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A note on recurrent strongly continuous semigroups of operators

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Abstract. The class of recurrent operators plays a significant role in the field of linear topological dynamics. In this note, we analyze recurrent strongly continuous semigroups of operators in Banach spaces. We establish several structural results and propose an open problem in this context.

1. Introduction and preliminaries

Suppose that $(X, \|\cdot\|)$ is a complex Banach space. Following G. Costakis, I. Parissis [7] and G. Costakis, A. Manoussos, I. Parissis [6], we say that a linear operator $T : X \to X$ is recurrent if and only if for every non-empty open subset U of X there exists $k \in \mathbb{N}$ such that $U \cap T^{-k}(U) \neq \emptyset$. A vector $x \in X$ is said to be recurrent for T if and only if there exists a strictly increasing sequence of positive integers $(k_n)_{n\in\mathbb{N}}$ such that $T^{k_n}x \to x$ as $n \to +\infty$; the set consisting of all reccurent vectors of T will be denoted by $Rec(T)$. A much stronger notion than the recurrence is the measure theoretic rigidity, introduced in the ergodic theoretic setting by H. Furstenberg and B. Weiss ([12]; see also the important research monograph [11] by H. Furstenberg). In the context of topological dynamical systems, this concept is known as the (uniform) rigidity, which was introduced by S. Glasner and D. Maon in [13]. We say that a bounded linear operator $T : X \to X$ is rigid if and only if there exists a strictly increasing sequence of positive integers $(k_n)_{n\in\mathbb{N}}$ such that $T^{k_n}x \to x$ as $n \to +\infty$, for every $x \in X$. A bounded linear operator $T : X \to X$ is called uniformly rigid if and only if there exists an increasing sequence of positive integers $(k_n)_{n\in\mathbb{N}}$ such that $\|T^{k_n} - I\| = \sup_{\|x\|\leq 1} \|T^{k_n}x - x\| \to 0$ as $n \to +\infty$. For more details about recurrent and rigid operators on Banach spaces, see the research articles [3] by J. Bés et al., [5] by C.-C. Chen, [10] by T. Eisner and S. Grivaux, [20] by Y. Puig de Dios, [21] by Z. Xin and [22] by Z. Yin, Y. Wei; for more details about topological dynamical systems, see the research monographs [8] by J. de Vries and [9] by T. Eisner et al.

On the other hand, a linear continuous operator T on X is said to be hypercyclic if and only if there exists an element $x \in X$ whose orbit $\{T^n x : n \in \mathbb{N}_0\}$ is dense in X; T is said to be topologically transitive,

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resp. topologically mixing, if and only if for every pair of open non-empty subsets $U,\ V$ of X, there exists $n_0 \in \mathbb{N}$ such that $T^{n_0}(U) \cap V \neq \emptyset$, resp. if for every pair of open non-empty subsets $U,\ V$ of X, there exists $n_0 \in \mathbb{N}$ such that, for every $n \in \mathbb{N}$ with $n \geq n_0$, $T^n(U) \cap V \neq \emptyset$. In recent years, considerable effort has been directed toward the hypercyclicity and mixing of various types of operators and operator semigroups. It is well known that any hypercyclic operator is recurrent as well as that the converse statement does not hold in general. For more details about hypercyclic operators and hypercyclic abstract partial differential equations, we refer the reader to the research monographs [2] by F. Bayart, É. Matheron, [14] by K.-G. Grosse-Erdmann, A. Peris and [17] by M. Kostić.

The organization of this note can be briefly described as follows. The notion of a recurrent set of operators has recently been introduced by M. Amouch and O. Benchiheb in [1]. Albeit can be clarified in this setting, Theorem 2.1 is given here with a complete proof, in which we use the strong continuity of the operator family under our consideration (see also Theorem 2.3 and Proposition 2.4, whose proofs are given for the sake of completeness and which can be extended in the general framework of [1]; Proposition 2.2 is new and not considered in [1]). The main result of this note is Theorem 2.5, in which we partially transfer the result of G. Costakis, A. Manoussos and I. Parissis [6, Proposition 2.3(ii)] for strongly continuous semigroups of operators. In the formulation of this theorem, we use the assumption that a strongly continuous semigroup $(T(t))_{t\geq 0}$ can be embedded into a strongly continuous group of operators $(T(t))_{t\in\mathbb{R}}$ and ask whether this assumption can be neglected.

We will use the following notion:

Definition 1.1. (see also [1]) Let $I = [0, \infty)$ or $I = \mathbb{R}$. We say that a family $(W(t))_{t\in I}$ of bounded linear operators on X is recurrent if and only if for every open non-empty set $U \subseteq X$ there exists some $t \in I$ such that $U \cap (W(t))^{-1}(U) \neq \emptyset$. A vector $x \in X$ is called a recurrent vector for $(W(t))_{t\in I}$ if and only if there exists an unbounded sequence of numbers (t_k) in I such that $W(t_k)x \to x$ as $k \to +\infty$. By $Rec(W(t))$ we denote the set consisting of all recurrent vectors for $(W(t))_{t\in I}$.

2. Recurrent strongly continuous semigroups of operators

Suppose that $\Delta = [0, \infty)$ or $\Delta = \mathbb{R}$. Let us recall that an operator family $(T(t))_{t\in\Delta} \subseteq L(X)$ is said to be a strongly continuous semigroup (if $\Delta = \mathbb{R}$, then we also say $(T(t))_{t\in\Delta}$ is a strongly continuous group of operators, with the meaning clear) if and only if the following holds:

(i) $T(0) = I$,

(ii) $T(t + s) = T(t)T(s),\ t,\ s \in \Delta$ and

(iii) the mapping $t \mapsto T(t)x,\ t \in \Delta$ is continuous for every fixed $x \in X$.

The linear operator

$$A := \left\{ (x, y) \in X \times X : \lim_{t\to 0+} \frac{T(t)x - x}{t} = y \right\}$$

is said to be the infinitesimal generator of $(T(t))_{t\in\Delta}$. A strongly continuous semigroup (group) $(T(t))_{t\in\Delta}$ is also said to be C_0-semigroup (group).

The following result is fundamental in our analysis:

Theorem 2.1. *Let $(T(t))_{t\in I}$ be a C_0-semigroup if $I = [0, \infty)$, resp. C_0-group if $I = \mathbb{R}$, of bounded linear operators on X. Then the following statements are equivalent:*

(i) $(T(t))_{t\in I}$ is recurrent.

(ii) $\overline{Rec(T(t))} = X$.

If this is the case, the set of recurrent vectors for $(T(t))_{t\in I}$ is a G_δ-subset of X.

Proof. First we will show that (ii) $\Rightarrow$ (i). Let $\overline{Rec(T(t))} = X$ and U be an arbitrary open non-empty subset in X. Let x be a recurrent vector and $\varepsilon > 0$ such that $B(x,\varepsilon) \subseteq U$, where $B(x,\varepsilon) = \{y \in X : \|x-y\| < \varepsilon\}$. Then there exists $t \in I$ such that $\|T(t)x - x\| < \varepsilon$. Thus $x \in U \cap T(t)(U) \neq \emptyset$, so $(T(t))_{t\in I}$ is recurrent. Now, we will show that (i) $\Rightarrow$ (ii). Let $(T(t))_{t\in I}$ be recurrent and let $B = B(x,\varepsilon)$ be an open ball in X, for fixed $x \in X$ and $\varepsilon < 1$. The proof will end if we show that there exists a recurrent vector in B. We use the recurrence property of $(T(t))_{t\in I}$. So, there exists $t_1 \in I$ such that $x_1 \in B \cap T(t_1)^{-1}(B)$, for some $x_1 \in E$. Since $(T(t))_{t\in I}$ is strongly continuous, we have that there exists $\varepsilon_1 < \frac{1}{2}$ such that $B_2 = B(x_1,\varepsilon_1) \subseteq B \cap T(t_1)^{-1}(B)$. Since $(T(t))_{t\in I}$ is recurrent, there exists $t_2 \in I$ with $|t_2| > |t_1|$ and some $x_2 \in E$ such that $x_2 \in B_2 \cap T(t_2)^{-1}(B_2)$. Using the same argument with strong continuity and recurrence of $(T(t))_{t\in I}$, we can inductively construct a sequence (x_n) in X, an unbounded sequence (t_n) in I and a decreasing sequence of positive real numbers (ε_n), such that for every integer $n \in \mathbb{N}$ one has $\varepsilon_n < 2^{-n}$,

$$B(x_n,\varepsilon_n) \subseteq B(x_{n-1},\varepsilon_{n-1}) \quad \text{and} \quad T(t_n)\big(B(x_n,\varepsilon_n)\big) \subseteq B(x_{n-1},\varepsilon_{n-1}).$$

By Cantor's theorem we have that

$$\bigcap_{n=1}^{\infty} B(x_n,\varepsilon_n) = \{y\},$$

for some $y \in X$. It is clear that $T(t_n)y \to y$ as $n \to +\infty$. Hence $y \in B$ is a recurrent vector in the open ball B, so the proof of (ii) $\Rightarrow$ (i) is finished. Let us prove that

$$Rec(T(t)) = \bigcap_{k=1}^{\infty} \bigcup_{n=1}^{\infty} \left\{ x \in X : \big\|T(q_n)x - x\big\| < \frac{1}{k} \right\} =: \mathrm{R}(T(t)), \tag{2.1}$$

where (q_n) denotes the sequence consisting of all rational numbers which do have the modulus strictly greater than 1. It simply follows that $Rec(T(t))$ is contained in the set $\mathrm{R}(T(t))$. For the opposite inclusion, for each element $x \in \mathrm{R}(T(t))$ and for each integer $k \in \mathbb{N}$ we can pick up a rational number q_k which do have the modulus strictly greater than 1 and for which $\|T(q_k)x - x\| < 1/k$. If the sequence (q_k) is unbounded, we have done. If not, then there exists a convergent subsequence (q_{n_k}) of (q_n) such that $\lim_{k\to\infty} q_{n_k} = q$ for some real number $q \in I$ such that $|q| \geq 1$. In this case, the strong continuity of $(T(t))_{t\in I}$ shows that $x = T(q)x$ so that clearly $x \in Rec(T(t))$ because, in this case, we have $T(nq)x = x$ for all $n \in \mathbb{N}$. Hence, (2.1) holds and $(T(t))_{t\in I}$ is a G_δ subset of E. $\square$

Using the representation formula (2.1) and the proof of [6, Proposition 2.6], it can be easily seen that the following result holds good:

Proposition 2.2. *Let $(T(t))_{t\in\mathbb{R}}$ be a C_0-group on X. Then $(T(t))_{t\geq 0}$ is recurrent if and only if $(T(-t))_{t\geq 0}$ is recurrent.*

Now we will state and prove the following continuous analogue of [6, Proposition 2.3(i)]:

Theorem 2.3. *Let $(T(t))_{t\in I}$ be a C_0-semigroup if $I = [0,\infty)$, resp. C_0-group if $I = \mathbb{R}$, of bounded linear operators on X. Then, for every $\lambda \in \mathbb{C}$ with $|\lambda| = 1$, we have $Rec(T(t)) = Rec(\lambda T(t))$.*

Proof. It is enough to show that $Rec(T(t)) \subseteq Rec(\lambda T(t))$. For $x \in Rec(T(t))$, we define the set $L := \big\{|\mu| = 1 : \lambda^n T(t_n)x \to \mu x, \text{for some unbounded sequence}\,(t_n) \text{ in } I\big\}$. To finish the proof, we have to prove that $1 \in L$. First of all, let us note that $L \neq \emptyset$. Since $x \in Rec(T(t))$, there exists an unbounded sequence (t_n) in I such that $T(t_n)x \to x$. There exists a subsequence of (t_n), denoted by (t_{n_k}), such that $\lambda^{t_{k_n}} \to \rho$ as $k \to \infty$, for some $|\rho| = 1$. Hence, we have $\lambda^{t_{k_n}} T(t_{n_k})x \to \rho x$ as $k \to \infty$, which means that $\rho \in L$. Let $\mu_1,\ \mu_2 \in L$ and $\varepsilon > 0$ be fixed. Since $\mu_1 \in L$, there exist a positive integer $n_1 \in \mathbb{N}$ and a real number $t_1 \in I$, with modulus sufficiently large, such that

$$\big\|\lambda^{n_1} T(t_1)x - \mu_1 x\big\| < \frac{\varepsilon}{2}.$$

Since $\mu_2 \in L$, there is a positive integer $n_2 \in \mathbb{N}$ and a real number $t_2 \in I$, with the modulus sufficiently large, such that

$$\left\|\lambda^{n_2}T(t_2)x - \mu_2 x\right\| < \frac{\varepsilon}{2\|T(t_1)\|}.$$

Hence,

$$\begin{aligned}\left\|\lambda^{n_1+n_2}T(t_1+t_2)x - \mu_1\mu_2 x\right\| &\leq \left\|\lambda^{n_1}T(t_1)(\lambda^{n_2}T(t_2)x - \mu_2 x)\right\| + \left\|\mu_2(\lambda^{n_1}T(t_1)x - \mu_1 x)\right\| \\ &\leq \left\|T(t_1)\right\|\left\|(\lambda^{n_2}T(t_2))x - \mu_2 x\right\| + \frac{\varepsilon}{2} < \varepsilon,\end{aligned}$$

so that $\mu_1\mu_2 \in L$. Hence, $\mu^n \in L$ for $\mu \in L$. If μ is a rational rotation, this means that $1 \in L$ and we are done. If μ is an irrational rotation, there is a strictly increasing sequence of positive integers (s_k) such that $\mu^{s_k} \to 1$. Since L is closed, it follows that $1 \in L$. $\square$

Proposition 2.4. *(see also [1, Proposition 2.10]) Let $(T(t))_{t\in I}$ be a C_0-semigroup if $I = [0,\infty)$, resp. C_0-group if $I = \mathbb{R}$, of bounded linear operators on X. If $(T(t) \oplus T(t))_{t\in I}$ is recurrent, then $(T(t))_{t\in I}$ is likewise recurrent.*

Proof. Let $x_1 \oplus x_2$ be a recurrent vector for $(T(t) \oplus T(t))_{t\in I}$. Then it is clear that x_1 and x_2 are recurrent vectors for $(T(t))_{t\in I}$; hence, $(T(t))_{t\in I}$ is recurrent. $\square$

The question whether the direct sum $(T(t) \oplus T(t))_{t\in I}$ of recurrent strongly continuous operator families $(T(t))_{t\in I}$ is recurrent is not simple. The answer is affirmative if $(T(t))_{t\in I}$ possesses some extra properties (see [6] for more details about the single-valued case). It is also worth noticing that we can simply clarify the comparision lemma for recurrent strongly continuous semigroups (see also [1, Corollary 2.9]).

As already mentioned, in the following theorem we will partially transfer the statement of [6, Proposition 2.3(ii)] for strongly continuous semigroups of operators:

Theorem 2.5. *Let $(T(t))_{t\in\mathbb{R}}$ be a C_0-group. Then the following assertions are equivalent:*

(i) $(T(t))_{t\geq 0}$ is recurrent.

(ii) For every $t_0 > 0$, the operator $T(t_0)$ is recurrent.

(iii) There exists $t_0 > 0$ such that the operator $T(t_0)$ is recurrent.

If this is the case, then for every $t_0 \in I \setminus \{0\}$, we have $Rec(T(t)) = Rec(T(t_0))$.

Proof. The only non-trivial part is that (i) implies (ii), with the equality $Rec(T(t)) = Rec(T(t_0))$ for any fixed number $t_0 > 0$. To see this, assume that $(T(t))_{t\geq 0}$ is a recurrent C_0-semigroup. Then it is clear that $Rec(T(t)) \supseteq Rec(T(t_0))$ and, owing to Theorem 2.1, all that we need to prove is that the preassumption $x \in Rec(T(t))$ implies $x \in Rec(T(t_0))$. Without loss of generality, we can assume that $t_0 = 1$. Indeed, we can consider the semigroup $(\tilde{T}(t))_{t\geq 0}$, with $\tilde{T}(t) := T(tt_0)$, for every $t \geq 0$. It is clear that x is a recurrent vector for $(\tilde{T}(t))_{t\geq 0}$ and $\tilde{T}(1) = T(t_0)$. Denote by $\mathbb{T}$ the unit sphere in $\mathbb{C}$ and define the mapping $\phi : [0,\infty) \to \mathbb{T}$ by $\phi(t) := e^{2\pi i t}$, $t \geq 0$. For every $u \in X$, we define the set

$$F_u := \Big\{\lambda \in \mathbb{T} : \ \exists (t_n)_n \in (0,\infty) \ \text{ s.t. } \lim_{n\to\infty} t_n = \infty, \lim_{n\to\infty} T(t_n)u = u \text{ and } \lim_{n\to\infty} \phi(t_n) = \lambda\Big\}.$$

Note that the set F_u is not empty by its definition and the recurrence property of the semigroup $(T(t))_{t\geq 0}$. The set F_u is closed for $u \in X$, as it can be easily checked. Next, we will prove that if $u \in X$ and $\lambda,\ \mu \in F_u$, then $\lambda\mu \in F_u$. Let U be an open balanced neighborhood of zero in X and $\varepsilon > 0$ arbitrary. Then we can find $t_1 > 0$ such that $\|T(t_1)u - \lambda u\| \leq \epsilon/2$ and $|\phi(t_1) - \mu| < \varepsilon/2$. Choose an open balanced neighborhood of zero V in X and number $t_2 > 0$ such that $T(t_1)(V) \subseteq U$, $T(t_2)u - \mu u \in V$ and $|\phi(t_2) - \lambda| < \varepsilon/2$. Hence,

$$\begin{aligned}T(t_1+t_2)u - \lambda\mu u &= T(t_1)\big(T(t_2)u - \mu u\big) + \mu\big(T(t_1)u - \lambda u\big) \\ &\in T(t_1)(V) + B(0,\epsilon/2) \subseteq U + B(0,\epsilon/2),\end{aligned}$$

so that

$$\big|\phi(t_1+t_2)-\lambda\mu\big| = \big|\phi(t_1)\phi(t_2)-\lambda\mu\big| \le \big|\phi(t_1)-\mu\big|\cdot\big|\phi(t_2)\big| + |\mu|\cdot\big|\phi(t_2)-\lambda\big| < \varepsilon.$$

This simply implies that $\lambda\mu \in F_u$ as claimed. Further on, it is clear that there exists $x \in (-\pi, \pi]$ such that $e^{ix} = \lambda \in F_u$. If x is rational, then using the fact that F_u is closed under multiplication immediately gives $1 \in F_u$. If x is not rational, then F_u is dense in $\mathbb{T}$ since it contains the set $\{e^{inx} : n \in \mathbb{N}\}$ so that $1 \in F_u$ again. Hence, $1 \in F_u$. Suppose now $u \in Rec(T(t))$. Then we have the existence of a sequence $(t_n)_{n\in\mathbb{N}}$ of positive real numbers tending to infinity such that $\lim_{n\to\infty} T(t_n)u = u$ and $\lim_{\to\infty} \phi(t_n) = 1$. Let (k_n) be a sequence of positive integers and $\varepsilon_n \in [-1, 1]$ such that $t_n = k_n + \varepsilon_n$, for all $n \in \mathbb{N}$. Obviously, $\lim_{n\to\infty} \varepsilon_n = 0$. Hence, $\|T(k_n)u-u\| \le \|T(-\varepsilon_n)[T(t_n)u-u]+[T(-\varepsilon_n)u-u]\| \le \sup_{\xi\in[-1,1]} \|T(\xi)\|\cdot\|T(t_n)u-u\|+\|T(-\varepsilon_n)u-u\| \to 0$ as $n \to +\infty$. As a consequence, we have $u \in Rec(T(1))$. $\square$

Remark 2.6. Condition that $(T(t))_{t\ge 0}$ can be extended to a C_0-group seems to be slightly redundant and we want to address the problem of removing this condition, if possible. Due to [19, Theorem 6.5, p. 24], this is the case provided that there exists a finite number $t_0 > 0$ such that $[T(t_0)]^{-1} \in L(X)$.

Suppose that $\Delta = [0, \infty)$ or $\Delta = \mathbb{R}$. A measurable function $\rho : \Delta \to (0, \infty)$ is said to be an admissible weight function if and only if there exist constants $M \ge 1$ and $\omega \in \mathbb{R}$ such that $\rho(t) \le Me^{\omega|t'|}\rho(t + t')$ for all $t,\ t' \in \Delta$. Let us introduce the Banach spaces

$$L^p_\rho(\Delta, \mathbb{C}) := \big\{u : \Delta \to \mathbb{C}\ ; u(\cdot) \text{ is measurable and } ||u||_p < \infty\big\},$$

where $p \in [1, \infty)$ and $||u||_p := (\int_\Delta |u(t)|^p \rho(t)\, dt)^{1/p}$, and

$$C_{0,\rho}(\Delta, \mathbb{C}) := \Big\{u : \Delta \to \mathbb{K}\ ; u(\cdot) \text{ is continuous and } \lim_{t\to\infty} u(t)\rho(t) = 0\Big\},$$

with $||u|| := \sup_{t\in\Delta} |u(t)\rho(t)|$. For any function $f : \Delta \to \mathbb{C}$, we define $T(t)f := f(\cdot + t)$, $t \in \Delta$. If $\rho(\cdot)$ is an admissible weight function and $\Delta = [0, \infty)$, resp. $\Delta = \mathbb{R}$, then the translation semigroup, resp. group, $(T(t))_{t\in\Delta}$ is strongly continuous on $L^p_\rho(\Delta, \mathbb{C})$ and $C_{0,\rho}(\Delta, \mathbb{C})$. Recently, Z. Yin and Y. Wei have considered the weak recurrence of translation operators on weighted Lebesgue spaces and weighted continuous function spaces ([22]). They have shown that the existence of a function $f \in X$, where $X = L^p_\rho([0, \infty), \mathbb{C})$ or $X = C_{0,\rho}([0, \infty), \mathbb{C})$, satisfying that there exists a strictly increasing sequence (α_n) of positive reals tending to plus infinity such that (compare with the equation (2.2) below)

$$\lim_{n\to+\infty} \big\|f(\cdot + \alpha_n) - f(\cdot)\big\|_X = 0$$

is equivalent to saying that $\liminf_{t\to+\infty} \rho(t) = 0$ (the hypercyclicity of $(T(t))_{t\ge 0}$); see also the preprint [4] by W. Brian and J. P. Kelly.

In connection with the notion of recurrent translation operators, we would like to finally mention the notion of a uniformly recurrent function. Let $I = \mathbb{R}$ or $I = [0, \infty)$. Following A. Haraux and P. Souplet [15], we say that a continuous function $f : I \to X$ is uniformly recurrent if and only if there exists a strictly increasing sequence (α_n) of positive real numbers such that $\lim_{n\to+\infty} \alpha_n = +\infty$ and

$$\lim_{n\to\infty} \sup_{t\in I} \big\|f(t + \alpha_n) - f(t)\big\| = 0. \tag{2.2}$$

It is well known that any almost periodic function is uniformly recurrent, while the converse statement is not true in general. In [15, Theorem 1.1], the authors have proved that the function $f : \mathbb{R} \to \mathbb{R}$, given by

$$f(t) := \sum_{n=1}^{\infty} \frac{1}{n} \sin^2\Big(\frac{t}{2^n}\Big)\, dt, \quad t \in \mathbb{R},$$

is uniformly continuous, uniformly recurrent and unbounded. For more details about uniformly recurrent functions, almost periodic functions and related problematic, we refer the reader to [18] and references cited therein.

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