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A Note on Unilateral Weighted Left Shift Operators

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Abstract. We give an estimation of the spectrum and the surjective spectrum of unilateral weighted left shifts $L : \ell_\infty \to \ell_\infty$, where ℓ_∞ is the space of all bounded complex sequences.

1. Introduction

Let ℓ_∞ be the Banach space of all bounded sequences of complex numbers with usual linear operations and sup norm. A unilateral weighted left shift $L : \ell_\infty \to \ell_\infty$ is defined by

$$L(\xi_1, \xi_2, \xi_3, \cdots) = (w_1\xi_2, w_2\xi_3, \cdots),$$

where $(\xi_1, \xi_2, \xi_3, \cdots) \in \ell_\infty$ and (w_n) is a sequence of complex numbers satisfying $|w_n| \le 1$ for every $n \in \mathbb{N}$. The objective of this note is to estimate the spectrum of L and the surjective spectrum of L by applying elementary arguments. In a particular situation when $w_n = 1$ for all $n \in \mathbb{N}$ we give a complete description of these spectra.

The following proposition summarizes the basic properties of L.

Proposition 1. *A weighted left shift $L : \ell_\infty \to \ell_\infty$ with the corresponding weight sequence (w_n) satisfying $|w_n| \le 1$, $n \in \mathbb{N}$, is a bounded linear operator. In addition, $\|L\| = \sup\limits_{i\in\mathbb{N}} |w_i|$.*

Proof. Let $x = (\xi_1, \xi_2, \xi_3, \cdots) \in \ell_\infty$. Since $|w_n\xi_{n+1}| = |w_n||\xi_{n+1}| \le \sup_{i\in\mathbb{N}} |\xi_i|$ for every $n \in \mathbb{N}$, it follows that $Lx = (w_1\xi_2, w_2\xi_3, \cdots) \in \ell_\infty$.

Obviously, L is linear. Moreover,

$$\|L(x)\| = \sup_{n\in\mathbb{N}} |w_n\xi_{n+1}| \le (\sup_{i\in\mathbb{N}} |w_i|)(\sup_{j\in\mathbb{N}} |\xi_j|) = (\sup_{i\in\mathbb{N}} |w_i|)\|x\|,$$

and hence $\|L\| \le \sup_{i\in\mathbb{N}} |w_i|$. Now, consider vectors $e_i = (0, \cdots, 0, 1, 0, \cdots)$, $i \in \mathbb{N}$, where 1 is in the ith position, 0 elsewhere. Clearly, $\|e_{i+1}\| = 1$ and $\|Le_{i+1}\| = |w_i|$ for all $i \in \mathbb{N}$, so we have

$$\|L\| \ge \sup_{i\in\mathbb{N}} \|Le_{i+1}\| = \sup_{i\in\mathbb{N}} |w_i|,$$

and consequently $\|L\| = \sup\limits_{i\in\mathbb{N}} |w_i|$. $\square$

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Let X be a Banach space and let T be a bounded linear operator on X. The spectrum of T and the surjective spectrum of T will be denoted by $\sigma(T)$ and $\sigma_{su}(T)$, respectively. The injectivity modulus of T is defined by

$$j(T) = \inf\left\{\frac{\|Tx\|}{\|x\|} : x \in X, x \neq 0\right\}.$$

We say that T is bounded below if $j(T) > 0$. The approximate point spectrum of T, denoted by $\sigma_{ap}(T)$, is the set of all $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is not bounded below. It is well known that $\partial\sigma(T) \subset \sigma_{su}(T) \cap \sigma_{ap}(T)$ [4, Theorem 12.11], where $\partial\sigma(T)$ is the boundary of the spectrum. We will use the symbol $\mathbb{D}(0, r)$ to denote the set $\mathbb{D}(0, r) = \{\lambda \in \mathbb{C} : |\lambda| \leq r\}$, $r > 0$. If $|\lambda| < j(T)$, $\lambda \in \mathbb{C}$, then $T - \lambda I$ is bounded below [4, Proposition 9.10]. Since $\sigma_{ap}(T) \subset \sigma(T) \subset \mathbb{D}(0, \|T\|)$, we obtain a rough description of the approximate point spectrum

$$\sigma_{ap}(T) \subset \{\lambda \in \mathbb{C} : j(T) \leq |\lambda| \leq \|T\|\}. \tag{1}$$

2. The main result

Theorem 2. *Let $L : \ell_\infty \to \ell_\infty$ be a unilateral weighted left shift with the corresponding weight sequence (w_n) satisfying $0 < \inf_{i\in\mathbb{N}} |w_i| \leq |w_n| \leq 1$ for every $n \in \mathbb{N}$. Then:*

(i) $\mathbb{D}(0, \inf_{i\in\mathbb{N}} |w_i|) \subset \sigma(L) \subset \mathbb{D}(0, \sup_{i\in\mathbb{N}} |w_i|)$;

(ii) $\sigma_{su}(L) \subset \{\lambda \in \mathbb{C} : \inf_{i\in\mathbb{N}} |w_i| \leq |\lambda| \leq \sup_{i\in\mathbb{N}} |w_i|\}$.

In particular, if $w_n = 1$ for all $n \in \mathbb{N}$ then $\sigma(T) = \sigma_{ap}(T) = \mathbb{D}(0, 1)$ and $\sigma_{su}(T) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$.

Proof. The proof will be divided into four steps.

Step 1: We solve the equation

$$(L - \lambda I)(\xi_1, \xi_2, \cdots) = (\eta_1, \eta_2, \cdots), \tag{2}$$

where $\lambda \in \mathbb{C}$, I is the identity operator on ℓ_∞, $y = (\eta_1, \eta_2, \cdots) \in \ell_\infty$ is a given vector, and $x = (\xi_1, \xi_2, \cdots)$ is unknown. Precisely, in this step the domain of L and I is extended to the space of all complex sequences and it is possible that (2) has a solution which is not necessarily in ℓ_∞.

Step 2: We apply Step 1 to prove that for $|\lambda| \leq \inf_{i\in\mathbb{N}} |w_i|$ there exists a non zero vector $x = (\xi_1, \xi_2, \cdots) \in \ell_\infty$ such that $(L - \lambda I)x$ is the zero vector.

Step 3: Using Step 1 we show that for $|\lambda| < \inf_{i\in\mathbb{N}} |w_i|$ the equation (2) has a solution $x \in \ell_\infty$ for every $(\eta_1, \eta_2, \cdots) \in \ell_\infty$.

Step 4: The result follows by applying Proposition 1 and Steps 2 and 3. Indeed, it is clear that $\sigma_{su}(L) \subset \sigma(L) \subset \mathbb{D}(0, \|L\|)$. According to Proposition 1, we have

$$\sigma_{su}(L) \subset \sigma(L) \subset \mathbb{D}(0, \sup_{i\in\mathbb{N}} |w_i|). \tag{3}$$

Further, from Step 2 we conclude that $L - \lambda I$ is not injective for $|\lambda| \leq \inf_{i\in\mathbb{N}} |w_i|$. Consequently, $L - \lambda I$ is not invertible for $|\lambda| \leq \inf_{i\in\mathbb{N}} |w_i|$, thus

$$\mathbb{D}(0, \inf_{i\in\mathbb{N}} |w_i|) \subset \sigma(L). \tag{4}$$

From (3) and (4) we obtain (i). Moreover, Step 3 implies that if $|\lambda| < \inf_{i\in\mathbb{N}} |w_i|$ then $\lambda \notin \sigma_{su}(L)$. Using this fact and (3) gives (ii).

The details are as follows.

Step 1: (2) can be rewritten in the form

$$(w_1\xi_2 - \lambda\xi_1, w_2\xi_3 - \lambda\xi_2, \cdots) = (\eta_1, \eta_2, \cdots),$$

or equivalently

$$\eta_n = w_n\xi_{n+1} - \lambda\xi_n,\ n \in \mathbb{N}. \tag{5}$$

If we put $n = 1$ in (5) we obtain $\eta_1 = w_1\xi_2 - \lambda\xi_1$, and hence

$$\xi_2 = \frac{\eta_1}{w_1} + \frac{\lambda\xi_1}{w_1} \tag{6}$$

Let $n = 2$. (5) becomes $\eta_2 = w_2\xi_3 - \lambda\xi_2$, and thus

$$\xi_3 = \frac{\eta_2}{w_2} + \frac{\lambda\xi_2}{w_2}. \tag{7}$$

Combining (6) with (7) gives

$$\xi_3 = \frac{\eta_2}{w_2} + \frac{\lambda\eta_1}{w_1w_2} + \frac{\lambda^2\xi_1}{w_1w_2}.$$

Proceeding further in this direction, we obtain

$$\xi_{n+1} = \frac{\eta_n}{w_n} + \frac{\lambda\eta_{n-1}}{w_{n-1}w_n} + \frac{\lambda^2\eta_{n-2}}{w_{n-2}w_{n-1}w_n} + \cdots + \frac{\lambda^{n-1}\eta_1}{w_1\cdots w_n} + \frac{\lambda^n\xi_1}{w_1\cdots w_n} \tag{8}$$

for every $n \in \mathbb{N}$. It follows that $x = (\xi_1, \xi_2, \cdots)$, where ξ_1 is arbitrary and ξ_n, $n \geq 2$, is as in (8), is a formal solution of the equation $(L - \lambda I)x = y$.

Step 2: Let $|\lambda| \leq \inf_{i\in\mathbb{N}} |w_i|$. Using Step 1 we see that the equation $(L - \lambda I)(\xi_1, \xi_2, \cdots) = (0, 0, \cdots)$ has a particular solution $x = (\xi_1, \xi_2, \cdots)$ such that

$$\xi_1 = 1,\quad \xi_{n+1} = \frac{\lambda^n}{w_1\cdots w_n},\ n \in \mathbb{N}.$$

Clearly, x is not the zero vector. Further,

$$|\xi_{n+1}| = \frac{|\lambda|^n}{|w_1|\cdots|w_n|} \leq \left(\frac{|\lambda|}{\inf_{i\in\mathbb{N}} |w_i|}\right)^n \leq 1,\ n \in \mathbb{N},$$

which proves that $x \in \ell_\infty$.

Step 3: Let $|\lambda| < \inf_{i\in\mathbb{N}} |w_i|$ and $y = (\eta_1, \eta_2, \cdots) \in \ell_\infty$. According to Step 1, $x = (\xi_1, \xi_2, \cdots)$, where

$$\xi_1 = 0,\ \xi_{n+1} = \frac{\eta_n}{w_n} + \frac{\lambda\eta_{n-1}}{w_{n-1}w_n} + \frac{\lambda^2\eta_{n-2}}{w_{n-2}w_{n-1}w_n} + \cdots + \frac{\lambda^{n-1}\eta_1}{w_1\cdots w_n},\ n \in \mathbb{N},$$

satisfies $(L - \lambda I)x = y$. We have the following estimation

$$|\xi_{n+1}| \leq \frac{\sup_{i\in\mathbb{N}} |\eta_i|}{|w_n|}\left[1 + \frac{|\lambda|}{|w_{n-1}|} + \frac{|\lambda|}{|w_{n-2}|}\frac{|\lambda|}{|w_{n-1}|} + \cdots + \frac{|\lambda|}{|w_1|}\frac{|\lambda|}{|w_2|}\cdots\frac{|\lambda|}{|w_{n-1}|}\right],\ n \geq 2.$$

Using $1/|w_k| \leq 1/\inf_{i\in\mathbb{N}} |w_i|$, $k \in \mathbb{N}$, we deduce

$$|\xi_{n+1}| \leq \frac{\sup_{i\in\mathbb{N}} |\eta_i|}{\inf_{i\in\mathbb{N}} |w_i|}\left[1 + \sum_{k=1}^{\infty}\left(\frac{|\lambda|}{\inf_{i\in\mathbb{N}} |w_i|}\right)^k\right],\ n \geq 2.$$

Since $\dfrac{|\lambda|}{\inf_{i\in\mathbb{N}} |w_i|} < 1$, the above series converges, and hence $x \in \ell_\infty$.

To prove the last statement, let $w_n = 1$ for all $n \in \mathbb{N}$. Since $\inf_{i\in\mathbb{N}} |w_i| = \sup_{i\in\mathbb{N}} |w_i| = 1$, $\sigma(L) = \mathbb{D}(0, 1)$ and $\sigma_{su}(L) \subset \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ by (i) and (ii). In addition, we have $\{\lambda \in \mathbb{C} : |\lambda| = 1\} = \partial\sigma(L) \subset \sigma_{su}(L)$, and consequently $\sigma_{su}(L) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. Furthermore, it is easy to see that $\sigma(L) = \sigma_{ap}(L) \cup \sigma_{su}(L)$ and $\partial\sigma(L) \subset \sigma_{ap}(L)$ imply $\sigma_{ap}(L) = \sigma(L) = \mathbb{D}(0, 1)$, which completes the proof. □

3. Remarks

A unilateral weighted right shift operator $R : \ell_\infty \to \ell_\infty$ is defined similarly as

$$R(\xi_1, \xi_2, \cdots) = (0, w_1\xi_1, w_2\xi_2, \cdots), \ \ (\xi_1, \xi_2, \cdots) \in \ell_\infty.$$

The following result provides a first insight into the spectral properties of R. For completeness of exposition, we include the proof.

Proposition 3. *Let $R : \ell_\infty \to \ell_\infty$ be a unilateral weighted right shift with the corresponding weight sequence (w_n) satisfying $0 < \inf_{i\in\mathbb{N}} |w_i| \le |w_n| \le 1$ for every $n \in \mathbb{N}$. Then:*

(i) $\mathbb{D}(0, \inf_{i\in\mathbb{N}} |w_i|) \subset \sigma(R) \subset \mathbb{D}(0, \sup_{i\in\mathbb{N}} |w_i|)$*;*

(ii) $\sigma_{ap}(R) \subset \{\lambda \in \mathbb{C} : \inf_{i\in\mathbb{N}} |w_i| \le |\lambda| \le \sup_{i\in\mathbb{N}} |w_i|\}$.

In particular, if $w_n = 1$ for all $n \in \mathbb{N}$ then $\sigma(R) = \sigma_{su}(R) = \mathbb{D}(0, 1)$ and $\sigma_{ap}(R) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$.

Proof. It is easily seen that $\|R\| = \sup_{i\in\mathbb{N}} |w_i|$ and $j(R) = \inf_{i\in\mathbb{N}} |w_i|$. The statement (ii) follows immediately from (1).

(i). It is clear that $\sigma(R) \subset \mathbb{D}(0, \sup_{i\in\mathbb{N}} |w_i|)$. Since R is not surjective, $0 \in \sigma(R)$. Let $0 < |\lambda| < \inf_{i\in\mathbb{N}} |w_i|$. We consider the equation $(R - \lambda I)(\xi_1, \xi_2, \cdots) = (-\lambda, 0, 0, \cdots)$. An easy computation shows that the only solution of this equation is

$$x = (\xi_1, \xi_2, \cdots), \quad \xi_1 = 1, \quad \xi_{n+1} = \frac{w_1 \cdots w_n}{\lambda^n}, \quad n \in \mathbb{N}.$$

From

$$|\xi_{n+1}| = \frac{|w_1|}{|\lambda|} \cdots \frac{|w_n|}{|\lambda|} \ge \left(\frac{\inf_{i\in\mathbb{N}} |w_i|}{|\lambda|}\right)^n \to \infty \ (n \to \infty),$$

we see that $x \notin \ell_\infty$. It follows that $R - \lambda I$ is not surjective, i.e. $\lambda \in \sigma(R)$. Since $\sigma(R)$ is closed, $\mathbb{D}(0, \inf_{i\in\mathbb{N}} |w_i|) \subset \sigma(R)$.

The remaining part follows by the same method as in Theorem 2. □

Unilateral weighted shifts (left and right) can be considered in other sequence spaces (say ℓ_p, $1 \le p < \infty$) and have been widely studied in the literature. It is worth noting that our primary goal is to localize the surjective spectrum and the spectrum of L using a simple approach. For a comprehensive treatment on the subject one may refer to [1, Problems 89-94], [2, Examples III-3.16, IV-5.3 and IV-5.4], [3, Section 1.6] and [5].

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