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Convergence Analysis for Multivalued Nonexpansive Mappings in Banach Spaces

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Abstract. The purpose of this paper is to study a new three-step iteration scheme for three multivalued nonexpansive mappings of Rafiq [12] type and establish some strong convergence theorems in the setting of Banach spaces. Our results extend and generalize several corresponding results from the existing literature.

1. Introduction and Preliminaries

Let $\mathcal{X}$ be a real Banach space. A subset $\mathcal{K}$ is called proximal if for each $x \in \mathcal{X}$, there exists an element $k \in \mathcal{K}$ such that

$$d(x, k) = \inf\{\|x - y\| : y \in \mathcal{K}\} = d(x, \mathcal{K}).$$

It is well known that a weakly compact convex subset of a Banach space and closed convex subsets of a uniformly convex Banach space are Proximal. We shall denote the family of all nonempty bounded proximal subsets of $\mathcal{K}$ by $P(\mathcal{K})$ and let $CB(\mathcal{K})$ be the class of all nonempty bounded and closed subsets of $\mathcal{K}$. Let H denote the Hausdorff metric induced by the metric d of $\mathcal{X}$, that is,

$$H(A, B) = \max\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\}$$

for every $A, B \in CB(\mathcal{X})$, where $d(x, B) = \inf\{\|x - y\| : y \in B\}$.

A multivalued mapping $\mathcal{T} : \mathcal{K} \to P(\mathcal{K})$ is said to be a *contraction* if there exists a constant $\lambda \in [0, 1)$ such that for any $x, y \in \mathcal{K}$,

$$H(\mathcal{T}x, \mathcal{T}y) \le \lambda \|x - y\|,$$

and $\mathcal{T}$ is said to be *nonexpansive* if

$$H(\mathcal{T}x, \mathcal{T}y) \le \|x - y\|,$$

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for all $x, y \in \mathcal{K}$. A point $x \in \mathcal{K}$ is called a fixed point of $\mathcal{T}$ if $x \in \mathcal{T}x$.

In 1969, Nadler [9] combined the ideas of multivalued mapping and Lipschitz mapping and proved some fixed theorems for multivalued contraction mappings. These results place no severe restrictions on the images of points and all that is required of the space is that it is a complete metric space. In the setting of multivalued mappings, the fundamental result of Nadler's theorem [9] is as follows:

Nadler's Theorem. Let $(\mathcal{X}, d)$ be a complete metric space and $\mathcal{T}\colon \mathcal{X} \to CB(\mathcal{X})$ be such that $H(\mathcal{T}x, \mathcal{T}y) \le a\,d(x, y)$ for all $x, y \in \mathcal{X}$ and some $a \in [0, 1)$, where $CB(\mathcal{X})$ denotes the family of all nonempty closed and bounded subsets of $\mathcal{X}$. Then $Fix(\mathcal{T})$ is nonempty, that is, there exists $x \in \mathcal{X}$ such that $x \in \mathcal{T}x$.

Later, an interesting and rich fixed point theory for such maps was developed which has applications in control theory, convex optimization, differential inclusion and economics (see [3] and references cited therein). Moreover, the existence of fixed points for multivalued nonexpansive mappings in uniformly convex Banach spaces was proved by Lim [7]. Many authors have studied the fixed point for multivalued mappings (e.g., see [2, 6, 8, 11, 16, 17, 20]).

In 2005, Sastry and Babu [13] obtained the convergence results from single valued mappings to multivalued mappings by defining Ishikawa and Mann iterates for multivalued mappings with a fixed point. They considered the following:
Let $\mathcal{K}$ be a nonempty convex subset of $\mathcal{X}$, $\mathcal{T}\colon \mathcal{K} \to P(\mathcal{K})$ is a multivalued mapping with $p \in \mathcal{T}p$.

(i) The sequence of Mann iterates is defined by

$$\begin{cases} x_1 = x \in \mathcal{K}, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n s_n, \ n \ge 1, \end{cases} \tag{1}$$

where $\{\alpha_n\}$ is a real sequence in $(0, 1)$ and $s_n \in \mathcal{T}x_n$ such that $\|s_n - p\| = d(p, \mathcal{T}x_n)$.

(ii) The sequence of Ishikawa iterates is defined by

$$\begin{cases} x_1 = x \in \mathcal{K}, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n r_n, \\ y_n = (1-\beta_n)x_n + \beta_n s_n, \ n \ge 1, \end{cases} \tag{2}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0, 1)$, $\|s_n - r_n\| = d(\mathcal{T}x_n, \mathcal{T}y_n)$ and $\|r_n - p\| = d(p, \mathcal{T}y_n)$ for $s_n \in \mathcal{T}x_n$ and $r_n \in \mathcal{T}y_n$. They established some strong and weak convergence results of the above iterates for multivalued nonexpansive mappings $\mathcal{T}$ under some appropriate conditions.

In 2007, Panyanak [11] generalized the results of Sastry and Babu [13] to uniformly convex Banach spaces and proved a convergence theorem of Mann iterates for a mapping defined on a noncompact domain. Later in 2008, Song and Wang [15] proved strong convergence theorems of Mann and Ishikawa iterates for multivalued nonexpansive mappings under some appropriate control conditions. Furthermore, they also gave an affirmative answer to Panyanak's open question in [11].

In 2000, Noor [10] introduced and studied the following iteration scheme: let $\mathcal{K}$ be a nonempty convex subset of a uniformly smooth Banach space E and $\mathcal{T}$ be a nonlinear mapping of $\mathcal{K}$ into itself. Then the sequence $\{x_n\}$ in $\mathcal{K}$ is defined by

$$\begin{cases} x_0 = x \in \mathcal{K}, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n \mathcal{T}y_n, \\ y_n = (1-\beta_n)x_n + \beta_n \mathcal{T}z_n, \\ z_n = (1-\gamma_n)x_n + \gamma_n \mathcal{T}x_n, \ n \ge 0, \end{cases} \tag{3}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are real sequences in $[0,1]$ satisfying some conditions.

In 2006, Rafiq [12] introduced the following modified three-step iteration scheme which include Noor, Ishikawa and Mann iterations as special case and used it to approximate the unique common fixed point of a family of strongly pseudocontractive operators.

Let $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 \colon \mathcal{K} \to \mathcal{K}$ be three mappings. Then the sequence $\{x_n\}$ in $\mathcal{K}$ is defined by

$$\begin{cases} x_0 = x \in \mathcal{K}, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n \mathcal{T}_1 y_n, \\ y_n = (1-\beta_n)x_n + \beta_n \mathcal{T}_2 z_n, \\ z_n = (1-\gamma_n)x_n + \gamma_n \mathcal{T}_3 x_n, \ n \geq 0, \end{cases} \tag{4}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are real sequences in $[0,1]$ satisfying some conditions.

Motivated by Sastry and Babu [13], Panyanak [11] and Song and Wang [15], we first give a multivalued version of the iteration scheme (4) of Rafiq [12] and then study its convergence analysis in the setting of Banach spaces. We define our iteration scheme as follows:

$$\begin{cases} x_1 = x \in \mathcal{K}, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n v_n, \\ y_n = (1-\beta_n)x_n + \beta_n w_n, \\ z_n = (1-\gamma_n)x_n + \gamma_n u_n, \ n \geq 1, \end{cases} \tag{5}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $(0,1)$, $u_n \in \mathcal{T}_3 x_n$, $v_n \in \mathcal{T}_1 y_n$ and $w_n \in \mathcal{T}_2 z_n$ such that $\|w_n - u_n\| = d(\mathcal{T}_2 z_n, \mathcal{T}_3 x_n)$, $\|v_n - w_n\| = d(\mathcal{T}_1 y_n, \mathcal{T}_2 z_n)$, $\|v_n - u_n\| = d(\mathcal{T}_1 y_n, \mathcal{T}_3 x_n)$, $\|u_{n+1} - v_n\| = d(\mathcal{T}_3 x_{n+1}, \mathcal{T}_1 y_n)$ and $\|u_{n+1} - w_n\| = d(\mathcal{T}_3 x_{n+1}, \mathcal{T}_2 z_n)$, respectively.

Now, we recall the following definitions.

Definition 1.1. *A multivalued nonexpansive mapping $\mathcal{T} \colon \mathcal{K} \to CB(\mathcal{K})$ where $\mathcal{K}$ a subset of $\mathcal{X}$ is said to satisfy condition (I) if there exists a nondecreasing function $f \colon [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(t) > 0$ for all $t \in (0,\infty)$ such that $d(x, \mathcal{T}x) \geq f(d(x, F(\mathcal{T})))$ for all $x \in \mathcal{K}$, where $F(\mathcal{T}) \neq \emptyset$ is the fixed point set of the multivalued mapping $\mathcal{T}$.*

Definition 1.2. *([5]) Let f be a nondecreasing self-map on $[0,\infty)$ with $f(0) = 0$, $f(t) > 0$ for all $t \in (0,\infty)$ and let $d(x, F(\mathcal{S},\mathcal{T})) = \inf\{d(x,y) : y \in F(\mathcal{S},\mathcal{T})\}$. Let $\mathcal{S},\mathcal{T} : \mathcal{K} \to P(\mathcal{K})$ be two multivalued maps with $F(\mathcal{S},\mathcal{T}) \neq \emptyset$. Then the two maps are said to satisfy condition (A') if*

$$d(x, \mathcal{T}x) \geq f(d(x, F(\mathcal{S},\mathcal{T})) \ \text{ or } \ d(x, \mathcal{S}x) \geq f(d(x, F(\mathcal{S},\mathcal{T})) \text{ for all } x \in \mathcal{K}.$$

Now, we generalize the above definition for three mappings.

Definition 1.3. *Let f be a nondecreasing self-map on $[0,\infty)$ with $f(0) = 0$, $f(t) > 0$ for all $t \in (0,\infty)$ and let $d(x,\mathcal{F}) = \inf\{d(x,y) : y \in \mathcal{F}\}$, where $\mathcal{F} = F(\mathcal{T}_1) \cap F(\mathcal{T}_2) \cap F(\mathcal{T}_3)$. Let $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 \colon \mathcal{K} \to P(\mathcal{K})$ be three multivalued maps with $\mathcal{F} \neq \emptyset$. Then the three maps are said to satisfy condition (GA') if*

$$d(x, \mathcal{T}_1 x) \geq f(d(x,\mathcal{F})) \ \text{ or } \ d(x, \mathcal{T}_2 x) \geq f(d(x,\mathcal{F})) \ \text{ or } \ d(x, \mathcal{T}_3 x) \geq f(d(x,\mathcal{F})),$$

for all $x \in \mathcal{K}$.

Definition 1.4. *([5]) A map $\mathcal{T} : \mathcal{K} \to P(\mathcal{K})$ is semi-compact if any bounded sequence $\{x_n\}$ satisfying $d(x_n, \mathcal{T}x_n) \to 0$ as $n \to \infty$ has a convergent subsequence.*

In 2008, Suzuki [18] introduced a condition which is weaker than nonexpansiveness. Suzuki's condition which was named by him the condition (C) reads as follows: a mapping $\mathcal{T}$ is said to satisfy the condition (C) on $\mathcal{X}$ if

$$\frac{1}{2}\|x - \mathcal{T}x\| \leq \|x - y\| \Rightarrow \|\mathcal{T}x - \mathcal{T}y\| \leq \|x - y\|, \ \forall\, x, y \in \mathcal{X}.$$

In 2010, Abkar and Eslamian [1] introduced Suzuki's condition for multivalued mappings. The definition is as follows:

Definition 1.5. *([1]) A multivalued mapping $\mathcal{T}\colon \mathcal{X} \to CB(\mathcal{X})$ is said to satisfy condition (C) provided that*

$$\frac{1}{2}d(x, \mathcal{T}x) \leq \|x - y\| \Rightarrow H(\mathcal{T}x, \mathcal{T}y) \leq \|x - y\|, \ x, y \in \mathcal{X}.$$

We mention that there exist single-valued and multi-valued mappings satisfying the condition (C) which are not nonexpansive, for example:

Example 1.6. *([18]) Define a mapping $\mathcal{T}$ on $[0, 3]$ by*

$$T(x) = \begin{cases} 0, & \text{if } x \neq 3, \\ 1, & \text{if } x = 3. \end{cases}$$

Then $\mathcal{T}$ is a single-valued mapping satisfying condition (C), but $\mathcal{T}$ is not nonexpansive.

Example 1.7. *([1]) Define a mapping $\mathcal{T}\colon [0, 5] \to [0, 5]$ by*

$$T(x) = \begin{cases} [0, \frac{x}{5}], & \text{if } x \neq 5, \\ \{1\}, & \text{if } x = 5, \end{cases}$$

then it is easy to show that $\mathcal{T}$ is a multi-valued mapping satisfying condition (C), but $\mathcal{T}$ is not nonexpansive.

We need the following Lemmas to prove our main results.

Lemma 1.8. *(See [19]) Let $\{p_n\}$, $\{q_n\}$, $\{r_n\}$ be three sequences of nonnegative real numbers satisfying the following conditions:*

$$p_{n+1} \leq (1 + q_n)p_n + r_n, \ n \geq 0, \quad \sum_{n=0}^{\infty} q_n < \infty, \quad \sum_{n=0}^{\infty} r_n < \infty.$$

Then
(1) $\lim_{n\to\infty} p_n$ exists.
(2) In addition, if $\liminf_{n\to\infty} p_n = 0$, then $\lim_{n\to\infty} p_n = 0$.

Lemma 1.9. *(See [14]) Let E be a uniformly convex Banach space and $0 < \alpha \leq t_n \leq \beta < 1$ for all $n \in \mathbb{N}$. Suppose further that $\{x_n\}$ and $\{y_n\}$ are sequences of E such that $\limsup_{n\to\infty} \|x_n\| \leq a$, $\limsup_{n\to\infty} \|y_n\| \leq a$ and $\lim_{n\to\infty} \|t_n x_n + (1 - t_n)y_n\| = a$ hold for some $a \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 1.10. *(See [17]) Let $\mathcal{T}\colon \mathcal{K} \to P(\mathcal{K})$ be a multivalued mapping and $P_{\mathcal{T}}(x) = \{y \in \mathcal{T}x : \|x - y\| = d(x, \mathcal{T}x)\}$. Then the following are equivalent.*
(1) $x \in F(\mathcal{T})$;
(2) $P_{\mathcal{T}}(x) = \{x\}$;
(3) $x \in F(P_{\mathcal{T}})$.
Moreover, $F(\mathcal{T}) = F(P_{\mathcal{T}})$.

Lemma 1.11. *(See [1]) Let $\mathcal{T}\colon \mathcal{X} \to CB(\mathcal{X})$ be a multi-valued mapping. If $\mathcal{T}$ is nonexpansive, then $\mathcal{T}$ satisfies the condition (C).*

2. Main Results

In this section, we prove some strong convergence theorems using iteration scheme (5). First, we need the following lemmas to prove main results. Let $\mathcal{F} = F(\mathcal{T}_1) \cap F(\mathcal{T}_2) \cap F(\mathcal{T}_3)$ denotes the set of all common fixed points of the mappings $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$.

Lemma 2.1. *Let $\mathcal{X}$ be a real Banach space and $\mathcal{K}$ be a nonempty closed and convex subset of $\mathcal{X}$. Let $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 \colon \mathcal{K} \to P(\mathcal{K})$ be three multivalued mappings such that $\mathcal{F} \neq \emptyset$ and $P_{\mathcal{T}_1}$, $P_{\mathcal{T}_2}$ and $P_{\mathcal{T}_3}$ are nonexpansive mappings. Let $\{x_n\}$ be the sequence defined by (5), where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $(0,1)$. Then $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in \mathcal{F}$.*

Proof. Let $p \in \mathcal{F}$. Then $p \in P_{\mathcal{T}_1}(p) = \{p\}$, $p \in P_{\mathcal{T}_2}(p) = \{p\}$ and $p \in P_{\mathcal{T}_3}(p) = \{p\}$ by Lemma 1.10. It follows from (5) that

$$\begin{aligned} \|z_n - p\| &\leq (1-\gamma_n)\|x_n - p\| + \gamma_n\|u_n - p\| \\ &\leq (1-\gamma_n)\|x_n - p\| + \gamma_n H(P_{\mathcal{T}_3}(x_n), P_{\mathcal{T}_3}(p)) \\ &\leq (1-\gamma_n)\|x_n - p\| + \gamma_n\|x_n - p\| \\ &= \|x_n - p\|. \end{aligned} \tag{6}$$

Again using (5) and (6), we obtain

$$\begin{aligned} \|y_n - p\| &\leq (1-\beta_n)\|x_n - p\| + \beta_n\|w_n - p\| \\ &\leq (1-\beta_n)\|x_n - p\| + \beta_n H(P_{\mathcal{T}_2}(z_n), P_{\mathcal{T}_2}(p)) \\ &\leq (1-\beta_n)\|x_n - p\| + \beta_n\|z_n - p\| \\ &\leq (1-\beta_n)\|x_n - p\| + \beta_n\|x_n - p\| \\ &= \|x_n - p\|. \end{aligned} \tag{7}$$

Now using (5), (6) and (7), we obtain

$$\begin{aligned} \|x_{n+1} - p\| &\leq (1-\alpha_n)\|x_n - p\| + \alpha_n\|v_n - p\| \\ &\leq (1-\alpha_n)\|x_n - p\| + \alpha_n H(P_{\mathcal{T}_1}(y_n), P_{\mathcal{T}_1}(p)) \\ &\leq (1-\alpha_n)\|x_n - p\| + \alpha_n\|y_n - p\| \\ &\leq (1-\alpha_n)\|x_n - p\| + \alpha_n\|x_n - p\| \\ &= \|x_n - p\|. \end{aligned} \tag{8}$$

It follows from Lemma 1.8 that $\lim_{n\to\infty} \|x_n - p\|$ exists for each $p \in \mathcal{F}$. This completes the proof. □

Lemma 2.2. *Let $\mathcal{X}$ be a uniformly convex Banach space and $\mathcal{K}$ be a nonempty closed and convex subset of $\mathcal{X}$. Let $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 \colon \mathcal{K} \to P(\mathcal{K})$ be three multivalued mappings such that $\mathcal{F} \neq \emptyset$ and $P_{\mathcal{T}_1}$, $P_{\mathcal{T}_2}$ and $P_{\mathcal{T}_3}$ are nonexpansive mappings. Let $\{x_n\}$ be the sequence defined by (5), where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $(0,1)$. Then $\lim_{n\to\infty} d(x_n, \mathcal{T}_1 y_n) = 0$, $\lim_{n\to\infty} d(x_n, \mathcal{T}_2 z_n) = 0$ and $\lim_{n\to\infty} d(x_n, \mathcal{T}_3 x_n) = 0$.*

Proof. From Lemma 2.1, $\lim_{n\to\infty} \|x_n - p\|$ exists for each $p \in \mathcal{F}$. We suppose that $\lim_{n\to\infty} \|x_n - p\| = a$ for some $a \geq 0$.

Since $\limsup_{n\to\infty} \|u_n - p\| \leq \limsup_{n\to\infty} H(\mathcal{T}_3 x_n, \mathcal{T}_3 p) \leq \limsup_{n\to\infty} \|x_n - p\| = a$,

so,

$$\limsup_{n\to\infty} \|u_n - p\| \leq a. \tag{9}$$

Again, since $\limsup_{n\to\infty} \|v_n - p\| \leq \limsup_{n\to\infty} H(\mathcal{T}_1 y_n, \mathcal{T}_1 p) \leq \limsup_{n\to\infty} \|y_n - p\| \leq \limsup_{n\to\infty} \|x_n - p\| = a$, so,

$$\limsup_{n\to\infty} \|v_n - p\| \le a. \tag{10}$$

Similarly,

$$\limsup_{n\to\infty} \|w_n - p\| \le a. \tag{11}$$

Applying Lemma 1.9, we get

$$\lim_{n\to\infty} \|u_n - v_n\| = 0, \tag{12}$$

$$\lim_{n\to\infty} \|w_n - u_n\| = 0, \tag{13}$$

and

$$\lim_{n\to\infty} \|v_n - w_n\| = 0. \tag{14}$$

Taking lim sup on both sides of (6) and (7), we obtain

$$\limsup_{n\to\infty} \|z_n - p\| \le a, \tag{15}$$

and

$$\limsup_{n\to\infty} \|y_n - p\| \le a. \tag{16}$$

Also

$$\begin{aligned} \|x_{n+1} - p\| &= \|(1-\alpha_n)x_n + \alpha_n v_n - p\| \\ &= \|(1-\alpha_n)(x_n - p) + \alpha_n(v_n - p)\| \\ &\le (1-\alpha_n)\|x_n - p\| + \alpha_n\|v_n - p\|, \end{aligned}$$

it implies that

$$\|x_n - p\| \le \frac{\|x_n - p\| - \|x_{n+1} - p\|}{\alpha_n} + \|v_n - p\|. \tag{17}$$

Taking the lim inf on both sides of above inequality, we obtain

$$a \le \liminf_{n\to\infty} \|v_n - p\|. \tag{18}$$

Combining (10) and (18), we obtain

$$\lim_{n\to\infty} \|v_n - p\| = a. \tag{19}$$

Thus

$$\begin{aligned} \|v_n - p\| &\le \|v_n - w_n\| + \|w_n - p\| \\ &\le \|v_n - w_n\| + H(\mathcal{T}_2 z_n, \mathcal{T}_2 p) \\ &\le \|v_n - w_n\| + \|z_n - p\|, \end{aligned}$$

gives

$$a \le \liminf_{n\to\infty} \|z_n - p\|, \tag{20}$$

and by virtue of (15), we obtain

$$\lim_{n\to\infty} \|z_n - p\| = a. \tag{21}$$

By Lemma 1.9, we have

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{22}$$

Now, note that

$$\|x_n - w_n\| \quad \leq \quad \|x_n - u_n\| + \|u_n - w_n\|.$$

Using (13) and (22), we obtain

$$\lim_{n\to\infty} \|x_n - w_n\| = 0. \tag{23}$$

Since

$$\|x_n - v_n\| \quad \leq \quad \|x_n - u_n\| + \|u_n - v_n\|.$$

Using (14) and (22), we obtain

$$\lim_{n\to\infty} \|x_n - v_n\| = 0. \tag{24}$$

Since $d(x_n, \mathcal{T}_3 x_n) \leq \|x_n - u_n\|$, we have

$$\lim_{n\to\infty} d(x_n, \mathcal{T}_3 x_n) = 0. \tag{25}$$

Again since $d(x_n, \mathcal{T}_1 y_n) \leq \|x_n - v_n\|$, we have

$$\lim_{n\to\infty} d(x_n, \mathcal{T}_1 y_n) = 0. \tag{26}$$

Similarly, since $d(x_n, \mathcal{T}_2 z_n) \leq \|x_n - w_n\|$, we have

$$\lim_{n\to\infty} d(x_n, \mathcal{T}_2 z_n) = 0. \tag{27}$$

This completes the proof. □

We now give some strong convergence theorems using iteration scheme (5) in real Banach spaces.

Theorem 2.3. *Let $\mathcal{X}$ be a uniformly convex Banach space and $\mathcal{K}$ be a nonempty closed and convex subset of $\mathcal{X}$. Let $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 \colon \mathcal{K} \to P(\mathcal{K})$ be three multivalued mappings such that $\mathcal{F} \neq \emptyset$ and $P_{\mathcal{T}_1}$, $P_{\mathcal{T}_2}$ and $P_{\mathcal{T}_3}$ are nonexpansive mappings. Let $\{x_n\}$ be the sequence defined by (5), where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $(0, 1)$. Then $\{x_n\}$ converges strongly to a common fixed point of $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ if and only if $\liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0$.*

Proof. The necessity is obvious. Conversely, suppose that $\liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0$. As proved in Lemma 2.1, we have

$$\|x_{n+1} - p\| \leq \|x_n - p\|,$$

which gives

$$d(x_{n+1}, \mathcal{F}) \leq d(x_n, \mathcal{F}).$$

This implies that $\lim_{n\to\infty} d(x_n, \mathcal{F})$ exists and so by the hypothesis, $\liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0$. Therefore, we must have $\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0$.

Next, we have to show that $\{x_n\}$ is a Cauchy sequence in $\mathcal{K}$. Let $\varepsilon > 0$ be arbitrary chosen. Since $\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0$, there exists a constant N_1 such that for all $n \geq N_1$ we have

$$d(x_n, \mathcal{F}) < \frac{\varepsilon}{4}.$$

In particular, $\inf\{\|x_{N_1} - q\| : q \in \mathcal{F}\} < \frac{\varepsilon}{4}$. There must exists a $q_1 \in \mathcal{F}$ such that

$$\|x_{N_1} - q_1\| < \frac{\varepsilon}{2}.$$

Now for $m, n \geq N_1$, we have

$$\begin{aligned} \|x_{n+m} - x_n\| &\leq \|x_{n+m} - q_1\| + \|x_n - q_1\| \\ &\leq 2\|x_{N_1} - q_1\| \\ &< 2\Big(\frac{\varepsilon}{2}\Big) = \varepsilon. \end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence in a closed subset $\mathcal{K}$ of a Banach space $\mathcal{X}$, and hence converges, say to $q_2 \in \mathcal{K}$. Now it is left to show that $q_2 \in \mathcal{F}$. Now

$$\begin{aligned} d(q_2, P_{\mathcal{T}_1}(q_2)) &\leq \|x_n - q_2\| + d(x_n, P_{\mathcal{T}_1}(y_n)) + H(P_{\mathcal{T}_1}(y_n), P_{\mathcal{T}_1}(q_2)) \\ &\leq \|x_n - q_2\| + \|x_n - v_n\| + \|y_n - q_2\| \\ &\leq \|x_n - q_2\| + \|x_n - v_n\| + \|x_n - q_2\| + \beta_n\|w_n - x_n\| \\ &\leq 2\|x_n - q_2\| + \|x_n - v_n\| + \|w_n - x_n\| \\ &\to 0 \text{ as } n \to \infty, \end{aligned}$$

which gives that $d(q_2, \mathcal{T}_1 q_2) = 0$ and

$$\begin{aligned} d(q_2, P_{\mathcal{T}_2}(q_2)) &\leq \|x_n - q_2\| + d(x_n, P_{\mathcal{T}_2}(z_n)) + H(P_{\mathcal{T}_2}(z_n), P_{\mathcal{T}_2}(q_2)) \\ &\leq \|x_n - q_2\| + \|x_n - w_n\| + \|z_n - q_2\| \\ &\leq \|x_n - q_2\| + \|x_n - w_n\| + \|x_n - q_2\| + \gamma_n\|u_n - x_n\| \\ &\leq 2\|x_n - q_2\| + \|x_n - w_n\| + \|u_n - x_n\| \\ &\to 0 \text{ as } n \to \infty, \end{aligned}$$

which gives that $d(q_2, \mathcal{T}_2 q_2) = 0$. Similarly, we have

$$\begin{aligned} d(q_2, P_{\mathcal{T}_3}(q_2)) &\leq \|x_n - q_2\| + d(x_n, P_{\mathcal{T}_3}(x_n)) + H(P_{\mathcal{T}_3}(x_n), P_{\mathcal{T}_3}(q_2)) \\ &\leq \|x_n - q_2\| + \|x_n - u_n\| + \|x_n - q_2\| \\ &= 2\|x_n - q_2\| + \|x_n - u_n\| \\ &\to 0 \text{ as } n \to \infty, \end{aligned}$$

which gives that $d(q_2, \mathcal{T}_3 q_2) = 0$. But $P_{\mathcal{T}_1}$, $P_{\mathcal{T}_2}$ and $P_{\mathcal{T}_3}$ are nonexpansive mappings and so $\mathcal{F}$ is closed. Therefore, $q_2 \in \mathcal{F}$ as required. Thus $\{x_n\}$ converges strongly to a common point of $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$. This completes the proof. □

Theorem 2.4. *Let $\mathcal{X}$ be a real Banach space and $\mathcal{K}$ be a nonempty compact convex subset of $\mathcal{X}$. Let $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 \colon \mathcal{K} \to P(\mathcal{K})$ be three multivalued mappings such that $\mathcal{F} \neq \emptyset$ and $P_{\mathcal{T}_1}$, $P_{\mathcal{T}_2}$ and $P_{\mathcal{T}_3}$ are nonexpansive mappings. Let $\{x_n\}$ be the sequence defined by (5), where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $(0, 1)$. Then $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converges strongly to a common point of $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$.*

Proof. By Lemma 2.2, we have $\lim_{n\to\infty} d(x_n, \mathcal{T}_3 x_n) = 0$. Since by hypothesis $\mathcal{K}$ be a nonempty compact convex subset of $\mathcal{X}$, so there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim_{k\to\infty} \|x_{n_k} - q'\| = 0$ for some $q' \in \mathcal{K}$. Thus

$$\begin{aligned} d(q', \mathcal{T}_3 q') &\leq \|x_{n_k} - q'\| + d(x_{n_k}, \mathcal{T}_3 x_{n_k}) + H(\mathcal{T}_3 x_{n_k}, \mathcal{T}_3 q') \\ &\leq 2\|x_{n_k} - q'\| + \|x_{n_k} - u_{n_k}\| \\ &\to 0 \text{ as } k \to \infty. \end{aligned}$$

This shows that q' is a fixed of $\mathcal{T}_3$. From Lemma 2.1, we get that $\lim_{n\to\infty}\|x_n - q'\| = 0$. Again from Lemma 2.2, we get that

$$\begin{aligned} \|y_n - x_n\| &= \|(1-\beta_n)x_n + \beta_n w_n - x_n\| \\ &\le \beta_n\|w_n - x_n\| \\ &\le \|w_n - x_n\| \\ &\to 0 \text{ as } n \to \infty, \end{aligned}$$

and

$$\begin{aligned} \|z_n - x_n\| &= \|(1-\gamma_n)x_n + \gamma_n u_n - x_n\| \\ &\le \gamma_n\|u_n - x_n\| \\ &\le \|u_n - x_n\| \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

It follows that $\lim_{n\to\infty}\|y_n - q'\| = 0$ and $\lim_{n\to\infty}\|z_n - q'\| = 0$. Thus the desired conclusion follows. This completes the proof. □

Now, applying Lemma 2.2 and Theorem 2.3, we can easily obtain the following results.

Theorem 2.5. *Let $\mathcal{X}$ be a uniformly convex Banach space and $\mathcal{K}$ be a nonempty closed and convex subset of $\mathcal{X}$. Let $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 \colon \mathcal{K} \to P(\mathcal{K})$ be three multivalued mappings such that $\mathcal{F} \neq \emptyset$ and $P_{\mathcal{T}_1}$, $P_{\mathcal{T}_2}$ and $P_{\mathcal{T}_3}$ are nonexpansive mappings. Let $\{x_n\}$ be the sequence defined by (5), where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $(0,1)$. Suppose that $P_{\mathcal{T}_1}$, $P_{\mathcal{T}_2}$ and $P_{\mathcal{T}_3}$ satisfies condition (GA'), then the sequence $\{x_n\}$ converges strongly to a common fixed point of $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$.*

Theorem 2.6. *Let $\mathcal{X}$ be a uniformly convex Banach space and $\mathcal{K}$ be a nonempty closed and convex subset of $\mathcal{X}$. Let $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 \colon \mathcal{K} \to P(\mathcal{K})$ be three multivalued mappings such that $\mathcal{F} \neq \emptyset$ and $P_{\mathcal{T}_1}$, $P_{\mathcal{T}_2}$ and $P_{\mathcal{T}_3}$ are nonexpansive mappings. Let $\{x_n\}$ be the sequence defined by (5), where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $(0,1)$. Suppose that one of the map in $P_{\mathcal{T}_1}$, $P_{\mathcal{T}_2}$ and $P_{\mathcal{T}_3}$ is semi-compact, then the sequence $\{x_n\}$ converges strongly to a common fixed point of $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$.*

Using condition (C), Lemma 1.11, Lemma 2.2 and Theorem 2.3, we can easily obtain the following results.

Theorem 2.7. *Let $\mathcal{X}$ be a real Banach space and $\mathcal{K}$ be a nonempty closed and convex subset of $\mathcal{X}$. Let $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 \colon \mathcal{K} \to P(\mathcal{K})$ be three multivalued mappings with $\mathcal{F} \neq \emptyset$ and such that $P_{\mathcal{T}_1}$, $P_{\mathcal{T}_2}$ and $P_{\mathcal{T}_3}$ satisfies the condition (C). Let $\{x_n\}$ be the sequence defined by (5), where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $(0,1)$. Then the sequence $\{x_n\}$ converges strongly to a common fixed point of $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ if and only if $\liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0$.*

Theorem 2.8. *Let $\mathcal{X}$ be a uniformly convex Banach space and $\mathcal{K}$ be a nonempty closed and convex subset of $\mathcal{X}$. Let $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 \colon \mathcal{K} \to P(\mathcal{K})$ be three multivalued mappings with $\mathcal{F} \neq \emptyset$ and such that $P_{\mathcal{T}_1}$, $P_{\mathcal{T}_2}$ and $P_{\mathcal{T}_3}$ satisfies the condition (C). Let $\{x_n\}$ be the sequence defined by (5), where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $(0,1)$. If the following condition is satisfied:*

(C_1) *there exists an increasing function $g \colon [0,\infty) \to [0,\infty)$ with $g(0) = 0$ and $g(r) > 0$, $\forall\, r > 0$ such that*

$$d(x_n, \mathcal{T}_1 x_n) \ge g(d(x_n, \mathcal{F})) \ \text{ or } \ d(x_n, \mathcal{T}_2 x_n) \ge g(d(x_n, \mathcal{F})) \ \text{ or } \ \sqrt{d(x_n, \mathcal{T}_3 x_n)} \ge g(d(x_n, \mathcal{F})),$$

then $\{x_n\}$ converges strongly to a common fixed point of $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$.

Theorem 2.9. *Let $\mathcal{X}$ be a uniformly convex Banach space and $\mathcal{K}$ be a nonempty closed and convex subset of $\mathcal{X}$. Let $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 \colon \mathcal{K} \to P(\mathcal{K})$ be three multivalued mappings with $\mathcal{F} \neq \emptyset$ and such that $P_{\mathcal{T}_1}$, $P_{\mathcal{T}_2}$ and $P_{\mathcal{T}_3}$ satisfies the condition (C). Let $\{x_n\}$ be the sequence defined by (5), where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $(0,1)$. Suppose that one of the map in $P_{\mathcal{T}_1}$, $P_{\mathcal{T}_2}$ and $P_{\mathcal{T}_3}$ is semi-compact, then the sequence $\{x_n\}$ converges strongly to a common fixed point of $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$.*

Example 2.10. *Let $\mathcal{K} = [0,1]$ be equipped with the Euclidean norm $\|.\| = |.|$. Let $\mathcal{S}, \mathcal{T} \colon \mathcal{K} \to CB(\mathcal{K})$ (family of closed and bounded subset of $\mathcal{K}$) be defined by $\mathcal{T}_1(x) = [0, \frac{x}{4}]$, $\mathcal{T}_2(x) = [0, \frac{x}{3}]$ and $\mathcal{T}_3(x) = [0, \frac{x}{2}]$. Then any $x, y \in \mathcal{K}$*

$$\begin{aligned} H(\mathcal{T}_1(x), \mathcal{T}_1(y)) &= \max\left\{\left|\frac{x}{4} - \frac{y}{4}\right|, 0\right\} = \left|\frac{x}{4} - \frac{y}{4}\right| = \left|\frac{x-y}{4}\right| \\ &\leq |x-y|. \end{aligned}$$

$$\begin{aligned} H(\mathcal{T}_3(x), \mathcal{T}_3(y)) &= \max\left\{\left|\frac{x}{2} - \frac{y}{2}\right|, 0\right\} = \left|\frac{x}{2} - \frac{y}{2}\right| = \left|\frac{x-y}{2}\right| \\ &\leq |x-y|. \end{aligned}$$

Similarly,

$$\begin{aligned} H(\mathcal{T}_2(x), \mathcal{T}_2(y)) &= \max\left\{\left|\frac{x}{3} - \frac{y}{3}\right|, 0\right\} = \left|\frac{x}{3} - \frac{y}{3}\right| = \left|\frac{x-y}{3}\right| \\ &\leq |x-y|. \end{aligned}$$

Thus $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ are multivalued nonexpansive mappings. Clearly, $F(\mathcal{T}_1) \cap F(\mathcal{T}_2) \cap F(\mathcal{T}_3) = \{0\}$. Hence, $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ have a unique common fixed point in $\mathcal{K}$.

3. Concluding remarks

In this paper, we establish some strong convergence theorems under some standard conditions applying on the space in the setting of real Banach spaces. Our results extend and generalize several results from the current existing literature (see, for example, [4, 11, 13, 15–17] and many others) to the case of three-step iteration scheme and three multivalued nonexpansive mappings.

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