\{x \in \mathcal{X} : \langle x, y\rangle = 0 \text{ for any } y \in M\Big\}$. If $T \in \mathcal{L}(\mathcal{X},\mathcal{Y})$ does not have closed range, then neither $\mathcal{N}(T)$ nor $\overline{\mathcal{R}(T)}$ needs to be orthogonally complemented. In addition, if $T \in \mathcal{L}(\mathcal{X},\mathcal{Y})$ and $\overline{\mathcal{R}(T^*)}$ is not orthogonally complemented, then it may happen that $\mathcal{N}(T)^\perp \neq \overline{\mathcal{R}(T^*)}$; see [9, 10]. The above facts show that the theory Hilbert C^*-modules are much different and more complicated than that of Hilbert spaces.

2. Main results

By Moore-Penrose properties and block matrix forms of C^*-modular operators we prove that $T \leqslant_* S$ is equivalent to $T \leqslant^* S$ that define ordering relation. When T and S have closed ranges, we give an explicit formula for Moore-Penrose product of $S^\dagger$ and T, in the case it is idempotent.

Conditions are stated in the following theorem that $(ST^\dagger)^* = TS^\dagger$ hold.

Theorem 2.1. *Let $\mathcal{X},\mathcal{Y}$ be Hilbert $\mathcal{A}$-modules and $T, S \in \mathcal{L}(\mathcal{X},\mathcal{Y})$ have closed ranges such that $T^*T = T^*S$ and $ST^\dagger = TT^\dagger$, then $(ST^\dagger)^* = TS^\dagger$.*

Proof. We have

$$ST^\dagger = SS^\dagger ST^\dagger = (SS^\dagger)^*ST^\dagger = (S^\dagger)^*S^*ST^\dagger = (S^\dagger)^*S^*TT^\dagger.$$

Taking adjoint we conclude that $(ST^\dagger)^* = (T^\dagger)^*T^*SS^\dagger = (T^\dagger)^*T^*TS^\dagger = TS^\dagger$. $\square$

Now, we give an explicit formula for Moore-Penrose product of $S^\dagger$ and T, in the case it is idempotent.

Theorem 2.2. *Suppose that $T, S \in \mathcal{L}(\mathcal{X},\mathcal{Y})$ and $S^\dagger T$ and $TS^\dagger$ have closed ranges. Then the following assertions hold.*

(i) If $TT^\dagger = ST^\dagger$ then $(S^\dagger T)^\dagger$ is idempotent and

$$(S^\dagger T)^\dagger = (S^\dagger T)^* - P_{\mathcal{R}(S^*)}[(1 - P_{\mathcal{R}(T^*)})(1 - P_{\mathcal{R}(S^*)})]^\dagger P_{\mathcal{R}(S^*)}.$$

(ii) If $T^*T = T^*S$ then $(TS^\dagger)^\dagger$ is idempotent and

$$(TS^\dagger)^\dagger = (TS^\dagger)^* - P_{\mathcal{R}(S)}[(1 - P_{\mathcal{R}(S)})(1 - P_{\mathcal{R}(T)})]^\dagger P_{\mathcal{R}(T)}.$$

Proof. (i) Suppose that $TT^\dagger = ST^\dagger$. Multiplying by T on the right we have $T = ST^\dagger T$. Multiplying $S^\dagger$ on the left yields $S^\dagger T = S^\dagger ST^\dagger T = P_{\mathcal{R}(S^*)}P_{\mathcal{R}(T^*)}$. Now, [11, Theorem 2.3] implies that $(S^\dagger T)^\dagger$ is idempotent and [11, Corollary 2.4] implies that

$$(S^\dagger T)^\dagger = (S^\dagger T)^* - P_{\mathcal{R}(T^*)}[(1 - P_{\mathcal{R}(T^*)})(1 - P_{\mathcal{R}(S^*)})]^\dagger P_{\mathcal{R}(S^*)}.$$

(ii) Since $T^*T = T^*S$, multiplying by $(T^*)^\dagger$ on the left we have $T = (T^*)^\dagger T^*T = (T^*)^\dagger T^*S = TT^\dagger S$. Multiplying $T = TT^\dagger S$ on the right by $S^\dagger$ yields $TS^\dagger = TT^\dagger SS^\dagger = P_{\mathcal{R}(T)}P_{\mathcal{R}(S)}$. Again by applying [11, Theorem 2.3] implies that $(TS^\dagger)^\dagger$ is idempotent and [11, Corollary 2.4] immediately implies that

$$(TS^\dagger)^\dagger = (TS^\dagger)^* - P_{\mathcal{R}(S)}[(1 - P_{\mathcal{R}(S)})(1 - P_{\mathcal{R}(T)})]^\dagger P_{\mathcal{R}(T)}.$$

□

Remark 2.3. *In Theorem 2.2 items (i) and (ii), respectively, conditions $TT^\dagger = ST^\dagger$ and $T^*T = T^*S$ can be replaced by $TT^* = ST^*$ and $T^\dagger T = T^\dagger S$.*

The following theorem is expressed that $\leqslant_*$ coincides with $\leqslant^*$.

Theorem 2.4. *Let $\mathcal{X}, \mathcal{Y}$ be Hilbert $\mathcal{A}$-modules and $T, S \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ be such that T has closed range. Then $T \leqslant_* S$ if and only if $T \leqslant^* S$.*

Proof. Since T, S have closed ranges, we have $\mathcal{X} = \mathcal{R}(S^*) \oplus \mathcal{N}(S)$ and $\mathcal{Y} = \mathcal{R}(T) \oplus \mathcal{N}(T^*)$. Hence, by using these complemented submodules, T and S admit the following matrix representations

$$\begin{aligned} T &= \begin{bmatrix} T_1 & T_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(S^*) \\ \mathcal{N}(S) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(T) \\ \mathcal{N}(T^*) \end{bmatrix}, \\ S &= \begin{bmatrix} S_1 & 0 \\ S_2 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(S^*) \\ \mathcal{N}(S) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(T) \\ \mathcal{N}(T^*) \end{bmatrix}. \end{aligned}$$

By matrix decompositions T and S, we obtain matrix representations T^*T, T^*S, TT^*, $ST^*, T^\dagger T, T^\dagger S, TT^\dagger$ and $ST^\dagger$ with the following

$$T^*T = \begin{bmatrix} T_1^* & 0 \\ T_2^* & 0 \end{bmatrix} \begin{bmatrix} T_1 & T_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} T_1^*T_1 & T_1^*T_2 \\ T_2^*T_1 & T_2^*T_2 \end{bmatrix}, \tag{5}$$

$$T^*S = \begin{bmatrix} T_1^* & 0 \\ T_2^* & 0 \end{bmatrix} \begin{bmatrix} S_1 & 0 \\ S_2 & 0 \end{bmatrix} = \begin{bmatrix} T_1^*S_1 & 0 \\ T_2^*S_1 & 0 \end{bmatrix}, \tag{6}$$

$$TT^* = \begin{bmatrix} T_1 & T_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_1^* & 0 \\ T_2^* & 0 \end{bmatrix} = \begin{bmatrix} T_1T_1^* + T_2T_2^* & 0 \\ 0 & 0 \end{bmatrix}, \tag{7}$$

$$ST^* = \begin{bmatrix} S_1 & 0 \\ S_2 & 0 \end{bmatrix} \begin{bmatrix} T_1^* & 0 \\ T_2^* & 0 \end{bmatrix} = \begin{bmatrix} S_1T_1^* & 0 \\ S_2T_1^* & 0 \end{bmatrix}, \tag{8}$$

by using [8, Lemma 2.4], we have

$$T^{\dagger}T = \begin{bmatrix} T_1^*E^{-1} & 0 \\ T_2^*E^{-1} & 0 \end{bmatrix}\begin{bmatrix} T_1 & T_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} T_1^*E^{-1}T_1 & T_1^*E^{-1}T_2 \\ T_2^*E^{-1}T_1 & T_2^*E^{-1}T_2 \end{bmatrix}, \tag{9}$$

$$T^{\dagger}S = \begin{bmatrix} T_1^*E^{-1} & 0 \\ T_2^*E^{-1} & 0 \end{bmatrix}\begin{bmatrix} S_1 & 0 \\ S_2 & 0 \end{bmatrix} = \begin{bmatrix} T_1^*E^{-1}S_1 & 0 \\ T_2^*E^{-1}S_1 & 0 \end{bmatrix}, \tag{10}$$

$$TT^{\dagger} = \begin{bmatrix} T_1 & T_2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} T_1^*E^{-1} & 0 \\ T_2^*E^{-1} & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \tag{11}$$

$$ST^{\dagger} = \begin{bmatrix} S_1 & 0 \\ S_2 & 0 \end{bmatrix}\begin{bmatrix} T_1^*E^{-1} & 0 \\ T_2^*E^{-1} & 0 \end{bmatrix} = \begin{bmatrix} S_1T_1^*E^{-1} & 0 \\ S_2T_1^*E^{-1} & 0 \end{bmatrix}, \tag{12}$$

where $E = T_1T_1^* + T_2T_2^*$ is invertible.

($\Rightarrow$) Now, suppose that $T \leqslant_* S$ or equivalently, $T^*T = T^*S$ and $TT^* = ST^*$. By the equations (5) and (6),

$$\begin{bmatrix} T_1^*T_1 & T_1^*T_2 \\ T_2^*T_1 & T_2^*T_2 \end{bmatrix} = \begin{bmatrix} T_1^*S_1 & 0 \\ T_2^*S_1 & 0 \end{bmatrix}$$

and consequently

$$T_1^*T_1 = T_1^*S_1, \tag{13}$$

$$T_2^*T_1 = T_2^*S_1, \tag{14}$$

$$T_1^*T_2 = 0,$$

$$T_2^*T_2 = 0. \tag{15}$$

Equation (15) implies that,

$$T_2 = 0. \tag{16}$$

Since $TT^* = ST^*$ then (16) implies that

$$\begin{bmatrix} T_1T_1^* & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} S_1T_1^* & 0 \\ S_2T_1^* & 0 \end{bmatrix}$$

that is

$$T_1T_1^* = S_1T_1^*, \tag{17}$$

$$S_2T_1^* = 0. \tag{18}$$

Since $E = T_1T_1^*$ is invertible, multiplying the equality (17) by $(T_1T_1^*)^{-1}$ from the right side, we obtain

$$S_1T_1^*(T_1T_1^*)^{-1} = 1. \tag{19}$$

Again, multiplying the equality (18) by E^{-1} from the right side, we get

$$S_2T_1^*E^{-1} = 0, \tag{20}$$

by applying (19) and (20) we conclude

$$ST^{\dagger} = \begin{bmatrix} S_1T_1^*E^{-1} & 0 \\ S_2T_1^*E^{-1} & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = TT^{\dagger}. \tag{21}$$

On the other hand, multiplying the equality (13) by $E^{-1}T_1$ from the left side, we obtain

$$(E^{-1}T_1T_1^*)T_1 = (E^{-1}T_1T_1^*)S_1,$$

$$T_1 = S_1. \tag{22}$$

Now, prove that $T^\dagger T = T^\dagger S$. By (16) and (22) we have

$$T^\dagger T = \begin{bmatrix} T_1^* E^{-1} T_1 & T_1^* E^{-1} T_2 \\ T_2^* E^{-1} T_1 & T_2^* E^{-1} T_2 \end{bmatrix} = \begin{bmatrix} T_1^* E^{-1} S_1 & 0 \\ T_2^* E^{-1} S_1 & 0 \end{bmatrix} = T^\dagger S. \tag{23}$$

Then (21) and (23) implies that, $T \leqslant^* S$.

Conversely, suppose that $T \leqslant^* S$ that is, $T^\dagger T = T^\dagger S$ and $TT^\dagger = ST^\dagger$. As $T^\dagger T = T^\dagger S$ by applying (9) and (10), we have

$$\begin{bmatrix} T_1^* E^{-1} T_1 & T_1^* E^{-1} T_2 \\ T_2^* E^{-1} T_1 & T_2^* E^{-1} T_2 \end{bmatrix} = \begin{bmatrix} T_1^* E^{-1} S_1 & 0 \\ T_2^* E^{-1} S_1 & 0 \end{bmatrix}$$

so we conclude that

$$T_1^* E^{-1} T_1 = T_1^* E^{-1} S_1, \tag{24}$$
$$T_2^* E^{-1} T_1 = T_2^* E^{-1} S_1, \tag{25}$$
$$T_1^* E^{-1} T_2 = 0, \tag{26}$$
$$T_2^* E^{-1} T_2 = 0. \tag{27}$$

By multiplication T_1 and T_2 on the left of equations (41) and (42), respectively, and additive obtained the equalities we achieve, $(T_1 T_1^* + T_2 T_2^*) E^{-1} T_1 = E E^{-1} S_1$, then $T_1 = S_1$.

Again, By multiplication T_1 and T_2 on the left of equations (26) and (27), respectively, we get $T_1 T_1^* E^{-1} T_2 = 0$ and $T_2 T_2^* E^{-1} T_2 = 0$. By additive the obtained equalities, we have $T_2 = 0$.

Also, since $TT^\dagger = ST^\dagger$ rewrite matrix forms (11) and (12)

$$TT^\dagger = \begin{bmatrix} (T_1 T_1^* + T_2 T_2^*) E^{-1} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} S_1 T_1^* E^{-1} & 0 \\ S_2 T_1^* E^{-1} & 0 \end{bmatrix} = ST^\dagger,$$

this conclude that, $S_1 T_1^* E^{-1} = 1$ and $S_2 T_1^* E^{-1} = 0$. Since E is invertible, we obtain

$$S_1 T_1^* = E, \tag{28}$$
$$S_2 T_1^* = 0. \tag{29}$$

By the equations $T_1 = S_1$ and $T_2 = 0$ and (29) we get

$$T^* T = \begin{bmatrix} T_1^* T_1 & T_1^* T_2 \\ T_2^* T_1 & T_2^* T_2 \end{bmatrix} = \begin{bmatrix} T_1^* S_1 & 0 \\ 0 & 0 \end{bmatrix} = T^* S, \tag{30}$$

$$TT^* = \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} S_1 T_1^* & 0 \\ S_2 T_1^* & 0 \end{bmatrix} = ST^*. \tag{31}$$

Hence, equations (30) and (31) implies that $T \leqslant_* S$. $\square$

Theorem 2.5. *Let $\mathcal{X}, \mathcal{Y}$ be Hilbert $\mathcal{A}$-modules and $T, S \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ such that T has closed range. If $T \leqslant_* S$ then $TS^* \leqslant_* ST^*$.*

Proof. Suppose that $T \leqslant_* S$, that is $T^* T = T^* S$ and $TT^* = ST^*$.
To show that $TS^*(TS^*)^* = ST^*(TS^*)^*$ and $(TS^*)^* TS^* = (TS^*)^* ST^*$, or equivalently,

$$ST^* TS^* = (ST^*)^2 = TS^* ST^*.$$

By using the complemented submodules from [8, Lemma 2.4] and matrix representations T, S and equation (12), we compute ST^*TS^*, $(ST^*)^2$ and TS^*ST^* with the following

$$
\begin{aligned}
ST^*TS^* &= \begin{bmatrix} S_1 & 0 \\ S_2 & 0 \end{bmatrix}\begin{bmatrix} T_1 & T_2 \\ 0 & 0 \end{bmatrix}^*\begin{bmatrix} T_1 & T_2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} S_1 & 0 \\ S_2 & 0 \end{bmatrix}^* \\
&= \begin{bmatrix} S_1 & 0 \\ S_2 & 0 \end{bmatrix}\begin{bmatrix} T_1^* & 0 \\ T_2^* & 0 \end{bmatrix}\begin{bmatrix} T_1 & T_2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} S_1^* & S_2^* \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} S_1T_1^*T_1S_1^* & S_1T_1^*T_1S_2^* \\ S_2T_1^*T_1S_1^* & S_2T_1^*T_1S_2^* \end{bmatrix} \\
\text{(by (18))} &= \begin{bmatrix} S_1T_1^*T_1S_1^* & 0 \\ 0 & 0 \end{bmatrix} \\
\text{(by (22))} &= \begin{bmatrix} (T_1T_1^*)^2 & 0 \\ 0 & 0 \end{bmatrix},
\end{aligned}
$$

$$
\begin{aligned}
(ST^*)^2 &= \begin{bmatrix} S_1T_1^* & 0 \\ S_2T_1^* & 0 \end{bmatrix}\begin{bmatrix} S_1T_1^* & 0 \\ S_2T_1^* & 0 \end{bmatrix} \\
&= \begin{bmatrix} S_1T_1^*S_1T_1^* & 0 \\ S_2T_1^*S_1T_1^* & 0 \end{bmatrix} \\
\text{(by (18))} &= \begin{bmatrix} S_1T_1^*S_1T_1^* & 0 \\ 0 & 0 \end{bmatrix} \\
\text{(by (22))} &= \begin{bmatrix} (T_1T_1^*)^2 & 0 \\ 0 & 0 \end{bmatrix}
\end{aligned}
$$

and

$$
\begin{aligned}
TS^*ST^* &= \begin{bmatrix} T_1S_1^* & T_1S_2^* \\ 0 & 0 \end{bmatrix}\begin{bmatrix} S_1T_1^* & 0 \\ S_2T_1^* & 0 \end{bmatrix} \\
&= \begin{bmatrix} T_1S_1^*S_1T_1^* + T_1S_2^*S_2T_1^* & 0 \\ 0 & 0 \end{bmatrix} \\
\text{(by (18))} &= \begin{bmatrix} T_1S_1^*S_1T_1^* & 0 \\ 0 & 0 \end{bmatrix} \\
\text{(by (22))} &= \begin{bmatrix} (T_1T_1^*)^2 & 0 \\ 0 & 0 \end{bmatrix}.
\end{aligned}
$$

Hence, we conclude that

$$ST^*TS^* = (ST^*)^2 = TS^*ST^*. \tag{32}$$

Equation (32) implies that $TS^*(TS^*)^* = ST^*(TS^*)^*$ and $(TS^*)^* = TS^*ST^*$, or, equivalently $TS^* \leqslant_* ST^*$. □

In the following theorem we show that $\leqslant_*$ has some inherited properties.

Theorem 2.6. *Let $\mathcal{X}, \mathcal{Y}$ be Hilbert $\mathcal{A}$-modules and $T, S \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ have closed ranges such that $T \leqslant_* S$. Then the following statements are hold:*

(i) $T^* \leqslant_* S^*$,
(ii) $T^\dagger \leqslant_* S^\dagger$,
(iii) $(T - S)^\dagger = T^\dagger - S^\dagger$,
(iv) $(TT^* - SS^*)^\dagger = (TT^*)^\dagger - (SS^*)^\dagger$,
(v) $(TT^\dagger - SS^\dagger)^\dagger = (TT^\dagger)^\dagger - (SS^\dagger)^\dagger$,

(vi) $(TS^* - ST^*)^\dagger = (TS^*)^\dagger - (ST^*)^\dagger$.

Proof. (i) Suppose that $T \leqslant^* S$, or, equivalently $T^*T = T^*S$ and $TT^* = ST^*$. By taking conjugates of previous equations, we get $T^*T = S^*T$ and $TT^* = TS^*$, respectively. Hence, implies that $T^* \leqslant_* S^*$.

(ii) Assuming the case is equivalent with $T^*T = T^*S$ and $TT^* = ST^*$. By multiplication S_1^* and S_2^* on the left of equations (28) and (29), respectively, and additive obtained the equalities we achieve,

$$FT_1^* = S_1^*E. \tag{33}$$

Where $F = S_1^*S_1 + S_2^*S_2$ is invertible. Multiplying (33) by E^{-1} on the right side and by $E^{-1}T_1F^{-1}$ on the left side, we obtain

$$E^{-1}T_1F^{-1}S_1^* = E^{-1}. \tag{34}$$

We show that $(T^\dagger)^*T^\dagger = (T^\dagger)^*S^\dagger$ and $T^\dagger(T^\dagger)^* = S^\dagger(T^\dagger)^*$ and conclude that $T^\dagger \leqslant_* S^\dagger$. Since $TS^\dagger = ST^\dagger$ then

$$\begin{bmatrix} T_1 & T_2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} F^{-1}S_1^* & F^{-1}S_1^* \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} S_1 & 0 \\ S_2 & 0 \end{bmatrix}\begin{bmatrix} T_1^*E^{-1} & 0 \\ T_2^*E^{-1} & 0 \end{bmatrix}$$

$$\begin{bmatrix} T_1F^{-1}S_1^* & T_1F^{-1}S_2^* \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} S_1T_1^*E^{-1} & 0 \\ S_2T_1^*E^{-1} & 0 \end{bmatrix}. \tag{35}$$

So we have $T_1F^{-1}S_2^* = 0$ and consequently,

$$E^{-1}T_1F^{-1}S_2^* = 0. \tag{36}$$

Using [8, Lemma 2.4], (34) and (36) we compute

$$\begin{aligned} (T^\dagger)^*T^\dagger &= \begin{bmatrix} E^{-1}T_1 & E^{-1}T_2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} T_1^*E^{-1} & 0 \\ T_2^*E^{-1} & 0 \end{bmatrix} \\ &= \begin{bmatrix} E^{-1} & 0 \\ 0 & 0 \end{bmatrix} \end{aligned} \tag{37}$$

and

$$\begin{aligned} (T^\dagger)^*S^\dagger &= \begin{bmatrix} E^{-1}T_1 & E^{-1}T_2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} F^{-1}S_1^* & F^{-1}S_2^* \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} E^{-1}T_1F^{-1}S_1^* & E^{-1}T_1F^{-1}S_2^* \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} E^{-1} & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned} \tag{38}$$

Hence, equations (37) and (38) implies that $(T^\dagger)^*T^\dagger = (T^\dagger)^*S^\dagger$.

In the same way, applying the equation (33) we derive that $T_1^*E^{-1} = F^{-1}S_1^*$. Again, by using [8, Lemma 2.4] and (16) we reach

$$\begin{aligned} T^\dagger(T^\dagger)^* &= \begin{bmatrix} T_1^*E^{-1} & 0 \\ T_2^*E^{-1} & 0 \end{bmatrix}\begin{bmatrix} E^{-1}T_1 & E^{-1}T_2 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} T_1^*(E^{-1})^2T_1 & T_1^*(E^{-1})^2T_2 \\ T_2^*(E^{-1})^2T_1 & T_2^*(E^{-1})^2T_2 \end{bmatrix} = \begin{bmatrix} T_1^*(E^{-1})^2T_1 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned} \tag{39}$$

and

$$\begin{aligned} S^\dagger(T^\dagger)^* &= \begin{bmatrix} F^{-1}S_1^* & F^{-1}S_2^* \\ 0 & 0 \end{bmatrix}\begin{bmatrix} E^{-1}T_1 & E^{-1}T_2 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} F^{-1}S_1^*E^{-1}T_1 & F^{-1}S_1^*E^{-1}T_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} T_1^*(E^{-1})^2T_1 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned} \tag{40}$$

Equations (39) and (40) implies that $T^\dagger T^{\dagger^*} = S^\dagger T^{\dagger^*}$.
Hence $T^\dagger \leqslant_* S^\dagger$.

(*iii*) By Theorem 2.4 since $T \leqslant_* S$, we have $T^\dagger T = T^\dagger S = S^\dagger T$ and $TT^\dagger = ST^\dagger = TS^\dagger$. Let $X = T^\dagger - S^\dagger$, since

$$\begin{aligned}
& (T-S)X(T-S) = (T-S)(T^\dagger - S^\dagger)(T-S) \\
= \ & TT^\dagger T - TT^\dagger S - TS^\dagger T + TS^\dagger S - ST^\dagger T + ST^\dagger S + SS^\dagger T - SS^\dagger S \\
= \ & T - ST^\dagger S - TT^\dagger S + ST^\dagger S - SS^\dagger T + ST^\dagger S + SS^\dagger T - S \\
= \ & T - TT^\dagger S + ST^\dagger S - S = T - ST^\dagger S + ST^\dagger S - S = T - S,
\end{aligned}$$

$$\begin{aligned}
& X(T-S)X = (T^\dagger - S^\dagger)(T-S)(T^\dagger - S^\dagger) \\
= \ & T^\dagger TT^\dagger - T^\dagger TS^\dagger - T^\dagger ST^\dagger + T^\dagger ST^\dagger - S^\dagger TT^\dagger + S^\dagger TS^\dagger + S^\dagger ST^\dagger - S^\dagger SS^\dagger \\
= \ & T^\dagger - T^\dagger TS^\dagger - T^\dagger ST^\dagger + T^\dagger ST^\dagger - S^\dagger TT^\dagger + S^\dagger TS^\dagger + S^\dagger ST^\dagger - S^\dagger \\
= \ & T^\dagger - S^\dagger TS^\dagger - S^\dagger TS^\dagger + S^\dagger TS^\dagger - S^\dagger ST^\dagger + S^\dagger TS^\dagger + S^\dagger ST^\dagger - S^\dagger \\
= \ & T^\dagger - S^\dagger = X,
\end{aligned}$$

$$\begin{aligned}
(T-S)X &= (T-S)(T^\dagger - S^\dagger) = TT^\dagger - TS^\dagger - ST^\dagger + SS^\dagger \\
&= TT^\dagger - TT^\dagger - TT^\dagger + SS^\dagger = SS^\dagger - TT^\dagger.
\end{aligned}$$

So $(T-S)X$ is hermitian. In the same way, prove that $X(T-S)$ is hermitian. By uniqueness of Moore-Penrose inverse, we have $(T-S)^\dagger = T^\dagger - S^\dagger$.

(*iv*) We know $TT^* \leqslant_* SS^*$ if and only if $(TT^*)^*TT^* = (TT^*)^*SS^*$ and $TT^*(TT^*)^* = SS^*(TT^*)^*$, or, equivalently,

$$(TT^*)^2 = TT^*SS^* = SS^*(TT^*)^*.$$

Since $T \leqslant_* S$ then

$$\begin{aligned}
T^*T &= T^*S, && (41) \\
TT^* &= ST^*. && (42)
\end{aligned}$$

Thus, we get

$$\begin{aligned}
TT^*SS^* &= T(T^*S)S^* \\
\text{(by (41))} \quad &= T(T^*T)S^* \\
&= (TT^*)(TS^*) \\
\text{(by (42))} \quad &= (TT^*)(TT^*) \\
&= (TT^*)^2. && (43)
\end{aligned}$$

Taking conjugate of (41), we obtain that $(TT^*)^2 = TT^*SS^* = SS^*(TT^*)^*$ is satisfied. By using the statement (*iii*) we conclude that $(TT^* - SS^*)^\dagger = (TT^*)^\dagger - (SS^*)^\dagger$.

(*v*) Suppose that $T \leqslant_* S$. By Theorems 2.1 and 2.4, we obtain $TT^\dagger = TS^\dagger$. So we get

$$TT^\dagger = TS^\dagger = TT^\dagger TS^\dagger = TT^\dagger SS^\dagger. \quad (44)$$

By taking conjugate of (44) we conclude that $TT^\dagger = SS^\dagger TT^\dagger$. Hence $TT^\dagger \leqslant_* SS^\dagger$. By using the statement (*iii*) we conclude that $(TT^\dagger - SS^\dagger)^\dagger = TT^\dagger - SS^\dagger$.

(*vi*) By using Theorem 2.1 and the statement (*iii*) we conclude that $(TS^* - ST^*)^\dagger = (TS^*)^\dagger - (ST^*)^\dagger$. □

Proposition 2.7. *Let $\mathcal{X}, \mathcal{Y}$ be Hilbert $\mathcal{A}$-modules and $T, S \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$. If $U \in \mathcal{L}(\mathcal{X})$ is a unitary operator, then $T \leqslant_* S$ if and only if $TU \leqslant_* SU$.*

Proof. Let U be an unitary operator with respect to the decomposition $X = \mathcal{R}(S^*) \oplus \mathcal{N}(S)$. Since $T \leqslant_* S$ then $T^*T = T^*S$ and $TT^* = ST^*$ therefore, we have

$$(TU)^*TU = U^*T^*TU = U^*T^*SU = (TU)^*SU.$$

Also, we compute

$$TU(TU)^* = TUU^*T^* = TT^* = ST^* = SUU^*T^* = SU(SU)^*.$$

Hence $TU \leqslant_* SU$.

Conversely, suppose that $TU \leqslant_* SU$, then we have $(TU)^*TU = (TU)^*SU$ and $TU(TU)^* = SU(SU)^*$. So we obtain

$$\begin{aligned} (TU)^*TU &= (TU)^*SU \\ U^*T^*TU &= U^*T^*SU \\ UU^*T^*TUU^* &= UU^*T^*SUU^* \\ T^*T &= T^*S. \end{aligned}$$

In the same way,

$$TT^* = TUU^*T^* = TU(TU)^* = SU(TU)^* = SUU^*T^* = ST^*.$$

Therefore, $TT^* = ST^*$ and $T^*T = T^*S$, or, equivalently $T \leqslant_* S$. $\square$

Theorem 2.8. *Let $\mathcal{X}, \mathcal{Y}$ be Hilbert $\mathcal{A}$-modules and $T, S \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, then the following statement are equivalent*

(i) $T^*T = T^*S$ *and* $ST^\dagger = TT^\dagger$,

(ii) $T \leqslant_* S$,

(iii) $T \leqslant^* S$.

Proof. $(i) \Rightarrow (ii)$ Since $T^*T = T^*S$, multiplying by $(T^\dagger)^*$ on the left we have $(T^\dagger)^*T^*T = (T^\dagger)^*T^*S$, therefore $T = TT^\dagger S$, multiplying by $T^\dagger$ on the left we get $T^\dagger T = T^\dagger S$. By this fact and assumption $ST^\dagger = TT^\dagger$ we desired the result.

$(ii) \Leftrightarrow (iii)$ By Theorem 2.4 is obvious.

$(iii) \Rightarrow (i)$ Since $T \leqslant^* S$ and $T \leqslant_* S$ hold, then $ST^\dagger = TT^\dagger$ and $T^*T = T^*S$. $\square$

Theorem 2.9. *Let $\mathcal{X}, \mathcal{Y}$ be Hilbert $\mathcal{A}$-modules and $T, S, U \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ have closed ranges. Then $\leqslant_*$ is an ordering relation.*

Proof. It is clear that $T \leqslant_* T$.

Now suppose that $T \leqslant_* S$ and $S \leqslant_* T$. Then we get $T = TT^\dagger T = TS^\dagger T = SS^\dagger T = SS^\dagger S = S$. Suppose that $T \leqslant_* S$ and $S \leqslant_* U$, we have

$$TT^\dagger = ST^\dagger = TS^\dagger, \tag{45}$$

$$T^\dagger T = T^\dagger S = S^\dagger T, \tag{46}$$

and

$$SS^\dagger = US^\dagger = SU^\dagger, \tag{47}$$

$$S^\dagger S = S^\dagger U = U^\dagger S. \tag{48}$$

Multiplying the equality (47) by S from the right side, leads to $S = US^\dagger S$ and consequently, we obtain $TT^\dagger = US^\dagger ST^\dagger = US^\dagger TT^\dagger = UT^\dagger TT^\dagger = UT^\dagger$. Also, multiplying the equality (48) by S from the left side, leads to $S = SS^\dagger U$ and consequently, we obtain $T^\dagger T = T^\dagger S = T^\dagger SS^\dagger U = T^\dagger TS^\dagger U = T^\dagger TT^\dagger U = T^\dagger U$.

$\square$

Remark 2.10. *As an application of star order we note that if $T, S \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ have closed ranges and $T \leqslant_* S$, since $TS^\dagger = TT^\dagger$. Then the system of operator equations $TXS = T = SXT$ is solvable if and only if $SS^\dagger TS^\dagger S = T$, in this case $T = S$. It is obviously $X = T^\dagger + V(1 - TT^\dagger) + (1 - T^\dagger T)W$, where $V, W \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ are arbitrary operators. Hence, proof [12, Theorem 3.8.] is clear.*

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