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Some Fixed Point Theorems for $\mathcal{T}_F$-type Contraction Under Implicit Relation in Partial Metric Spaces

G. S. Saluja[a]

[a]*Department of Mathematics, Govt. K. P. G. College Jagdalpur, Jagdalpur - 494001 (C.G.), India.*

Abstract. This paper aims to study $\mathcal{T}_F$-type contraction in partial metric spaces and establish some fixed point theorems using implicit relation for the said space. The results presented in this paper extend and generalize the corresponding results of Kir and Kiziltunc [11] to the setting of ψ-implicit contraction. The results also extend and generalize several results from the existing literature.

1. Introduction and Preliminaries

The Banach contraction mappings principle is the opening and vital result in the direction of fixed point theory. Subsequently, several authors have devoted their concentration to expanding and improving this theory (see, e.g., [6, 7, 17, 18] and many others).

Matthews ([13, 14]) launched the notion of partial metric space and proved equivalent result of Banach's theorem in such spaces. A partial metric is an extension of metric by replacing the condition $d(x,x) = 0$ of the (usual) metric with the inequality $d(x,x) \leq d(x,y)$ for all x, y. Afterwards, a multitude of results was obtained in these spaces (see, e.g., [2, 3, 9, 10, 17, 20]). Also, the concept of PMS provides to study denotational semantics of dataflow networks [13, 14, 19, 21].

Matthews [13] introduced the notion of partial metric spaces as follows:

Definition 1.1. *([13]) Let X be a nonempty set and let $p\colon X \times X \to \mathbb{R}^+$ be a function satisfy*
$(P1)$ $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$,
$(P2)$ $p(x,x) \leq p(x,y)$,
$(P3)$ $p(x,y) = p(y,x)$,
$(P4)$ $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$,
for all $x, y, z \in X$. Then p is called partial metric on X and the pair (X, p) is called partial metric space.

It is clear that if $p(x,y) = 0$, then from $(P1)$ and $(P2)$ we obtain $x = y$. But if $x = y$, $p(x,y)$ may not be zero. Various applications of this space has been extensively investigated by many authors (see [12], [20] for details).

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Email address: saluja1963@gmail.com (G. S. Saluja)

Example 1.2. *([4]) Let $X = \mathbb{R}^+$ and $p\colon X \times X \to \mathbb{R}^+$ given by $p(x, y) = \max\{x, y\}$ for all $x, y \in \mathbb{R}^+$. Then $(\mathbb{R}^+, p)$ is a partial metric space.*

Example 1.3. *([4]) Let $X = \{[a, b] : a, b \in \mathbb{R}, a \leq b\}$. Then $p\big([a, b], [c, d]\big) = \max\{b, d\} - \min\{a, c\}$ defines a partial metric p on X.*

Remark 1.4. *([8]) Let (X, p) be a partial metric space.*

$(A1)$ *The function $d_M\colon X \times X \to \mathbb{R}^+$ defined as $d_M(x, y) = 2p(x, y) - p(x, x) - p(y, y)$ is a (usual) metric on X and (X, d_M) is a (usual) metric space.*

$(A2)$ *The function $d_S\colon X \times X \to \mathbb{R}^+$ defined as $d_S(x, y) = \max\{p(x, y) - p(x, x), p(x, y) - p(y, y)\}$ is a (usual) metric on X and (X, d_S) is a (usual) metric space.*

Note also that each partial metric p on X generates a T_0 topology τ_p on X, whose base is a family of open p-balls $\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\}$ where $B_p(x, \varepsilon) = \{y \in X : p(x, y) \leq p(x, x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

On a partial metric space the notions of convergence, the Cauchy sequence, completeness and continuity are defined as follows [13].

Definition 1.5. *([13]) Let (X, p) be a partial metric space. Then*

$(B1)$ *a sequence $\{x_n\}$ in (X, p) is said to be convergent to a point $x \in X$ if and only if $p(x, x) = \lim_{n\to\infty} p(x_n, x)$,*

$(B2)$ *a sequence $\{x_n\}$ is called a Cauchy sequence if $\lim_{m,n\to\infty} p(x_m, x_n)$ exists and finite,*

$(B3)$ *(X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges to a point $x \in X$ with respect to τ_p. Furthermore,*

$$\lim_{m,n\to\infty} p(x_m, x_n) = \lim_{n\to\infty} p(x_n, x) = p(x, x).$$

$(B4)$ *A mapping $F\colon X \to X$ is said to be continuous at $x_0 \in X$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $F\big(B_p(x_0), \delta)\big) \subset B_p\big(F(x_0), \varepsilon\big)$.*

Definition 1.6. *([16]) Let (X, p) be a partial metric space. Then*

$(C1)$ *a sequence $\{x_n\}$ in (X, p) is called 0-Cauchy if $\lim_{m,n\to\infty} p(x_m, x_n) = 0$,*

$(C2)$ *(X, p) is said to be 0-complete if every 0-Cauchy sequence $\{x_n\}$ in X converges to a point $x \in X$, such that $p(x, x) = 0$.*

Moradi and Beiranvand [15] introduced the following notion.

Definition 1.7. *([15]) Let (X, d) be a metric space and $f, \mathcal{T}\colon X \to X$ be two mappings. The mapping f is said to be a $\mathcal{T}_F$-contraction if there exists $a \in [0, 1)$ such that for all $x, y \in X$*

$$F\big(d(\mathcal{T}fx, \mathcal{T}fy)\big) \leq a\, F\big(d(\mathcal{T}x, \mathcal{T}y)\big)$$

where

1) *$F\colon [0, \infty) \to [0, \infty)$, F is nondecreasing continuous from the right and $F^{-1}(0) = \{0\}$.*
2) *$\mathcal{T}$ is one to one and graph closed.*

Now, we define the following.

Definition 1.8. *Let (X, p) be a partial metric space and $\mathcal{T}: X \to X$ be a mapping. The mapping $\mathcal{T}$ is said to be a $\mathcal{T}_F$-type contraction if there exists $a \in [0, 1)$ such that for all $x, y \in X$ and*

$$F\Big(p(\mathcal{T}x, \mathcal{T}y)\Big) \leq a\, F\Big(p(x, y)\Big) \tag{1}$$

where $F\colon [0, \infty) \to [0, \infty)$ is a nondecreasing continuous function from the right and $F^{-1}(0) = \{0\}$.

Remark 1.9. *If we take $F(t) = t$ in equation (1), then we obtain Banach contraction condition in partial metric space (X, p) and if X is complete then T has a unique fixed point.*

Now, an implicit relation has been introduced to investigate some fixed point theorems in partial metric spaces.

Definition 1.10. *(***Implicit Relation***) Let Ψ be the family of all real valued continuous functions $\psi\colon \mathbb{R}_+^3 \to \mathbb{R}_+$, for three variables. For some $h \in [0, 1)$, we consider the following conditions.*
$(r1)$ For $x, y \in \mathbb{R}_+$, if $y \leq \psi(x, x, y)$, then $y \leq hx$.
$(r2)$ For $y \in \mathbb{R}_+$, if $y \leq \psi(0, 0, y)$, then $y = 0$.
$(r3)$ For $y \in \mathbb{R}_+$, if $y \leq \psi(y, 0, 0)$, then $y = 0$, since $h \in [0, 1)$.

Lemma 1.11. *([13, 14]) Let (X, p) be a partial metric space. Then*

(D1) a sequence $\{x_n\}$ in (X, p) is a Cauchy sequence if and only if it is a Cauchy sequence in the metric space (X, d_M),

(D2) (X, p) is complete if and only if the metric space (X, d_M) is complete,

(D3) a subset E of a partial metric space (X, p) is closed if a sequence $\{x_n\}$ in E such that $\{x_n\}$ converges to some $x \in X$, then $x \in E$.

Lemma 1.12. *([1]) Assume that $x_n \to z$ as $n \to \infty$ in a partial metric space (X, p) such that $p(z, z) = 0$. Then $\lim_{n\to\infty} p(x_n, y) = p(z, y)$ for every $y \in X$.*

The purpose of this paper is to study $\mathcal{T}_F$-type contraction in partial metric space and establish some fixed point theorems using implicit relation. The results of findings extend and generalize several results from the existing literature.

2. Main Results

In this section, some fixed point theorems shall be proved for $\mathcal{T}_F$-type contraction using implicit relation in the framework of partial metric spaces.

Theorem 2.1. *Let (X, p) be a complete partial metric space and $\mathcal{T}: X \to X$ be a mapping satisfying the inequality*

$$F\Big(p(\mathcal{T}x, \mathcal{T}y)\Big) \leq \psi\Big\{F\Big(p(x, y)\Big), F\Big(p(x, \mathcal{T}x)\Big), F\Big(p(y, \mathcal{T}y)\Big)\Big\} \tag{2}$$

for all $x, y \in X$, where $F\colon [0, \infty) \to [0, \infty)$ is nondecreasing continuous function and $F(t) = 0$ if and only if $t = 0$ and some $\psi \in \Psi$. Then we have the following.
(1) If ψ satisfies the conditions $(r1)$ and $(r2)$, then $\mathcal{T}$ has a fixed point.
(2) If ψ satisfies the condition $(r3)$ and $\mathcal{T}$ has a fixed point, then the fixed point is unique.

Proof. (1) Let $x_0 \in X$ be an arbitrary point and $x_n = \mathcal{T}x_{n-1} = \mathcal{T}^n x_0$, $n = 1, 2, \ldots$. It follows from (2) that

$$\begin{aligned} F\Big(p(x_n, x_{n+1})\Big) &= F\Big(p(\mathcal{T}x_{n-1}, \mathcal{T}x_n)\Big) \\ &\leq \psi\Big\{F\Big(p(x_{n-1}, x_n)\Big), F\Big(p(x_{n-1}, \mathcal{T}x_{n-1})\Big), F\Big(p(x_n, \mathcal{T}x_n)\Big)\Big\} \\ &= \psi\Big\{F\Big(p(x_{n-1}, x_n)\Big), F\Big(p(x_{n-1}, x_n)\Big), F\Big(p(x_n, x_{n+1})\Big)\Big\}. \end{aligned} \tag{3}$$

Since ψ satisfies the condition $(r1)$, there exists $h \in [0, 1)$ such that

$$F\Big(p(x_n, x_{n+1})\Big) \leq h\, F\Big(p(x_{n-1}, x_n)\Big) \leq h^n\, F\Big(p(x_0, x_1)\Big). \tag{4}$$

Set $V_n = F\Big(p(x_n, x_{n+1})\Big)$ and $V_{n-1} = F\Big(p(x_{n-1}, x_n)\Big)$, then from (4), we obtain

$$V_n \leq h\, V_{n-1} \leq h^2\, V_{n-2} \leq \cdots \leq h^n\, V_0.$$

Also for $m, n \in \mathbb{N}$ with $m > n$, then by using $(P4)$ and equation (4), we have

$$\begin{aligned}
F\Big(p(x_n, x_m)\Big) &\leq F\Big(p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + \dots \\
&\quad + p(x_{n+m-1}, x_m) - p(x_{n+1}, x_{n+1}) \\
&\quad - p(x_{n+2}, x_{n+2}) - \cdots - p(x_{n+m-1}, x_{n+m-1})\Big) \\
&\leq F\Big(h^n p(x_0, x_1) + h^{n+1} p(x_0, x_1) + \dots \\
&\quad + h^{n+m-1} p(x_0, x_1)\Big) \\
&= F\Big[h^n\Big(p(x_0, x_1) + hp(x_0, x_1) + \dots \\
&\quad + h^{m-1} p(x_0, x_1)\Big)\Big] \\
&= F\Big[h^n(1 + h + \cdots + h^{m-1})V_0\Big] \\
&\leq F\Big[h^n\Big(\frac{1 - h^{m-1}}{1 - h}\Big)V_0\Big].
\end{aligned}$$

Taking $n, m \to \infty$ in the above inequality and using the property of F, we get $F\Big(p(x_n, x_m)\Big) \to 0^+$ as $n, m \to \infty$, since $0 < h < 1$. As F is continuous, we obtain

$$\lim_{m,n\to\infty} p(x_n, x_m) = 0. \tag{5}$$

Hence $\{x_n\}$ is a Cauchy sequence in X. Thus by Lemma 1.11 this sequence will also Cauchy in (X, d_M). In addition, since (X, p) is a complete metric space, then (X, d_M) is also complete metric space. Thus there exists $z \in X$ such that $\{x_n\}$ converges to $z \in X$. Moreover by Lemma 1.12,

$$p(z, z) = \lim_{n\to\infty} p(z, x_n) = \lim_{n,m\to\infty} p(x_n, x_m) = 0, \tag{6}$$

implies

$$\lim_{n\to\infty} d_M(z, x_n) = 0. \tag{7}$$

Now, we will show that $z \in X$ is a fixed point of $\mathcal{T}$. Notice that due to (6), we have $p(z, z) = 0$. By using inequality (2), we get

$$\begin{aligned}
F\Big(p(x_{n+1}, \mathcal{T}z)\Big) &= F\Big(p(\mathcal{T}x_n, \mathcal{T}z)\Big) \\
&\leq \psi\Big\{F\Big(p(x_n, z)\Big), F\Big(p(x_n, \mathcal{T}x_n)\Big), F\Big(p(z, \mathcal{T}z)\Big)\Big\} \\
&= \psi\Big\{F\Big(p(x_n, z)\Big), F\Big(p(x_n, x_{n+1})\Big), F\Big(p(z, \mathcal{T}z)\Big)\Big\}.
\end{aligned}$$

Indeed, as F is continuous and note that $\psi \in \Psi$, then using (7), Lemma 1.12, property of F and taking the limit as $n \to \infty$, we get

$$F\Big(p(z, \mathcal{T}z)\Big) \leq \psi\Big\{0, 0, F\Big(p(z, \mathcal{T}z)\Big)\Big\}.$$

Since ψ satisfies the condition ($r2$), then $F\big(p(z, \mathcal{T}z)\big) \leq h.0 = 0$. This implies that $p(z, \mathcal{T}z) = 0$. Thus, $z = \mathcal{T}z$. Hence z is a fixed point of $\mathcal{T}$.

(2) Let x_1, x_2 be fixed points of $\mathcal{T}$ with $x_1 \neq x_2$. We shall prove that $x_1 = x_2$. It follows from equation (2), (6) and property of F that

$$\begin{aligned} F\big(p(x_1, x_2)\big) &= F\big(p(\mathcal{T}x_1, \mathcal{T}x_2)\big) \\ &\leq \psi\Big\{F\big(p(x_1, x_2)\big), F\big(p(x_1, \mathcal{T}x_1)\big), F\big(p(x_2, \mathcal{T}x_2)\big)\Big\} \\ &= \psi\Big\{F\big(p(x_1, x_2)\big), F\big(p(x_1, x_1)\big), F\big(p(x_2, x_2)\big)\Big\} \\ &= \psi\Big\{F\big(p(x_1, x_2)\big), 0, 0\Big\}. \end{aligned}$$

Since ψ satisfies the condition ($r3$), then we get

$$\begin{aligned} F\big(p(x_1, x_2)\big) &\leq h\,F\big(p(x_1, x_2)\big) \\ &\Rightarrow F\big(p(x_1, x_2)\big) = 0, \text{ since } 0 < h < 1. \end{aligned}$$

Indeed, as F is continuous, this implies that $p(x_1, x_2) = 0$. Hence, $x_1 = x_2$. Thus, the fixed point of $\mathcal{T}$ is unique. This completes the proof. □

Next, we give an analogues of fixed point theorems by combining Theorem 2.1 with $\psi \in \Psi$ and ψ satisfies conditions ($r1$), ($r2$) and ($r3$). The following results are due to Kir and Kiziltunc [11].

Corollary 2.2. *([11], Corollary 4) Let (X, p) be a complete partial metric space and $\mathcal{T}\colon X \to X$ be a mapping. If for each $\alpha \in [0, \frac{1}{2})$ and $x, y \in X$*

$$F\big(p(\mathcal{T}x, \mathcal{T}y)\big) \leq \alpha\Big[F\big(p(x, \mathcal{T}x)\big) + F\big(p(y, \mathcal{T}y)\big)\Big], \tag{8}$$

where $F\colon [0, \infty) \to [0, \infty)$ is a nondecreasing continuous function and $F(t) = 0$ if and only if $t = 0$. Then $\mathcal{T}$ has a unique fixed point in X.

Proof. The assertion follows using Theorem 2.1 with $\psi(a_1, a_2, a_3) = \alpha(a_2 + a_3)$ for some $\alpha \in [0, \frac{1}{2})$ and all $a_1, a_2, a_3 \in \mathbb{R}_+$. Indeed, ψ is continuous. First, we have $\psi(x, x, y) = \alpha(x + y)$. So, if $y \leq \psi(x, x, y)$, then $y \leq \big(\frac{\alpha}{1-\alpha}\big)x$ with $\big(\frac{\alpha}{1-\alpha}\big) < 1$. Thus, $\mathcal{T}$ satisfies the condition ($r1$).

Next, if $y \leq \psi(0, 0, y) = \alpha(0 + y) = \alpha\, y$, then $y = 0$, since $\alpha < \frac{1}{2} < 1$. Thus, $\mathcal{T}$ satisfies the condition ($r2$).

Finally, if $y \leq \psi(y, 0, 0) = \alpha.0 = 0$, then $y = 0$. Thus, $\mathcal{T}$ satisfies the condition ($r3$). □

If we take $F(t) = t$ in Corollary 2.2, then we have the following result.

Corollary 2.3. *([11], Corollary 5) Let (X, p) be a complete partial metric space and $\mathcal{T}\colon X \to X$ be a mapping. If for each $\alpha \in [0, \frac{1}{2})$ and $x, y \in X$*

$$p(\mathcal{T}x, \mathcal{T}y) \leq \alpha\Big[p(x, \mathcal{T}x) + p(y, \mathcal{T}y)\Big]. \tag{9}$$

Then $\mathcal{T}$ has a unique fixed point in X.

Remark 2.4. *Corollary 2.3 extends the well-known Kannan's contraction [7] from complete metric space to the setting of complete partial metric space.*

The following corollary is an analogue of S. Reich's type result [18].

Corollary 2.5. *Let (X, p) be a complete partial metric space and $\mathcal{T}\colon X \to X$ be a mapping. If for each $x, y \in X$ and*

$$F\Big(p(\mathcal{T}x, \mathcal{T}y)\Big) \leq q_1 F\Big(p(x, y)\Big) + q_2 F\Big(p(x, \mathcal{T}x)\Big) + q_3 F\Big(p(y, \mathcal{T}y)\Big), \tag{10}$$

where $q_1, q_2, q_3 \geq 0$ are constants with $q_1 + q_2 + q_3 < 1$, $F\colon [0, \infty) \to [0, \infty)$ is a nondecreasing continuous function and $F(t) = 0$ if and only if $t = 0$. Then $\mathcal{T}$ has a unique fixed point in X.

Proof. The assertion follows using Theorem 2.1 with $\psi(a_1, a_2, a_3) = q_1a_1 + q_2a_2 + q_3a_3$ for some $q_1, q_2, q_3 \geq 0$ are constants with $q_1 + q_2 + q_3 < 1$ and all $a_1, a_2, a_3 \in \mathbb{R}_+$. Indeed, ψ is continuous. First, we have $\psi(x, x, y) = q_1x + q_2x + q_3y$. So, if $y \leq \psi(x, x, y)$, then $y \leq \left(\frac{q_1+q_2}{1-q_3}\right)x$ with $\left(\frac{q_1+q_2}{1-q_3}\right) < 1$. Thus, $\mathcal{T}$ satisfies the condition $(r1)$.

Next, if $y \leq \psi(0, 0, y) = q_1.0 + q_2.0 + q_3.y = q_3y$, then $y = 0$ since $q_3 < 1$. Thus, $\mathcal{T}$ satisfies the condition $(r2)$.

Finally, if $y \leq \psi(y, 0, 0) = q_1.y + q_2.0 + q_3.0 = q_1y$, then $y = 0$ since $q_1 < 1$. Thus, $\mathcal{T}$ satisfies the condition $(r3)$. □

If we take $F(t) = t$ in Corollary 2.5, then we have the following result.

Corollary 2.6. *Let (X, p) be a complete partial metric space and $\mathcal{T}\colon X \to X$ be a mapping. If for each $x, y \in X$ and*

$$p(\mathcal{T}x, \mathcal{T}y) \leq q_1 p(x, y) + q_2 p(x, \mathcal{T}x) + q_3 p(y, \mathcal{T}y), \tag{11}$$

where $q_1, q_2, q_3 \geq 0$ are constants with $q_1 + q_2 + q_3 < 1$. Then $\mathcal{T}$ has a unique fixed point in X.

Remark 2.7. *Corollary 2.6 extends the well-known Reich's contraction [18] from complete metric space to the setting of complete partial metric space.*

In Theorem 2.1, if we take $F(t) = t$, then we obtain the following result.

Corollary 2.8. *Let (X, p) be a complete partial metric space and $\mathcal{T}\colon X \to X$ be a mapping satisfying the inequality*

$$p(\mathcal{T}x, \mathcal{T}y) \leq \psi\Big\{p(x, y), p(x, \mathcal{T}x), p(y, \mathcal{T}y)\Big\}, \tag{12}$$

for all $x, y \in X$ and some $\psi \in \Psi$. If ψ satisfies the conditions $(r1)$, $(r2)$ and $(r3)$, then $\mathcal{T}$ has a unique fixed point in X.

Again, we give analogue of fixed point theorem in metric spaces for partial metric spaces by combining Corollary 2.8 with $\psi \in \Psi$ and ψ satisfies conditions $(r1)$, $(r2)$ and $(r3)$. The following corollary is an analogue of Banach's type contraction principle.

Corollary 2.9. *([14], Theorem 5.3, [11], Corollary 2) Let (X, p) be a complete partial metric space. Suppose that the mapping $\mathcal{T}\colon X \to X$ satisfies the following condition:*

$$p(\mathcal{T}x, \mathcal{T}y) \leq r\, p(x, y), \tag{13}$$

for all $x, y \in X$, where $r \in [0, 1)$ is a constant. Then $\mathcal{T}$ has a unique fixed point in X. Moreover, $\mathcal{T}$ is continuous at the fixed point.

Proof. The assertion follows using Corollary 2.8 with $\psi(a_1, a_2, a_3) = ra_1$ for some $r \in [0, 1)$ and all $a_1, a_2, a_3 \in \mathbb{R}_+$. □

Remark 2.10. *Corollary 2.9 extends the well-known Banach fixed point theorem [5] from complete metric space to the setting of complete partial metric space.*

Example 2.11. *Let $X = [0,1]$. Define $p\colon X\times X \to \mathbb{R}^+$ as $p(x,y) = \max\{x,y\}$ with $\mathcal{T}\colon X \to X$ by $\mathcal{T}(x) = \frac{x}{4}$. Clearly (X,p) is a partial metric space. Now, let $x \le y$. Then choose $x = \frac{1}{2}$ and $y = 1$, we have $p(\mathcal{T}x,\mathcal{T}y) = \frac{y}{4}$, $p(x,y) = y$, $p(x,\mathcal{T}x) = x$, $p(y,\mathcal{T}y) = y$, $p(x,\mathcal{T}y) = x$, $p(y,\mathcal{T}x) = y$.*

(i) Now, consider inequality (13), we have

$$p(\mathcal{T}x,\mathcal{T}y) = \frac{y}{4} \le ry,$$

or $r \ge \frac{1}{4}$. If we take $0 < r < 1$, then $\mathcal{T}$ satisfies all the conditions of Corollary 2.9. Hence, applying Corollary 2.9, $\mathcal{T}$ has a unique fixed point. Here it is seen that $0 \in X$ is the unique fixed point of $\mathcal{T}$.

(ii) Now, consider inequality (9), we have

$$p(\mathcal{T}x,\mathcal{T}y) = \frac{y}{4} \le \alpha(x+y),$$

putting $x = \frac{1}{2}$ and $y = 1$ in the above inequality, we get

$$\frac{1}{4} \le \frac{3}{2}\alpha,$$

or $\alpha \ge \frac{1}{6}$. If we take $0 < \alpha < \frac{1}{2}$, then $\mathcal{T}$ satisfies all the conditions of Corollary 2.3. Hence, applying Corollary 2.3, $\mathcal{T}$ has a unique fixed point and the unique fixed point of $\mathcal{T}$ is $0 \in X$.

(iii) Now, consider inequality (11), we have

$$p(\mathcal{T}x,\mathcal{T}y) = \frac{y}{4} \le q_1 y + q_2 x + q_3 y,$$

putting $x = \frac{1}{2}$ and $y = 1$ in the above inequality, we get

$$\frac{1}{4} \le q_1 + \frac{1}{2}q_2 + q_3.$$

If we take (1) $q_1 = \frac{1}{3}$, $q_2 = \frac{1}{2}$ and $q_3 = 0$, (2) $q_1 = \frac{1}{2}$, $q_2 = 0$ and $q_3 = \frac{1}{3}$ and (3) $q_1 = 0$, $q_2 = \frac{1}{4}$ and $q_3 = \frac{1}{5}$, then $\mathcal{T}$ satisfies all the conditions of Corollary 2.6. Hence, applying Corollary 2.6, $\mathcal{T}$ has a unique fixed point and it is $0 \in X$.

(iv) Now, consider inequality (12), we have

$$p(\mathcal{T}x,\mathcal{T}y) = \frac{y}{4} \le \psi\Big\{y,x,y\Big\}.$$

Since ψ satisfies the condition $(r1)$, there exists $h \in [0,1)$ such that

$$\frac{y}{4} \le h\,y,$$

or

$$h \ge \frac{1}{4}.$$

If we take $0 < h < 1$, then $\mathcal{T}$ satisfies the condition $(r1)$. Similarly, we can show that $\mathcal{T}$ satisfies the conditions $(r2)$ and $(r3)$. Thus, $\mathcal{T}$ satisfies all the conditions of Corollary 2.8. Hence, applying Corollary 2.8, $\mathcal{T}$ has a unique fixed point. Here, note that $'0'$ is the unique fixed point of $\mathcal{T}$.

3. Conclusion

In this paper, we establish some fixed point theorems for $\mathcal{T}_F$-type contraction using implicit relation in the framework of complete partial metric spaces. Also, we obtain the results of Kir and Kiziltunc [11], the well-known Banach contraction principle, Kannan contraction and Reich contraction as corollaries to the results. The results also extend, unify and generalize several results from the existing literature to the setting of a more general class of metric spaces and contraction conditions.

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