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Interactions in Infinite Dimensions

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Abstract. The noncommuting variables of quantum theory find a theoretical basis in an infinite-dimensional classical formalism. The connection between theories of objects with dimension and classical point-particle models is clarified. It is demonstrated that the interaction region in the reduction of scattering amplitudes in quantum field theory is restricted to a region of a radius that scales as $d^{\frac{1}{2}}$ in d dimensions. The existence of a mass gap for a scalar field in the massless limit in the interior of a finite interaction region is proven when there are no bound state diagrams.

1. Introduction

It is known that the sum of the vacuum diagrams of superstring theory in ten-dimensional flat space over all finite orders of the perturbative expansion equals zero [**?**]. Unitarity requires a non-zero value which must be generated by boundary states. These boundary states might be attached as open string states or arise as ideal boundaries of infinite-genus surfaces. The inclusion of effectively closed infinite-genus surfaces has the property of not affecting the counting of the incoming and outgoing states. There exist an infinite number of complex structures on these surfaces, and the amplitude would be given an integral over an infinite-dimensional space.

The consistency of the quantum theory of strings then requires the introduction of infinite-dimensional spaces. It has been suggested that a theoretical basis for the dimension of space-time might be provided by a theory of quantum gravity. Given the known dimension space-time, together with the dimension of compact spaces in unified theories of the gravity and the elementary particle interactions, it is necessary to deduce a sequence of fundamental quantum theories in order of decreasing dimension.

It will be demonstrated that there is a difference between quantum field theories in finite and infinite dimensions. The reduction formalism for the evaluation of amplitudes is valid if the radius of the scattering region is finite in finite dimensions and scales as $d^{\frac{1}{2}}$ with the dimension. Therefore, the expression in terms of incoming and outgoing free fields at spatial infinity is exact only in the limit of infinite number of dimensions.

The extension of this result to field theories with a global symmetry follows from over the group-invariant Haar measure in the analogue of the Levy-Khintchine formula. An evaluation of the self-energy diagrams in a scalar field theory yields the radiative corrections to the mass, which can be combined with

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nonperturbative terms in finite regions. The mass shift may be estimated and bounded away from zero. The mass gap for the scalar field theory therefore would related to the effective radius of the interaction region.

2. Interacting Theory

In infinite dimensions, an action of the form

$$\int d^{\infty}x \left[\frac{1}{2}\dot{\phi}^2 - \frac{1}{2}|\nabla\phi|^2 - V(\phi)\right] \tag{1}$$

would diverge for a rotationally invariant configuration because $\nabla_i\phi = \nabla_j\phi$ implies that $\sum_i(\nabla_i\phi)(\nabla_i\phi)$ is infinite. Convergence requires that $|\nabla_n\phi|^2 \sim \frac{1}{n^{1+\epsilon}}\chi$ such that the system is not rotationally invariant. The spatial gradient term therefore should be omitted for rotational invariance, and the potential must be unchanged by an interchange of coordinate axes.

The values of a function along the coordinate axes can be listed as $f_1, f_2, f_3, ...$ Because the inclusion of an additional coordinate does not affect the cardinality of the set of axes in an infinite number of dimensions, the values could be defined to be $f_0, f_1, f_2, f_3, ...$ in place of $f_1, f_2, f_2, f_3, ...$ Therefore, the existence of functions in infinite dimensions unchanged by nontrivial mappings of the axes is compatible with rotational invariance. It would follows that isotropy is a characteristic of scalar theories formulated in terms of such functions.

Supertranslation groups may be defined through the series expansion of the parameters of the transformations between foliations tangent at either null or spatial infinity [**?**][**?**][**?**]. These supertranslations might be regarded as an ordering of translations in an infinite-dimensional space-time, where the isotropy property has been demonstrated. Extending the argument to space-times with the past infinity [**?**], the observed isotropy at cosmological scales could be traced to an inherent property of the gravitational system at the origin of its expansion, with the subsequent infinitesimal anisotropy resulting from fluctuations in the quantum expectation values of the fields.

Consider a classical field theory in infinite dimensions,

$$\mathcal{L}_{\infty} = \int d^{\infty}x\left[\frac{1}{2}\dot{x}^2 - V(x)\right]. \tag{2}$$

The Hamiltonian formalism would be based on

$$\mathcal{H}_{\infty} = p\dot{x} - \mathcal{L}_{\infty} \tag{3}$$

where

$$p = \frac{\partial\mathcal{L}_{\infty}}{\partial\dot{x}} = \dot{x} \tag{4}$$

such that

$$\mathcal{H}_{\infty} = \frac{1}{2}\dot{x}^2 + V(x). \tag{5}$$

The classical variables p and x may be multiplied and commute in a finite number of dimensions. However, in infinite dimensions, these product of these variables can be represented by matrices which have an inhomogeneous term in the commutator. This inhomogeneous term may be interpreted as the equivalent of a term containing Planck's constant in quantum mechanics. Therefore, the formulation of theories in infinite dimensions might provide a connection between classical and quantum theories.

It appears that the fundamental physical theories of elementary particles and gravity necessarily must introduce infinite-dimensional spaces. This is evident in a formulation of string theory based on universal Teichmüller space or $Diff(S^1)/S^1$. The connection between noncommuting variables in quantum theory and the existence of a size of the objects described by the field variables and the classical point particle field theory then can be deduced.

A grand partition function over different dimensions may be constructed for a quantum theory of gravity for which the dimension of space-time is theoretically predicted. Since an integration over dimension could include ∞ as an upper limit, the interaction of spheres in infinite dimensions may be investigated. These spheres contain zero volume in the Euclidean measure in $\mathbb{R}^\infty$, and the quantum theory of the interactions may provide a link between point particles and particles with dimension.

The volume inside a hypersphere satisfies the relation

$$V_n = \frac{2\pi r^2}{n} V_{n-2}, \tag{6}$$

which implies that the volume inside a sphere S^{n-1} decreases to zero in the Euclidean measure of the embedding space $\mathbb{R}^n$ as $n \to \infty$. The interactions of spherical objects in infinite-dimensions therefore can be compared to point-particle interactions in a finite number of dimensions.

Furthermore, the formalism in infinite-dimensions yields a regularization procedure for the computation of cross-sections of processes in point-particle field theories. Known divergences in the point-particle theories could be recast in the infinite-dimensional formalism and a regularization parameter given by a monotonic function of the inverse of the dimension could be used.

3. The Formalism for Interacting Quantum Fields

Given two independent systems, the existence of a local interaction can be established if the following inequality

$$\int_{\mathbb{R}^d} \int_0^1 \left(|\hat{v}_\mu(z) d\mu| \right)^2 dz < \infty \tag{7}$$

where $\hat{v}_\mu$ is a stochastic distribution [**?**]. The Levy-Khintchine distribution [**?**][**?**][**?**] is

$$\begin{aligned} &\hat{v}_\mu(z) = e^{-\mu\chi(z)} \\ &\chi(z) = a^2|z|^2 - iz \cdot b - \int_{\mathbb{R}^d} (e^{iz\cdot x} - | - iz \cdot x|)\sigma(dx). \end{aligned} \tag{8}$$

Since

$$\int_{\mathbb{R}^d} \int_0^1 e^{-2\mu\chi} d\mu dz = \frac{1}{2} \int_{\mathbb{R}^d} \frac{1 - e^{2\chi(z)}}{\chi} dz \tag{9}$$

and

$$\begin{aligned} \frac{1 - e^{-2\chi}}{2\chi} & \underset{\chi \to \infty}{\longrightarrow} \frac{1}{2\chi} \\ \frac{1}{(1+\chi)^2} & \underset{\chi \to \infty}{\longrightarrow} \frac{1}{\chi^2} \\ lim_{\chi\to 0} \frac{1 - e^{-2\chi}}{2\chi} &= lim_{\chi \to 0} \frac{1}{(1+\chi)^2} = 1. \end{aligned} \tag{10}$$

the function $\frac{1}{(1+\chi)^2}$ is majorized by $\frac{1}{2}\frac{1-e^{-2\chi}}{\chi}$ and the finiteness condition $\int_{\mathbb{R}^d} \frac{dz}{(1+\chi(z))^2} < \infty$ is a consequence of Eq.(3.1).

When $a \neq 0$, $\chi(z) \sim a^2|z|^2$ for large z and

$$\int_{\mathbb{R}^d} \frac{1}{(1+\chi(z))^2} dz \sim \int_{\mathbb{R}^d} \frac{1}{(1+a^2|z|^2)^2} dz \tag{11}$$
$$= \int_{\mathbb{R}^d} \frac{1}{(1+a^2|z|^2)^2} |z|^{d-1} d|z| d\Omega^{(d-1)}.$$

which converges if $d < 4$. In an infinite number of dimensions, this integral would be divergent. However, a monotonic function of the form $\chi(z) \sim a^2|z|^{\frac{d}{2}}$ would be sufficient to achieve convergence the spherical integral tends to zero as $d \to \infty$.

Theorem 1. *The condition for a finite stochastic measure for local interactions requires a radius of the integration region that increases with the dimension as $d^{\frac{1}{2}}$. Boundary terms would occur in the formula for a quantum field theoretic amplitude. Since plane-wave states representing particles of definite momentum must be replaced by wave packets, the validity of this result is limited to systems of larger size, which undergo random motion.*

Proof. In the initial measure (3.5), determined by $\chi(z) \sim a^2|z|^2$, if the radial integration range is restricted to $\{0 \leq |z| \leq R\}$, or equivalently, $\{0 \leq |\tilde{z}| \leq \frac{R}{R_0}\}$, where $\tilde{z} = \frac{z}{R_0}$.

$$\int_0^{\frac{R}{R_0}} \frac{|\tilde{z}|^{d-1} d|\tilde{z}|}{(1+a^2|\tilde{z}|^2)^2} \tag{12}$$
$$= \begin{cases} \frac{1}{a^d}\left[\frac{1}{d-4}\left(\left(1+\frac{a^2R^2}{R_0^2}\right)^{\frac{d-4}{2}} - 1\right) + \ldots + \frac{1}{2}(-1)^{\frac{d-4}{2}}\left(\frac{1}{1+\frac{a^2R^2}{R_0^2}} - 1\right)\right] & d > 4,\ d\ even \\ \frac{1}{a^4} ln\left(1+\frac{a^2R^2}{R_0^2}\right)^{\frac{1}{2}} + \frac{1}{2a^4}\left[\frac{1}{1+\frac{a^2R^2}{R_0^2}} - 1\right] & d = 4 \end{cases}$$
$$\longrightarrow \frac{1}{a^4}\begin{cases} \frac{\left(\frac{R}{R_0}\right)^{d-4}}{d-4} & d > 4,\ d\ even \\ ln\left(\frac{R}{R_0}\right) & d = 4 \end{cases}.$$

Since the surface area of the sphere is

$$A(S^{d-1}) = \begin{cases} \frac{(2\pi)^{\frac{d}{2}}}{(d-2)!!} & d\ is\ even \\ \frac{2(2\pi)^{\frac{d-1}{2}}}{(d-2)!!} & d\ is\ odd \end{cases} \tag{13}$$

the integral (3.5) equals

$$\frac{(2\pi)^{\frac{d}{2}}}{(d-2)!!}\frac{1}{a^d}\frac{1}{d-4}\left(\left(1+\frac{a^2R^2}{R_0^2}\right)^{\frac{d-4}{2}-1}\right) \longrightarrow \frac{(2\pi)^{\frac{d}{2}}}{(d-2)!!}\frac{1}{a^4}\frac{\left(\frac{R}{R_0}\right)^{d-4}}{d-4} \tag{14}$$

in the large-radius limit for d even.

As

$$(d-2)!! \approx 2^{\frac{d}{2}-1}\sqrt{\pi}\sqrt{d-2}\left(\frac{d}{2}-1\right)^{\frac{d}{2}-1} e^{-\left(\frac{d}{2}-1\right)}, \tag{15}$$

the limit of large radius and dimension equals

$$lim_{\substack{R\to\infty\\ d\to\infty}} \frac{1}{a^4}\frac{\left(\frac{R}{R_0}\right)^{d-4}}{d-4}\frac{(2\pi)^{\frac{d}{2}}}{2^{\frac{d}{2}-1}\sqrt{\pi}\sqrt{d-2}\left(\frac{d}{2}-1\right)^{\frac{d}{2}-1}e^{-\left(\frac{d}{2}-1\right)}} \tag{16}$$

$$= lim_{\substack{R\to\infty\\ d\to\infty}} \sqrt{\frac{2}{\pi}}\frac{1}{a^4}\frac{\left(\frac{aR}{R_0}\right)^{d-4}\pi^{\frac{d}{2}}e^{\left(\frac{d}{2}-1\right)}}{(d-4)\left(\frac{d}{2}-1\right)^{\frac{d-1}{2}}}.$$

For a finite limit when $d > 4$,

$$lim_{d\to\infty}\left(\frac{R}{R_0}\right)^{d-4} = lim_{d\to\infty}\sqrt{\frac{\pi}{2}}a^4K\frac{(d-4)\left(\frac{d}{2}-1\right)^{\frac{d-1}{2}}}{\pi^{\frac{d}{2}}e^{\left(\frac{d}{2}-1\right)}}, \tag{17}$$

which can be achieved with a dependence of the radius on the dimension given by

$$lim_{d\to\infty}R = lim_{d\to\infty}\left[\sqrt{\frac{\pi}{2}}a^4K\frac{(d-4)\left(\frac{d}{2}-1\right)^{\frac{d-1}{2}}}{\pi^{\frac{d}{2}}e^{\left(\frac{d}{2}-1\right)}}\right]^{\frac{1}{d-4}}R_0 \tag{18}$$

$$= lim_{d\to\infty}\left(\frac{d-2}{2\pi e}\right)^{\frac{1}{2}} R_0.$$

Finiteness of the integral (3.5) in an odd number of dimensions requires

$$\frac{1}{a^4}\frac{(R/R_0)^{d-4}}{d-4}\frac{2(2\pi)^{\frac{d-1}{2}}}{(d-2)!!} = K. \tag{19}$$

Then

$$\left(\frac{R}{R_0}\right)^{d-4} = \frac{a^4K}{2(2\pi)^{\frac{d-1}{2}}}\,(d-4)\,(d-2)!! \tag{20}$$

Since $(d-2)!! = \frac{(d-2)!}{(d-3)!!}$, where $d-3$ is even,

$$(d-2)!! \approx \sqrt{\frac{2(d-2)}{e(d-3)}}\frac{(d-2)^{d-2}}{(d-3)^{\frac{d-3}{2}}}e^{-\frac{d-2}{2}} \tag{21}$$

and

$$\frac{\left(\frac{R}{R_0}\right)^{d-4} \approx [a^2K]}{2(2\pi)^{\frac{d-1}{2}}(d-4)\sqrt{\frac{2(d-2)}{e(d-3)}}\frac{(d-2)^{d-2}}{(d-3)^{\frac{d-3}{2}}}e^{-\frac{d-2}{2}}} \tag{22}$$

$$\frac{R}{R_0} \approx \left[\sqrt{\frac{2(d-2)}{e(d-3)}}\frac{(d-2)^{d-2}}{(d-3)^{\frac{d-3}{2}}}e^{-\frac{d-2}{2}}\right]^{\frac{1}{d-4}} \tag{22}$$

$$\underset{d\to\infty}{\longrightarrow} \frac{1}{(2\pi e)^{\frac{1}{2}}}\left(\frac{d-2}{(d-3)^{\frac{1}{2}}}\right) \underset{d\to\infty}{\longrightarrow} \left(\frac{d-2}{2\pi e}\right)^{\frac{1}{2}}$$

□

This result would imply that, if the number of dimensions is finite, the size of the interaction region of point-particle field theories must be restricted. Therefore, the conditions for the reduction formalism for quantum field theory and the definition of asymptotic states strictly require an infinite number of dimensions.

In the Heisenberg picture, the asymptotic states are free fields and

$$\phi_H(x) \sim \begin{cases} \sqrt{Z}\phi_{in}(x) & t \to -\infty \\ \sqrt{Z}\phi_{out}(x) & t \to \infty \end{cases}. \tag{23}$$

where Z is the multiplicative factor resulting from renormlization through interactions.

The reduction formula [?] for a scalar field theory in four dimensions

$$\begin{aligned} \langle \beta_{out}|\alpha_{in}\rangle = \left(\frac{i}{\sqrt{Z}}\right)^{m+n} \int \prod_{i=1}^{n} d^4x_i e^{-ip_i\cdot x_i}(\partial^2_{x_i} + m^2) \int \prod_{j=1}^{m} d^4y_j e^{iq_j\cdot y_j}(\partial^2_{y_j} + m^2) \\ G(y_1, ..., y_m, x_1, ...x_n) \\ + \ extra\ terms \\ |\alpha_{in}\rangle = |p_1, ..., p_n\rangle \qquad |\beta_{out}\rangle = |q_1, ..., q_m\rangle \end{aligned} \tag{24}$$

where $G(x_1, ..., x_n) = \langle 0|T\{\phi_H(x_1)...\phi_H(x_n)\}|0\rangle$ is the time-ordered correlation function of interacting fields and the extra terms include delta functions between initial and final momenta, similar expressionw with fewer arguments and the sum $2i\hbar\left(\frac{i}{\sqrt{Z}}\right)^{m+n}\sum'_{(i,j)}(-p_i^2+m^2)(2\pi)^4\delta^4(p_i-q_j^0)-i\hbar\left(\frac{i}{\sqrt{Z}}\right)^{m+n}\sum'_{(i,j)}(-((p_i)^0)^2+m^2)(2\pi)^4\delta^4(p_i-q_j)\delta_{\beta_{out},\alpha_{in}}$ [?], with $\sum'_{(i,j)}$ defined over pairs (i, j) with each of the indices i and j counted only once. It depends on the equality of the integral of a total derivative with a three-dimensional integral over fields related to $\phi_{in}(x)$ and $\phi_{out}(x)$ through a renormalization constant. For a four-dimensional integral of finite extent, defined over surfaces of finite radius, the difference of the three-dimensional integrals would be

$$\begin{aligned} &\langle \beta_f|a_i^\dagger(p_1)|\alpha_i'\rangle - \langle \beta_f|a_f^\dagger(p_1)|\alpha_i'\rangle \\ &= -i\int_{\Sigma_i} e^{-ip_1\cdot x}\overleftrightarrow{\partial}_0\langle \beta_f|\phi_i(x)|\alpha_i'\rangle d^3x\Big|_{t=t_i} + i\int_{\Sigma_f} e^{-ip_1\cdot x}\overleftrightarrow{\partial}_0\langle \beta_f|\phi_f(x)|\alpha_i'\rangle\Big|_{t=t_2}, \end{aligned} \tag{25}$$

where $|\alpha_i'\rangle$ is a state without the particle of momentum p_1, $a_i^\dagger(p_1)$ is the creation operator for an initial state with momentum p_1 and $a_f^\dagger(p_1)$ is the creation operator for a final state with momentum p_1, while Σ_f and Σ_f are hypersurfaces are defined in terms of $R(d)$. If the limits of the Heisenberg fields are defined to be

$$\begin{aligned} lim_{t\to t_i}\phi_H(x) &= \sqrt{Z_i}\phi_i \\ lim_{t\to t_f}\phi_H(x) &= \sqrt{Z_f}\phi_f, \end{aligned} \tag{26}$$

and the above expression equals

$$\frac{i}{\sqrt{Z}_f}\int_{\Sigma_f} e^{-ip_1\cdot x}\overleftrightarrow{\partial}_0\langle \beta_f|\phi_H(x)|\alpha_i'\rangle d^3x - \frac{i}{\sqrt{Z}_i}\int_{\Sigma_i} e^{-ip_1\cdot x}\overleftrightarrow{\partial}_0\langle \beta_f|\phi_H(x)|\alpha_i'\rangle d^3x. \tag{27}$$

All scalar fields and quasi-free fields may expanded in terms of plane waves of definite momentum, and for large dimensions, when $R(d) \to \infty$, Z_i, $Z_f \to Z$. When the dimension is bounded, Z_i and Z_f can have different values from Z. The derivation of the formula for the inner product of the quantum states also depends on the existence of free fields at spatial infinity, and the quasi-free fields ϕ_i and ϕ_f only satisfy $(\partial^2 + m^2)\phi_i \approx F_i$ and $(\partial^2 + m^2)\phi_f \approx F_f$ for some nontrivial functions F_i and F_f. Instead of states of definite momentum, $|\alpha_i\rangle$ and $|\beta_f\rangle$ would be defined interms of distributional integral over the momenta, and the formula must be generalized such that $e^{-ip\cdot x}$ is replaced by a wave packet $\int d^4k\ \mathcal{U}(k)e^{-ik\cdot x}$. Consequently, the point-particle of definite momentum must be replaced by a nonlocalized wave packet, which provides an indication of the larger scale required for the application of this stochastic measure to quantum theory. □

The integral of a stochastic probability measure would be consistently set equal to one. Therefore, this theorem provides an upper bound on the size of the integration region in a finite dimension.

The proportionality of the radius of the interaction region with $d^{\frac{1}{2}}$ also can be derived from a stochastic differential equation for the magnitude of separation of particles undergoing Brownian motion. For point-particle scattering, which exist at a microscopic scale different from that of molecules, another result is derived.

Theorem 2. *The radius of the region of traversed in Brownian motion of two particles increases as $d^{\frac{1}{2}}$ with the dimension in the limit $d \to \infty$. The size of the interaction region of freely travelling point particles increases linearly with respect to the dimension.*

Proof. The differential element [?]

$$dx = \frac{1}{2}(d-1)\frac{\alpha_{\perp,\epsilon}(x^2)}{x}dt + \sqrt{\alpha_\epsilon(x^2)}\beta(dt) \tag{28}$$

where $\beta(t)$ is one-dimensional Brownian motion and $\alpha_{\perp,\epsilon}$ and α_ϵ are scalar functions resulting from frame-indifference. Then $\alpha_\epsilon,\ \alpha_{\perp,\epsilon} : \mathbb{R}_+ \to \mathbb{R}$ satisfy $\alpha_\epsilon(0) = \alpha_{\perp,\epsilon}(0) = 0$ and $lim_{|r|\to\infty}\alpha_{\perp,\epsilon}(|r|^2) = c_\epsilon$, where $|r|$ is the Euclidean norm in $\mathbb{R}^d$. For large $|r|$,

$$xdx = \frac{1}{2}(d-1)\alpha_{\perp,\epsilon}dt + \sqrt{\alpha_\epsilon}x\beta(dt) \tag{29}$$

and

$$\int^{2R} xdx = 2R^2 = \frac{1}{2}(d-1)\alpha_{\perp,\epsilon}\Delta t + \sqrt{\alpha_\epsilon}\int x\beta(dt). \tag{30}$$

When $d = 1$, two particles undergoing Brownian motion coalesce, and $\{0\}$ is the attractor for $r(t)$,

$$\int x\beta(dt) \to 0 \qquad t \to \infty, \tag{31}$$

whereas, if $d = 2$ and $x(0) \neq 0$, the solution can describe an infinite number of motions of attraction and separation [?]. The integral in the second term is

$$\int x\beta(dt) = \sum_{n=1}^{n_R} I_n \tag{32}$$

with I_n being the integral of the relative position coordinate between the n^{th} and $(n+1)^{th}$ coalescence of the particles.

In the limit of vanishing string size, surfaces of arbitrary genus can be included in the interactions of two particles of this kind. The existence of a point-particle limit with recurrent interactions would be sufficient to restrict the intrinsic coordinates of the interaction to two dimensions, since the distance between two Brownian particles tends to ∞ with probability 1 for $d \geq 3$ [?]

From the analogue of Eq.(3.23) for random motion which occupies a stationary interaction region, it follows that

$$2R^2 = \frac{1}{2}(d-1)\alpha_{\perp,\epsilon}\Delta t + \sqrt{\alpha_\epsilon}\cdot 2R\Delta t \tag{33}$$

and

$$R \approx \frac{1}{2}\Big[\alpha_{\perp,\epsilon}\Delta t\Big]^{\frac{1}{2}}(d-1)^{\frac{1}{2}}, \tag{34}$$

which is compatible with the result in Theorem 1.

For Brownian motion, the extent of the random walk increases as the square root of the elapsed time in three dimensions [?]. In higher dimensions [?]

$$x(t) = (2\pi)^{\frac{d}{2}}\left(\frac{d}{2}-1\right)t^{\frac{d}{2}-1}. \tag{35}$$

Since

$$t = \left[\frac{1}{(2\pi)^{\frac{d}{2}}}\frac{1}{\left(\frac{d}{2}-1\right)}\right]^{\frac{1}{\frac{d}{2}-1}} x(t)^{\frac{1}{\frac{d}{2}-1}}, \tag{36}$$

$$dt = \frac{1}{(2\pi)^{\frac{d}{d-2}}}\frac{1}{\left(\frac{d}{2}-1\right)^{\frac{d}{d-2}}}x^{\frac{4-d}{d-2}}dx \tag{37}$$

and

$$\int^{2R} x\beta(dt) = \frac{1}{(\pi(d-2))^{\frac{d}{d-2}}}(2R)^{\frac{d}{d-2}}. \tag{38}$$

By Eq.(3.25),

$$2R^2 = \frac{1}{2}(d-1)\alpha_{\perp,\epsilon}\cdot\frac{1}{(2\pi)^{\frac{d}{d-2}}}\frac{1}{\left(\frac{d-2}{2}\right)^{\frac{2}{d-2}}}(2R)^{\frac{2}{d-2}} + \frac{\sqrt{\alpha_\epsilon}}{(\pi(d-2))^{\frac{d}{d-2}}(2R)^{\frac{d}{d-2}}}. \tag{39}$$

In the limit $d \to \infty$,

$$2R^2 - \frac{2\sqrt{\alpha_\epsilon}}{\pi(d-2)}R - \frac{1}{4\pi}(d-1)\alpha_{\perp,\epsilon} \approx 0 \tag{40}$$

and the positive root is

$$\begin{aligned} R &\approx \frac{\sqrt{\alpha_\epsilon}}{2\pi(d-2)} + \frac{1}{2}\left[\frac{\alpha_\epsilon}{\pi^2(d-2)^2} + \frac{1}{2\pi}(d-1)\alpha_{\perp,\epsilon}\right]^{\frac{1}{2}} \\ &\underset{d\to\infty}{\longrightarrow} \frac{1}{2\sqrt{2\pi}}\alpha_{\perp,\epsilon}^{\frac{1}{2}}(d-1)^{\frac{1}{2}}. \end{aligned} \tag{41}$$

□

The size of the interaction region for the scattering of freely travelling point particles, by contrast, increases linearly with respect to the dimension. For point-particle scattering, $x = v\Delta t$, and

$$\left(2 - \frac{2\sqrt{\alpha_\epsilon}}{v}\right)R^2 = \frac{1}{2}(d-1)\alpha_{\perp,\epsilon}\frac{2R}{v} \tag{42}$$

with the solution

$$R = \frac{\frac{(d-1)\alpha_{\perp,\epsilon}}{v}}{2\Big[1-\frac{\sqrt{\alpha_\epsilon}}{v}\Big]}. \tag{43}$$

Infinitely divisible distributions on $\mathbb{R}^d$ can be characterized by a Levy measure [**?**][**?**][**?**]. This result is sufficient for the proof of Theorem 1 to be consistent with stochastic quantum theory.

The stochastic motion of particles is similar to the random paths of electrons in a large circuit. Since Brownian motion is characterized only by a drift velocity, without any additional force transporting electrons in a current, the lower bound for the size of the region traversed by the electron is proportional to $d^{\frac{1}{2}}$.

While S-matrices can be defined through analyticity and unitarity conditions, without the reduction formalism, a theoretical basis for this approach to Regge trajectories required string theory [**?**] [**?**][**?**][**?**]. The previous discussion and the conclusions then must be generalized.

The embedding of higher-genus surfaces in a finite interaction region would be relevant for string dynamics.

Theorem 3. *For real processes based on the creation and annihilation of nonlocal matter undergoing Brownian motion, the maximal genus in a finite interaction region depends on the dimension as* $\left(\frac{R_0}{\ell_{min}}\right)^d \frac{1}{\sqrt{\pi d}}\left(1-\frac{2}{d}\right)^{\frac{d}{2}}$ *as* $d \to \infty$*, which requires that* $\ell_{min} < R_0$ *for positive genus. For virtual processes, surfaces of infinite genus can be embedded in the interaction region.*

Proof. An even distribution of handles with a minimal thickness ℓ_{min}, in a disk would have a genus proportional to R_2^2, where R_2 is the radius of the disk. In the two-sphere, the maximum genus of an array of handles of this kind would be

$$\frac{V_3}{\ell_{min}^3} = \frac{4}{3}\frac{\pi R_2^3}{\ell_{min}^3} \tag{44}$$

Again, the volume in a sphere S^{d-1} would contain a maximal genus equal to

$$\frac{V_d}{\ell_{min}^d} = \begin{cases} \frac{1}{\ell_{min}^d}\frac{1}{\left(\frac{d}{2}\right)!}\pi^{\frac{d}{2}}R^d & d \text{ even} \\ \frac{1}{\ell_{min}^d}\frac{2^{\frac{(d+1)}{2}}}{d!!}\pi^{\frac{d-1}{2}}R^d & d \text{ odd} \end{cases} \tag{45}$$

The radius R_2 of the disk is equal to the radius of the R_d of the sphere S^{d-1}. For a large number of dimensions, these ratios equal

$$\frac{1}{\ell_{min}^d}\frac{1}{\left(\frac{d}{2}\right)!}\pi^{\frac{d}{2}}\left(\frac{d-2}{2\pi e}\right)^{\frac{d}{2}} R_0^d \underset{d\to\infty}{\longrightarrow} \left(\frac{R_0}{\ell_{min}}\right)^d \frac{1}{\sqrt{\pi d}}\left(1-\frac{2}{d}\right)^{\frac{d}{2}} \qquad d \text{ even} \tag{46}$$

and

$$\begin{aligned}\frac{1}{\ell_{min}^d}\frac{2^{\frac{(d+1)}{2}}}{d!!}\pi^{\frac{d-1}{2}}\left(\frac{d-2}{2\pi e}\right)^{\frac{d}{2}} R_0^d &\approx \left(\frac{R_0}{\ell_{min}}\right)^d \sqrt{\frac{e}{\pi d}}\left(1-\frac{1}{d}\right)^{\frac{d}{2}}\left(1-\frac{2}{d}\right)^{\frac{d}{2}} \\ &\underset{d\to\infty}{\longrightarrow} \left(\frac{R_0}{\ell_{min}}\right)^d \frac{1}{\sqrt{\pi d}}\left(1-\frac{2}{d}\right)^{\frac{d}{2}} \qquad d \text{ odd}\end{aligned} \tag{47}$$

The minimal length resulting from the analogue of the uncertainty principle to string theory would yield this dependence for the maximal genus enclosed in the interaction region for observable processes.

For virtual processes, the minimal bound is not imposed on the handles and surfaces of arbitrary genus may be included in the expansion of the amplitude. It is well established that a conformal transformation exists such that the infinite-genus surface can be embedded in a finite region in the a higher-dimensional space. □

Identification of the handles of a surface with the elements in an array yields an exponential dependence on the dimension which is valid also for the number of gate pairs separated by a horizontal distance ℓ. It equals, for example, $M(\ell) = m^2\left(\frac{\ell^3}{3} - 2\ell^2\sqrt{\frac{N}{m}} + 2\ell\frac{N}{m}\right)$, $1 \leq \ell \leq \sqrt{\frac{N}{m}}$, with m gates in each cell and $\sqrt{\frac{N}{M}} \times \sqrt{\frac{N}{m}}$ cells in the array, in three dimensions [?].

Connections with circuit theory also may be derived since the extent of a circuit models that of an interaction region in a scattering process. The increase of the size of this region with the dimension would be reflected for circuits in the ratio of the diameters of circuits with the same power supply and equally spaced elements located entirely in two and three dimensions [?].

A description of interacting quantum fields may be given through moment functionals that satisfy most of the Osterwalder-Schrader axioms under restrictions on the stochastic measure [?]. Inequalities necessary for the proof of properties of correlation functions in axiomatic field theory require a positive mass for particle states [?]. It is known also that analyticity of Schwinger functions in the coupling, which is necessary for the cluster expansion, depends on the finiteness parameter $\frac{\epsilon}{m_0^2}$ for non-vacuum states, where m_0 would be the particle mass [?]. The cluster expansion is used in the proof of the infinite-volume limit of distributions in field theory. If there is no positive lower bound for the particle masses, the analyticity required for the cluster expansion is not valid as the volume becomes infinite and the finite-volume formalism is necessary. Finite-volume corrections to the mass of a stable state have been shown to be proportional to an exponential function of the negative of the product of the mass and the linear extent [?], and this technique is likely to yield physically consistent results.

4. The Levy Measure for Field Theories with a Group Symmetry

The stochastic measure may be extended to Yang-Mills theories which describe known elementary particle interactions. The Levy-Khintchine measure may be generalized to principal bundles by considering the local trivialization $\mathbb{R}^d \times G$ where G is the structure group, and using the formula $\int_G \Phi(g)\mu(dg) = e^{-\eta_\Phi}$ where $\Phi(g)$ is a spherical function on the group [?], which can be projected to symmetric spaces G/K with the horospherical functions defined by the Iwasasa decomposition of G with respect to K [?]. In a local neighbourhood of a point in $\mathbb{R}^d$, the analogue of the Levy-Khintchine formula would be

$$\int_{\mathbb{R}^d}\int_G \mu(dx)\mu(dg)e^{iu\cdot x}\Phi(g) = e^{-\eta(u)-\eta_\Phi}. \tag{48}$$

Projection from the total space of the bundle to the base space would require $\pi : (x_0, g) \to x_0$, $g \in G$ and factorization by the integral over G through $\int_{\pi(x_0,G)} d\mu(g)$
$\Phi(g) = \Phi(e)$ which is a constant, yielding the conventional formula for the measure.

Therefore, the integral inequality for the Levy-Khintchine measure of field theories with a global symmetry or gauge fields will be equivalent to that for a particle in $\mathbb{R}^d$. The conditions of convergence for a set of terms in the quantum measure [?] also would be unchanged, and the the integral is finite over an unbounded region in d dimensions only if $d < 4$. When $d \geq 4$, the interaction region must be finite. Since

$$\frac{1}{a^4}ln\left(1 + \frac{a^2R^2}{R_0^2}\right)^{\frac{1}{2}} + \frac{1}{2a^4}\left[\frac{1}{1+\frac{a^2R^2}{R_0^2}} - 1\right] \underset{\frac{R}{R_0}\to\infty}{\longrightarrow} \frac{1}{a^4}ln\left(\frac{aR}{R_0}\right) \tag{49}$$

and the finiteness condition is

$$\frac{2\pi^2}{a^4}\,ln\left(\frac{aR}{R_0}\right) = K \tag{50}$$

or

$$R = \frac{R_0}{a}e^{\frac{Ka^4}{2\pi^2}}. \tag{51}$$

It follows from the formula $\Delta m = -\frac{3}{16\pi m^2 R}\lambda^2 e^{-\frac{\sqrt{3}}{2}mR} - \frac{3}{16\pi^2 mR}\int_{\mathbb{B}} dy e^{-\sqrt{m^2+y^2}R}$ $F(iy) + O\left(\frac{e^{-\bar{m}R}}{m^2 R}\right)$, where λ is the coupling in a interacting scalar field theory, m is the mass in the infinite-volume limit, $F(iy)$ is the forward scattering amplitude with $\nu = iy = \frac{s-u}{4m}$, $\mathbb{B} = \{(q_0, q_\perp) \in \mathbb{R}^3 | q_0^2 + q_\perp^2 \leq \frac{3}{4}m^2\}$ and $\bar{m} > m$ [?], there is a shift of the non-zero masses of the particle states. The function $F(\nu)$ can have a dependence of m^2, although there may be correcting factors resulting from the asymptotic dependence of the Bessel function or the form factor [?]. A mass gap for states in the finite-volume field theory arises if either initial term can be expanded in a series in m yielding a term independent of m. The first term arises from a bound state in the evaluation of the self-energy diagrams. An evaluation of the first two terms yields a negative value of Δm. It would be consistent with the positivity of the mass spectrum if the actual mass is set equal to the mass observed at large distances from the interaction region in the infinite-volume limit. Suppose that the mass of the particle in the finite interaction region,m$_{fin}$ is set equal to zero. It follows that $m = m_{fin} - \Delta m > 0$ if Δm can be bounded away from zero. The first term in the formula for Δm is derived from bound states and can be discarded in the zero-mass limit. The parameter a in the Levy-Khintchine formula may be adapted to the scalar field and would be proportional to the mass in a first-order approximation by dimensional analysis [?], because the expansion must have the form $a = \kappa_0 m + \kappa_1 m^2 L + \kappa_2 m^3 L^2 + ...$, where L is a length scale, with the coefficients of L^{-n}, $n > 0$, vanishing for the short-distance limit $L \to 0$ not to be divergent. Then $a \approx \kappa_0 m$, and $\sqrt{m^2 + y^2}$ can be set equal to $\bar{\kappa}_B m$, where $\bar{\kappa}_B$ is an average value derived from the integral over the ball $\mathbb{B}$ and the equality $e^{-\bar{\kappa}_B mR}F(i\bar{y}) \equiv e^{-\bar{\kappa}_B mR}F(\bar{\nu})$ for some $\bar{y} \in \mathbb{B}$. Then the second term is

$$
\begin{aligned}
&-\frac{3}{16\pi^2 m}\kappa_0 m e^{-\frac{\bar{\kappa}_B}{\kappa_0}} e^{-\frac{K\kappa_0^4}{2\pi^2}m^4 e^{-\frac{\bar{\kappa}_B}{\kappa_0}}} F(\bar{\nu}) \\
&= -\frac{3\kappa_0}{16\pi^2}\left[e^{-\frac{\bar{\kappa}_B}{\kappa_0}} - \frac{K\kappa_0^4}{2\pi^2}m^4 \sum_{k=0}^{\infty}(-1)^k \frac{k+1}{k!}\left(\frac{\bar{\kappa}_B}{\kappa_0}\right)^k + O\left(\left(\frac{K\kappa_0^4}{2\pi^2}m^4\right)^2\right)\right] \\
&= -\frac{3\kappa_0}{16\pi^2}\left[e^{-\frac{\bar{\kappa}_B}{\kappa_0}} - \frac{K\kappa_0^4}{2\pi^2}m^4 \left(\left(1 - \frac{\bar{\kappa}_B}{\kappa_0}\right)e^{-\frac{\bar{\kappa}_B[R_0]}{\kappa_0}}\right) + O\left(\left(\frac{K\kappa_0^4}{2\pi^2}m^4\right)^2\right)\right]
\end{aligned}
\tag{52}
$$

Then

$$
\begin{aligned}
m_{nbs} = m_{fin} + \frac{\kappa_0}{16\pi^2}\Bigg[e^{-\frac{\bar{\kappa}_B}{\kappa_0}} - \frac{K\kappa_0^4}{2\pi^2}m^4\left(1 - \frac{\bar{\kappa}_B}{\kappa_0}\right)e^{-\frac{\bar{\kappa}_B}{\kappa_0}} \\
+ O\left(\left(\frac{K\kappa_0^4}{2\pi^2}m^4\right)^2\right)\Bigg]
\end{aligned}
\tag{53}
$$

which verifies the existence of a mass gap in the massless limit for the particle in the finite interaction region with no bound state diagrams.

It has been established that the Yang-Mills measure on a compact space is a probability measure [?][?]. String theory may be regarded as a theory of gauge fields defined by the bundle $E(S^1, Diff(S^1), \pi)$ and existence theorems for solutions to stochastic differential equations on the diffeomorphism group have been proven [?][?][?]. A probabilistic formulation of string theory [?] then would follow.

5. Conclusion

The development of an interacting quantum field theory requires the existence of interactions of independent quantum systems propagating in the space-time. Given a stochastic distribution for such processes, the existence of local interactions is found to require a restriction on the radius of the scattering region.

The scaling of the radius of the interaction region for scattering amplitudes in a quantum field theory in d dimension is given by $d^{\frac{1}{2}}$

It is found that, only in an infinite number of dimensions, the asymptotic states and the reduction formula can be used. In an alternative approach, S-matrix elements are determined from unitarity and analyticity conditions without the reduction formalism, which has been found eventually to be practicable in string theory. A unification of the two methods might be best considered in an infinite-dimensional setting for a nonlocal theory such as string theory because of the link between the infinite-dimensional limit of the spherical volume and point particles.

The restriction to a finite interaction region is relevant in the formulation of string scattering amplitudes, because a conformal transformation would be required to reduce the larger surface to the spatial limits. At infinite genus, surfaces with boundaries that can be identified with the strings would yield matrix elements for the scattering of additional strings, and unitarity of the theory would not be preserved. While it is conceivable that such surfaces might be reduced to a finite interaction region through a rescaling of the coordinates, the conformal equivalence with a surface which generates additional string states at spatial infinity requires the exclusion from the sum over string histories in a finite number of dimensions. These results confirm necessity of restricting the domain of perturbation theory to the quasiconformally invariant subset of effectively closed surfaces which do not admit another source for the Green function in a finite-dimensional space-time. The condition of finiteness of the integral of the stochastic measure in four dimensions sets the range of the interactions in a scalar field theory and a shift in the nonperturbative mass, which supports the existence of a finite interaction region in string perturbation theory.

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