Functional Analysis,
Approximation and
Computation
12 (2) (2020), 31–38

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/faac

Parallel Summable Range Symmetric Matrices with Reference to Indefinite Inner Product

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Abstract. Let $\mathscr{I}$ be a indefinite inner product space. In this paper we investigate the results related inequalities on parallel summable range symmetric matrices with respect to indefinite inner product. The results include the relation between the invertible, range symmetric, parallel sum and positive semidefinite matrices. Some new properties are obtained and some known results are extended.

1. Introduction

An indefinite inner product is a conjugate symmetric sesquilinear form $[x, y]$ together with the regularity condition that $[x, y] = 0, \forall y \in \mathbb{C}^n$ only when $x = 0$. Any indefinite inner product is associated with a unique invertible complex matrix J (called a weight) such that $[x, y] = \langle x, Jy \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product on $\mathbb{C}^n$. We also make an additional assumption on J, that is, $J^2 = I$.

Investigations of linear maps on indefinite inner product utilize the usual multiplication of matrices which is induced by the Euclidean inner product of vectors ([2],[12]). This causes a problem as there are two different values for dot product of vectors. To overcome this difficulty, Ramanathan et al. introduced a new matrix product called indefinite matrix multiplication and investigated some of its properties in [12]. More precisely, the indefinite matrix product of two matrices A and B of sizes $m \times n$ and $n \times l$ complex matrices, respectively, is defined to be the matrix $A \circ B = AJ_nB$. The adjoint of A, denoted by $A^{[*]}$ is defined to be the matrix $J_nA^*J_m$, where J_m and J_n are weights. Further indefinite matrix product with properties are established. This concept was discussed further by many researcher in ([3], [5], [6], [7], [8], [10], [11]).

2. Preliminaries

We first recall the notion of an indefinite multiplication of matrices.

Definition 2.1. *[12] Let $A \in M_{(m,n)}(\mathbb{C})$, $B \in M_{(n,k)}(\mathbb{C})$. Let J_n be an arbitrary but fixed $n \times n$ complex matrix such that $J_n = J_n^* = J_n^{-1}$. The indefinite matrix product of A and B (relative to J_n) is defined by $A \circ B = AJ_nB$.*

2010 *Mathematics Subject Classification*. Primary 46C20; Secondary 15A09.
Keywords. Indefinite matrix product; Indefinite inner product; Range symmetric matrix.
Received: 6 March 2020; Accepted: 29 July 2020
Communicated by Dragan S. Djordjević
Supported by UGC-RGNF Award No: F1-17.1/2016-17/RGNF-2015-17-SC-TAM-17791
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Definition 2.2. *[12] For $A \in M_{(m,n)}(\mathbb{C})$, $A^{[*]} = J_n A^* J_m$ is the adjoint of A relative to J_n and J_m.*

Definition 2.3. *[12] A matrix $A \in M_n(\mathbb{C})$ is said to be J-invertible if there exists $X \in M_n(\mathbb{C})$, such that $A \circ X = X \circ A = J_n$ such an X is denoted by $A^{[-1]} = JA^{-1}J$.*

Definition 2.4. *[4] A matrix $A \in M_n(\mathbb{C})$ is said to be J-unitary if $A \circ A^{[*]} = A^{[*]} \circ A = J_n$.*

Definition 2.5. *[9] A matrix $A \in M_n(\mathbb{C})$ is said to be J-symmetric if $A = A^{[*]}$.*

Definition 2.6. *[7] For $A \in M_{(m,n)}(\mathbb{C})$, a matrix X satisfying $A \circ X \circ A = A$ is called a generalized inverse of A relative to the weight J. $A_J\{1\}$ is the set of all generalized inverses of A relative to the weight J.*

Remark 2.7. *[7] For the identity matrix J, it reduces to a generalized inverse of A and $A_J\{1\} = A\{1\}$. It can be easily verified that X is a generalized inverse of A under the indefinite matrix product if and only if $J_n X J_m$ is a generalized inverse of A under the usual product of matrices. Hence $A_J\{1\} = \{X : J_n X J_m$ is a generalized inverse of $A\}$.*

Definition 2.8. *[3] For $A \in M_{(m,n)}(\mathbb{C})$, and $X \in M_{(n,m)}(\mathbb{C})$ is called the Moore-Penrose inverse of A if it satisfies the following equations:*
(i) $A \circ X \circ A = A$.
(ii) $X \circ A \circ X = X$.
(iii) $(A \circ X)^{[*]} = A \circ X$.
(iv) $(X \circ A)^{[*]} = X \circ A$.
such an X is denoted by $A^{[\dagger]}$ and represented as $A^{[\dagger]} = J_n A^{\dagger} J_m$.

Definition 2.9. *[12] The range space of $A \in M_{(m,n)}(\mathbb{C})$ is defined by $Ra(A) = \{y = A \circ x \in \mathbb{C}^m : x \in \mathbb{C}^n\}$. The null space of $A \in M_{(m,n)}(\mathbb{C})$ is defined by $Nu(A) = \{x \in \mathbb{C}^n : A \circ x = 0\}$.*

Property 2.10. *[6] Let $A \in M_{(m,n)}(\mathbb{C})$. Then*
(i) $(A^{[*]})^{[*]} = A$.
(ii) $(A^{[\dagger]})^{[\dagger]} = A$.
(iii) $(AB)^{[*]} = B^{[*]}A^{[*]}$.
(iv) $Ra(A^{[*]}) = Ra(A^{[\dagger]})$.
(v) $Ra(A \circ A^{[*]}) = Ra(A)$, $Ra(A^{[*]} \circ A) = Ra(A^{[*]})$.
(vi) $Nu(A \circ A^{[*]}) = Nu(A^{[*]})$, $Nu(A^{[*]} \circ A) = Nu(A)$.

Definition 2.11. *[6] $A \in M_n(\mathbb{C})$ is range symmetric(J-EP) in $\mathscr{I}$ if and only if $Ra(A) = Ra(A^{[*]})$ $(A \circ A^{[\dagger]} = A^{[\dagger]} \circ A)$.*

Remark 2.12. *[6] In particular for $J = I_n$, this reduces to the definition of range symmetric matrix in unitary space (or) equivalently to an EP matrix.*

Theorem 2.13. *[6] For $A \in M_n(\mathbb{C})$, the following are equivalent:*
(i) A is range symmetric in $\mathscr{I}$.
(ii) AJ is EP.
(iii) JA is EP.
(iv) $N(A) = N(A^{[*]})$.
(v) $N(A^*) = N(AJ)$.
(vi) $A^{[*]} = HA = AK$, for some invertible matrices H and K.
(vii) $R(A^*) = R(JA)$.

Definition 2.14. *[4] A matrix $A \in M_n(\mathbb{C})$ is said to be positive semidefinite in $\mathscr{I}$ denoted as $A \geq_{\mathscr{I}} 0 \Leftrightarrow$ A is $J - EP$ and $[Ax, x] \geq 0$, for all $x \in \mathbb{C}^n$.*

Theorem 2.15. *[4] For $A \in M_n(\mathbb{C})$, $A \underset{\mathscr{I}}{\geq} 0 \Leftrightarrow A^{[\dagger]} \underset{\mathscr{I}}{\geq} 0$.*

Definition 2.16. *[13] A_1 and A_2 are said to be parallel summable if $N(A_1 + A_2) \subseteq N(A_1)$ and $N((A_1 + A_2)^*) \subseteq N(A_1^*)$.*

Definition 2.17. *[13] If A_1 and A_2 are parallel summable then parallel sum of A_1 and A_2 denoted by $A_1 : A_2$ is defined as $A_1 : A_2 = A_1(A_1 + A_2)^- A_2$. The product $A_1(A_1 + A_2)^- A_2$ is invariant for all choices of generalized inverse $(A_1 + A_2)^-$ of $(A_1 + A_2)$ under the conditions that A_1 and A_2 are parallel summable.*

Definition 2.18. *[5] A pair of matrices A_1 and A_2 are said to be parallel summable in $\mathscr{I}$ if $N(A_1 + A_2) \subseteq N(A_2)$ and $Nu((A_1 + A_2)^{[*]}) \subseteq Nu(A_2^{[*]})$ or equivalently $N(A_1 + A_2) \subseteq N(A_1)$ and $Nu((A_1 + A_2)^{[*]}) \subseteq Nu(A_1^{[*]})$.*

Theorem 2.19. *[5] Let A_1 and A_2 are parallel summable range symmetric in $\mathscr{I}$. Then $(A_1 : A_2)$ and $(A_1 + A_2)$ are range symmetric in $\mathscr{I}$.*

Property 2.20. *[5] Let A_1 and A_2 be a pair of parallel summable matrices in $\mathscr{I}$. Then the following hold:*
(i) $(A_1 : A_2)^{[*]} = (A_2 : A_1)^{[*]}$
(ii) $A_1^{[*]}$ and $A_2^{[*]}$ are parallel summable and $(A_1 : A_2)^{[*]} = A_1^{[*]} : A_2^{[*]}$.

3. Inequalities on parallel summable range symmetric matrices in $\mathscr{I}$

Lemma 3.1. *Let A and B be parallel summable range symmetric matrices in $\mathscr{I}$ such that $A + B$ is J-symmetric in $\mathscr{I}$. Then there exist non-singular matrices H and K such that $(HA : HB)^{[*]} = KA : KB = K(A : B)$.*

Proof. Since A and B are parallel summable range symmetric matrices in $\mathscr{I}$, by Theorem 2.13(vi), there exists non-singular matrices H and K such that $A^{[*]} = HA$ and $B^{[*]} = KB$. Now

$$\begin{aligned}
HA : HB &= HA(HA + HB)^- HB \\
&= HA((HA)^- + (HB)^-)HB \\
&= HA((HA)^- HB + (HB)^- HB) \\
&= HA(HA)^- HB + HA(HB)^- HB) \\
&= HB + HA \\
&= HAA^- B + HAB^- B \\
&= H(AA^- B + AB^- B) \\
&= H(A(A^- + B^-)B) \\
&= H(A(A + B)^- B) \\
&= H(A : B)
\end{aligned}$$

$$HA : HB = H(A : B) \tag{1}$$

Taking adjoint on bothsides

$$\begin{aligned}
(HA : HB)^{[*]} &= [H(A : B)]^{[*]} \\
&= (A : B)^{[*]} H^{[*]} \\
&= (B : A)^{[*]} H^{[*]} && \text{(By Property 2.20 (i))} \\
&= (B^{[*]} : A^{[*]}) H^{[*]} && \text{(By Property 2.20 (ii))} \\
&= B^{[*]}(B^{[*]} + A^{[*]})^- A^{[*]} H^{[*]} \\
&= B^{[*]}[(B + A)^{[*]}]^- A^{[*]} H^{[*]} \\
&= B^{[*]}(B + A)^- A^{[*]} H^{[*]} && \text{(By A+B is J-symmetric)} \\
&= KB(B + A)^- A && \text{(By Theorem 2.13(vi))} \\
&= K(B : A) \\
&= K(A : B) && \text{(By Property 2.20 (i))} \\
(HA : HB)^{[*]} &= KA : KB && \text{(By (1))}
\end{aligned}$$

□

Theorem 3.2. *Let A and B be parallel summable range symmetric matrices in $\mathscr{I}$ such that $A \underset{\mathscr{I}}{\geq} -B$. Then the following hold*
(i) $A \underset{\mathscr{I}}{\geq} KA : KB$, for some $K \in M_n(\mathbb{C})$
(ii) $A + B \underset{\mathscr{I}}{\geq} T(A : B)$, for some $T = K + H \in M_n(\mathbb{C})$.

Proof. Since A and B are parallel summable range symmetric matrices in $\mathscr{I}$. By Theorem 2.13(vi), there exists non-singular matrices H and K such that $A^{[*]} = HA$ and $B^{[*]} = KB$. Since $A \underset{\mathscr{I}}{\geq} -B \Rightarrow A + B \underset{\mathscr{I}}{\geq} 0$. Again by Theorem 2.15 $(A + B)^{[\dagger]} \underset{\mathscr{I}}{\geq} 0$.

Now $A - (A : B) = A - A(A + B)^{[\dagger]}B$
$= A - A(A + B)^{[\dagger]}(A + B) + A(A + B)^{[\dagger]}A$
$A - (A : B) = A(A + B)^{[\dagger]}A$
Pre multiplying both sides by H we get
$HA - H(A : B) = HA(A + B)^{[\dagger]}A$
$A^{[*]} - H(A : B) = A^{[*]}(A + B)^{[\dagger]}A$ (By Theorem 2.13(vi))
$= A^{[*]}(A + B)^{[\dagger]}A \underset{\mathscr{I}}{\geq} 0$ (By Theorem 2.15)
$A^{[*]} - H(A : B) \underset{\mathscr{I}}{\geq} 0$
Thus $A^{[*]} \underset{\mathscr{I}}{\geq} H(A : B)$.
Now taking adjoint on both sides we get $(A^{[*]})^{[*]} \underset{\mathscr{I}}{\geq} [H(A : B)]^{[*]}$

$$A \underset{\mathscr{I}}{\geq} KA : KB, \text{ for some } K \in M_n(\mathbb{C}) \tag{2}$$

(By Lemma 3.1). Thus (i) is proved.
Similarly, $B - (B : A) = B - B(A + B)^{[\dagger]}A$
$= B - B(A + B)^{[\dagger]}(A + B) + B(A + B)^{[\dagger]}B$
$B - (B : A) = B(A + B)^{[\dagger]}B$
Pre multiplying by K on both sides, we get $KB - K(B : A) = KB(A + B)^{[\dagger]}B$
$B^{[*]} - K(B : A) = B^{[*]}(A + B)^{[\dagger]}B$ (By Theorem 2.13(vi))
$= B^{[*]}(A + B)^{[\dagger]}B \underset{\mathscr{I}}{\geq} 0$ (By Theorem 2.15)
$B^{[*]} - K(B : A) \underset{\mathscr{I}}{\geq} 0$
$B^{[*]} \underset{\mathscr{I}}{\geq} K(B : A) = K(A : B)$ (By Property 2.20 (i))
Taking adjoint on both sides, we get $(B^{[*]})^{[*]} \underset{\mathscr{I}}{\geq} [K(A : B)]^{[*]}$

$$B \underset{\mathscr{I}}{\geq} HA : HB \text{ for some } H \in M_n(\mathbb{C}) \tag{3}$$

(By Lemma 3.1)

Now adding (2) and (3) we get $A + B \underset{\mathscr{I}}{\geq} KA : KB + HA : HB$
$\underset{\mathscr{I}}{\geq} K(A : B) + H(A : B)$ (By (1))
$= (H + K)(A : B)$
$A + B \underset{\mathscr{I}}{\geq} T(A : B)$, where $T = K + H \in M_n(\mathbb{C})$
Thus (ii) is proved. Hence the Theorem. □

Remark 3.3. *In particular when A and B are J-symmetric semi-definite, then the condition in the Theorem 3.2, automatically holds, and $H = K = I$. Then Theorem 3.2 reduces to $A \underset{\mathscr{I}}{\geq} A : B$, $B \underset{\mathscr{I}}{\geq} A : B$ and $A + B \underset{\mathscr{I}}{\geq} 2(A : B)$. We will show in a subsequent result that actually $A + B \underset{\mathscr{I}}{\geq} 4(A : B)$.*

For a range symmetric matrix $A \in M_n(\mathbb{C})$ in $\mathscr{I}$, by Theorem 2.13(ii), JA is EP implies $JA(JA)^{\dagger} = (JA)^{\dagger}JA$
$JAA^{\dagger}J = A^{\dagger}A$

$$JAA^{\dagger}J = A^{\dagger}A. \tag{4}$$

Let $P_A = AA^{\dagger}$ and $P_{A^*} = A^{\dagger}A$ be the orthogonal projectors onto $R(A)$ and $R(A^*)$ in unitary space.

Theorem 3.4. *Let A and B be parallel summable range symmetric matrices in $\mathscr{I}$ such that $A \underset{\mathscr{I}}{\geq} -B$. For $x, y, z \in \mathbb{C}^n$ if $z = x + y$ then there exist non-singular matrices H and K such that*
$$(A \circ x, x) + (B \circ y, y) \underset{\mathscr{I}}{\geq} (K(A : B) \circ x, z) + (H(A : B) \circ y, z).$$

Proof. For any $z = x + y$, let us define $x' = P_{J(A+B)}x$, $y' = P_{J(A+B)}y$ and $z' = P_{J(A+B)}z$. Since A and B are parallel summable range symmetric in $\mathscr{I}$. By Theorem 2.19 $(A + B)$ is range symmetric in $\mathscr{I}$. Again by Theorem 2.13(ii) $J(A + B)$ is EP. By using (4),

$$J(A + B)(A + B)^{\dagger}J = (A + B)^{\dagger}(A + B). \tag{5}$$

The given conditions, $A \underset{\mathscr{I}}{\geq} -B$ implies $A + B \underset{\mathscr{I}}{\geq} 0$ and hence by Theorem 2.15, $(A + B)^{[\dagger]} \underset{\mathscr{I}}{\geq} 0$.
Let $x_o' = (A + B)^{[\dagger]} \circ B \circ z$, $y_o' = (A + B)^{[\dagger]} \circ A \circ z$

$$\begin{aligned}
x_o' + y_o' &= (A + B)^{[\dagger]} \circ (A + B) \circ z \\
&= J(A + B)^{\dagger}JJ(A + B)Jz && (\text{By (5)}) \\
&= J(A + B)^{\dagger}(A + B)Jz && (\text{since } J^2 = I) \\
&= P_{J(A+B)}z \\
&= P_{J(A+B)}(x + y) \\
&= P_{J(A+B)}x + P_{J(A+B)}y \\
&= x' + y'.
\end{aligned}$$

Thus $x_o' + y_o' = x' + y' = z'$.
Let $x' = x_o' + t$ and $y' = y_o' - t$, where t is suitably so chosen that $x_o' + y_o' = x' + y'$, $A \circ t = A \circ x' - A \circ x_o'$ and $B \circ t = B \circ y_o' - B \circ y'$.
From the Definition of z', $A \circ z' = A \circ P_{J(A+B)}z = A \circ J(A + B)^{\dagger}(A + B)Jz$
$= A(A + B)^{\dagger}(A + B) \circ z = A \circ z.$

$$\text{Hence } A \circ z' = A \circ z. \text{ Thus } (A : B) \circ z' = (A : B) \circ z. \tag{6}$$

$$\begin{aligned}
\text{Now } (A \circ x', x') &= (A \circ x_o' + A \circ t, x_o' + t) \\
&= (A \circ x_o', x_o') + (A \circ x_o', t) + (A \circ t, x_o') + (A \circ t, t) \\
&= (A \circ x_o', x_o') + (A \circ x_o', t) + (A \circ x' - A \circ x_o', x_o') + (A \circ t, t) \\
&= (A \circ x_o', x_o') + (A \circ x_o', t) + (A \circ x', x_o') - (A \circ x_o', x_o') + (A \circ t, t)
\end{aligned}$$

$$(A \circ x', x') = (A \circ x_o', t) + (A \circ x', x_o') + (A \circ t, t). \tag{7}$$

$$\text{Similarly } (B \circ y', y') = -(B \circ y_o', t) + (B \circ y', y_o') + (B \circ t, t). \tag{8}$$

By Definition of x_o', y_o'
$A \circ x_o' = A \circ (A + B)^{[\dagger]} \circ B \circ z = (A : B) \circ z = (A : B) \circ z'$

$$B \circ y_o' = B \circ (A + B)^{[\dagger]} \circ B \circ z = (B : A) \circ z = (A : B) \circ z = (A : B) \circ z' \tag{9}$$

Now adding (7) and (8) and using (9) we get

$$\begin{aligned}
(A \circ x', x') + (B \circ y', y') &= (A \circ x_o', t) + (A \circ x', x_o') + (A \circ t, t) \\
&\quad - (B \circ y_o', t) + (B \circ y', y_o') + (B \circ t, t)
\end{aligned}$$

$$(A \circ x', x') + (B \circ y', y') = (A \circ x', x_o') + (B \circ y', y_o') + ((A + B) \circ t, t)$$

By hypothesis, $A \underset{\mathscr{I}}{\geq} -B \;\Rightarrow\; A + B \underset{\mathscr{I}}{\geq} 0$
$\Rightarrow ((A + B) \circ t, t) \underset{\mathscr{I}}{\geq} 0$

$$\text{Hence } (A \circ x', x') + (B \circ y', y') \underset{\mathscr{I}}{\geq} (A \circ x', x'_o) + (B \circ y', y'_o) \tag{10}$$

From the Definition of x' and y' we get

$$A \circ x' = A \circ P_{J(A+B)}x = A \circ x \tag{11}$$

$$\text{Similarly, } B \circ y' = B \circ y. \text{Thus } A \circ x' = A \circ x \text{ and } B \circ y' = B \circ y \tag{12}$$

$$\begin{aligned}
(A \circ x', x') + (B \circ y', y') &\underset{\mathscr{I}}{\geq} (A \circ x, x'_o) + (B \circ y, y'_o) \\
&= (x, A^{[*]} \circ x'_o) + (y, B^{[*]} \circ y'_o) \\
&= (x, HA \circ x'_o) + (y, KB \circ y'_o) \quad (\text{ By Theorem 2.13(vi) }) \\
&= (x, H(A : B) \circ z) + (y, K(A : B) \circ z) \quad (\text{ By (9) }) \\
&= ((H(A : B))^{[*]} \circ x, z) + ((K(A : B))^{[*]} \circ y, z) \\
&= ((KA : KB) \circ x, z) + ((HA : HB) \circ y, z) \\
&\qquad\qquad (\text{ By Lemma 3.1 })
\end{aligned}$$

$$(A \circ x', x') + (B \circ y', y') \underset{\mathscr{I}}{\geq} (K(A : B) \circ x, z) + (H(A : B) \circ y, z) \tag{13}$$

$$\begin{aligned}
(A \circ x', x') &= \left(A \circ P_{J(A+B)}(x), P_{J(A+B)}(x)\right) \\
&= \left(A \circ x, P_{J(A+B)}(x)\right) && (\text{ By (11) }) \\
&= (x, A^{[*]} \circ P_{J(A+B)}(x)) \\
&= \left(x, HA \circ P_{J(A+B)}(x)\right) && (\text{ By Theorem 2.13 (vi) }) \\
&= (x, HA \circ x) && (\text{ By (11) }) \\
&= (x, A^{[*]} \circ x) && (\text{ By Theorem 2.13(vi) }) \\
&= (A \circ x, x).
\end{aligned}$$

Similarly, $(B \circ y', y') = (B \circ y, y)$ can be proved. Thus (13) reduces to $(A \circ x, x) + (B \circ y, y) \underset{\mathscr{I}}{\geq} (H(A : B) \circ y, z) + (K(A : B) \circ x, z)$. Hence the Theorem. □

Remark 3.5. *In particular $x = 0$ implies $B \underset{\mathscr{I}}{\geq} H(A : B)$, $y = 0$ implies $A \underset{\mathscr{I}}{\geq} K(A : B)$ and $x = y \Rightarrow A + B \underset{\mathscr{I}}{\geq} 2T(A : B)$, where $T = H + K$. This is a stronger result than $A + B \underset{\mathscr{I}}{\geq} T(A + B)$. Thus in Remark 3.3, $A + B \underset{\mathscr{I}}{\geq} 4(A : B)$.*

Corollary 3.6. *Let A and B be parallel summable J-symmetric matrices such that $A \underset{\mathscr{I}}{\geq} -B$. For any $x, y, z \in \mathbb{C}^n$, if $z = x + y$ then $(A \circ x, x) + (B \circ y, y) \underset{\mathscr{I}}{\geq} ((A : B) \circ z, z)$.*

Proof. Since A, B are J-symmetric this follows from Theorem 3.4, by putting $H = K = I$. □

Remark 3.7. *In the special case, when A and B are J-symmetric semi-definite, the condition $A \underset{\mathscr{I}}{\geq} -B$ in Corollary 3.6 automatically hold, then for any $x, y, z \in \mathbb{C}^n$ if $z = x + y$, $(A \circ x, x) + (B \circ y, y) \underset{\mathscr{I}}{\geq} ((A : B) \circ z, z)$.*

Theorem 3.8. *Let A and B be parallel summable range symmetric matrices. If $z \in Ra(A) + Ra(B)$ with $x = (A + B)^{[\dagger]} \circ B \circ z$ and $y = (A + B)^{[\dagger]} \circ A \circ z$, then $z = x + y$ and $(A \circ x, x) + (B \circ y, y) = ((A : B) \circ z, z)$.*

Proof. Since A and B are parallel summable by Definition 2.18, $Ra(A) + Ra(B) \subseteq Ra(A + B)$. For any A and B, $Ra(A+B) \subseteq Ra(A)+Ra(B)$. Therefore $Ra(A+B) = Ra(A)+Ra(B)$. Since $z \in Ra(A)+Ra(B) = Ra(A+B)$ implies, $z = (A + B) \circ (A + B)^{[\dagger]} \circ z$. By Theorem 2.19, A and B are parallel summable range symmetric implies $A + B$ is range symmetric in $\mathscr{I}$. Hence by Theorem 2.13(ii) and by using (5), we get $J(A + B)$ is range symmetric in $\mathscr{I}$
$x + y = (A + B)^{[\dagger]} \circ (A + B) \circ z = J(A + B)(A + B)^{\dagger} Jz$
$= P_{J(A+B)}z = P_{A+B} \circ z = (A + B) \circ (A + B)^{[\dagger]} \circ z$
(By $A + B$ is range symmetric in $\mathscr{I}$)
$A \circ x = A \circ (A + B)^{[\dagger]} \circ B \circ z = (A : B) \circ z$
$B \circ y = B \circ (A + B)^{[\dagger]} \circ A \circ z = (B : A) \circ z = (A : B) \circ z$ (By Property 2.20 (i))
Thus $A \circ x = B \circ y = (A : B) \circ z$ and
$(A \circ x, x) + (B \circ y, y) = ((A : B) \circ z, x) + ((A : B) \circ z, y)$
$= ((A : B) \circ z, x + y)$
$(A \circ x, x) + (B \circ y, y) = ((A : B) \circ z, z)$ (By $z = x + y$)
Hence the Theorem. □

Theorem 3.9. *Let A and B be parallel summable J-symmetric matrices such that $A \underset{\mathscr{I}}{\geq} -B$. Then $(A \circ z, z) : (B \circ z, z) \underset{\mathscr{I}}{\geq} ((A : B) \circ z, z)$.*

Proof. If $((A : B) \circ z, z) = 0$, then the result is trivial. If not let us define
$x = \frac{(B \circ z,z) \circ z}{((A+B) \circ z,z)}$, $y = \frac{(A \circ z,z) \circ z}{((A+B) \circ z,z)}$ then $x + y = z$.

$$\begin{aligned}
(A \circ x, x) + (B \circ y, y) &= \left(\frac{A \circ (B \circ z, z) \circ z}{((A + B) \circ z, z)}, \frac{(B \circ z, z) \circ z}{((A + B) \circ z, z)}\right) \\
&\quad + \left(\frac{B \circ (A \circ z, z) \circ z}{((A + B) \circ z, z)}, \frac{(A \circ z, z) \circ z}{((A + B) \circ z, z)}\right) \\
&= \frac{(B \circ z, z) \circ \overline{(B \circ z, z)} \circ (A \circ z, z)}{((A + B) \circ z, z)} \\
&\quad + \frac{(A \circ z, z) \circ \overline{(A \circ z, z)} \circ (B \circ z, z)}{((A + B) \circ z, z)} \\
&= \frac{(A \circ z, z) \circ (B \circ z, z)}{((A + B) \circ z, z)^2} \circ \left[\overline{(\mathrm{A} \circ \mathrm{z}, \mathrm{z})} + \overline{(\mathrm{B} \circ \mathrm{z}, \mathrm{z})}\right] \\
&= \frac{(A \circ z, z) \circ (B \circ z, z)}{((A + B) \circ z, z)} \circ \frac{\overline{((A + B) \circ z, z)}}{((A + B) \circ z, z)} \\
&= \frac{(A \circ z, z) \circ (B \circ z, z)}{((A + B) \circ z, z)} \\
&= (A \circ z, z) \circ ((A + B) \circ z, z)^{[\dagger]} \circ (B \circ z, z) \\
(A \circ x, x) + (B \circ y, y) &= (A \circ z, z) : (B \circ z, z).
\end{aligned}$$

Since A and B are parallel summable J-symmetric and by using Corollary 3.6, we get, $(A \circ x, x) + (B \circ y, y) = (A \circ z, z) : (B \circ z, z) \underset{\mathscr{I}}{\geq} ((A : B) \circ z, z)$. Hence the Theorem. □

Corollary 3.10. *Let A, B, C, D be parallel summable J-symmetric matrices such that $A \underset{\mathscr{I}}{\geq} -C$ and $B \underset{\mathscr{I}}{\geq} -D$ then $(A + B) : (C + D) \underset{\mathscr{I}}{\geq} (A : C) + (B : D)$.*

Proof. Since A, B, C, D are parallel summable J-symmetric in $\mathscr{I}$, then by Theorem 2.19, $A + B + C + D$ is range symmetric in $\mathscr{I}$ and it suffices to consider $z \in Ra(A + B) + Ra(C + D)$. Since $(A + B)$ and $(C + D)$ are parallel summable J-symmetric by Theorem 3.8, for suitable choice of x_o, y_o, such that $z = x_o + y_o$,
$((A + B) : (C + D) \circ z, z) = ((A + B) \circ x_o) + ((C + D) \circ y_o, y_o)$
$= (A \circ x_o, x_o) + (C \circ y_o, y_o) + (B \circ x_o, x_o) + (D \circ y_o, y_o)$.

Since A and C are parallel summable J-symmetric such that $A \underset{\mathscr{J}}{\geq} -C$ and $z = x_o + y_o$, by Corollary 3.6, we get $(A \circ x_o, x_o) + (C \circ y_o, y_o) \underset{\mathscr{J}}{\geq} ((A : C)z, z)$.

Similarly using B and D are parallel summable J-symmetric such that $B \underset{\mathscr{J}}{\geq} -D$ by Corollary 3.6, we get $(B \circ x_o, x_o) + (D \circ y_o, y_o) \underset{\mathscr{J}}{\geq} ((B : D) \circ z, z)$.

Thus $((A + B) : (C + D) \circ z, z) \underset{\mathscr{J}}{\geq} (((A : C) + (B : D)) \circ z, z)$

$(A + B) : (C + D) \underset{\mathscr{J}}{\geq} (A : C) + (B : D)$

Hence the Theorem. □

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