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The modulus of nondensifiable convexity and its applications

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Abstract. Based in the so called degree of nondensifiability, denoted by ϕ, we introduce and analyze the concepts of modulus of nondensifiable convexity and the nearly uniform convexity characteristic of a given Banach space X associated to ϕ, denoted by Δ_X^{ϕ} and $\varepsilon^{\phi}(X)$, respectively. Although ϕ is not a measure of noncompactness (MNC), we prove that Δ_X^{ϕ} and $\varepsilon^{\phi}(X)$ are, respectively, a lower and upper bound for the modulus of noncompact convexity and the nearly uniform convexity characteristic of X associated to an arbitrary MNC. Also, we characterize the normal structure of X in terms of ϕ and, by using $\varepsilon^{\phi}(X)$, we give a sufficient condition for X has the weak fixed point property.

1. Introduction

To set the notation, $(X, \|\cdot\|)$ will be a Banach space, and U_X its closed unit ball. As usual, $\overline{B}$ and $\mathrm{Conv}(B)$ denote the closure and the convex hull of a non-empty set $B \subset X$, respectively. Also, $\mathcal{B}(X)$ is the class of non-empty and bounded subsets of X.

It is known that many geometric properties of X can be, in a suitable sense, measured by certain functions (often called *moduli*) and constants; see [3, 4, 10, 11, 19, 30] and references therein. We recall here some definitions related with the notion of convexity of X (see, for instance, [3, 11]).

Definition 1.1. *The Banach space X is said to be:*

(i) Strictly convex (SC) if whenever $x, y \in X$ are not collinear then $\|x + y\| < \|x\| + \|y\|$.

(ii) Uniformly convex (UC) if for each $\varepsilon \in [0, 2)$ there is $\delta > 0$ such that for all $x, y \in X$ with

$$x, y \in U_X \text{ and } \|x - y\| \geq \varepsilon, \quad \text{then} \quad 1 - \frac{\|x + y\|}{2} > \delta.$$

(iii) Nearly uniformly convex (NUC) if for every $\varepsilon > 0$ there is $\delta > 0$ such that if $(x_n)_{n\geq 1} \subset X$ is a sequence with

$$(x_n)_{n\geq 1} \subset U_X \text{ and } \inf\Big\{\|x_n - x_m\| : n \neq m\Big\} > \varepsilon, \quad \text{then}$$

$$\mathrm{Conv}\Big(\{x_n : n \geq 1\}\Big) \cap (1 - \delta)U_X \neq \emptyset.$$

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Is clear that (UC)⇒(NUC) and (UC)⇒(SC). Here, we are interested mainly in the analysis of the uniform convexity of X. The first modulus for the uniform convexity was introduced in 1936 by Clarkson [8]:

Definition 1.2. *The modulus of convexity of X, $\delta_X : [0,2] \longrightarrow I := [0,1]$, is defined by*

$$\delta_X(\varepsilon) := \inf\Big\{1 - \|\frac{x+y}{2}\| : x, y \in U_X, \|x-y\| \geq \varepsilon\Big\}, \quad \textit{for all } \varepsilon \in [0,2].$$

Furthermore, Goebel [17] in 1970 introduced the following:

Definition 1.3. *The characteristic of convexity of X is the constant*

$$\varepsilon_0(X) := \sup\Big\{\varepsilon \in [0,2] : \delta_X(\varepsilon) = 0\Big\}.$$

It is a well known fact (see, for instance, [11]) that some of the concepts given in Definition 1.1 are intimately related with $\delta_X(\varepsilon)$ and $\varepsilon_0(X)$. For instance, X uniformly convex if, and only if, $\varepsilon_0(X) = 0$, and X is strictly convex if, and only if, $\delta_X(2) = 1$.

Before to introduce another modulus related with the convexity of X (and more specifically, with the nearly uniform convexity of X), based on the so called *measures of noncompactness*, in short MNCs, we recall three of such MNCs, as well as their basic properties, that we will need later. For a detailed exposition of the MNCs, we refer to [1–3].

The Hausdorff MNC $\chi : \mathcal{B}(X) \longrightarrow \mathbb{R}_+ := [0,\infty)$ is defined as

$$\chi(B) := \inf\Big\{\varepsilon > 0 : B \text{ can be covered by finitely many balls with radii } \leq \varepsilon\Big\}$$

and the Kuratowski MNC $\kappa : \mathcal{B}(X) \longrightarrow \mathbb{R}_+$ by

$$\kappa(B) := \inf\Big\{\varepsilon > 0 : B \text{ can be covered by finitely many sets of diameter } \leq \varepsilon\Big\}.$$

for every $B \in \mathcal{B}(X)$.

Example 1.4. *(see [1, 3]) If X is finite dimensional then $\chi(U_X) = \kappa(U_X) = 0$. Otherwise, $\chi(U_X) = 1$ and $\kappa(U_X) = 2$.*

By recalling that $B \in \mathcal{B}(X)$ is said to be *r-separated* (or B is an *r-separation of* X) if $\|x-y\| \geq r$ for all $x, y \in B$, $x \neq y$, the Istrăţescu or separation MNC $\beta : \mathcal{B}(X) \longrightarrow \mathbb{R}_+$ is defined as

$$\beta(B) := \sup\Big\{r > 0 : B \text{ has an infinite } r-\text{separation}\Big\} =$$

$$\inf\Big\{r > 0 : B \text{ does not have an infinite } r-\text{separation}\Big\}, \quad \text{for all } B \in \mathcal{B}(X).$$

It is not difficult to prove, from the involved definitions, the inequalities

$$\chi(B) \leq \beta(B) \leq \kappa(B) \leq 2\chi(B) \quad \text{for all } B \in \mathcal{B}(X). \tag{1}$$

An essential difference between χ, κ and β is that $\beta(U_X)$ depends of the space X, as we show in the following example (see, for instance, [3, Chapter II]).

Example 1.5. *(see [3]) Fixed $1 \leq p < \infty$, let ℓ_p be the Banach space of the (real) sequences $(x_n)_{n\geq 1}$ such that $\sum_{n\geq 1} |x_n|^p < \infty$, and L^p the Banach space of the functions defined on I such that $\int_0^1 |x(t)|^p dt < \infty$, the integral means in the Lebesgue sense, both endowed their usual norms. Then, we have*

$$\beta(U_{\ell_p}) = 2^{\frac{1}{p}}, \quad \beta(U_{L^p}) = \begin{cases} 2^{\frac{1}{p}}, & \textit{for} \quad 1 \leq p \leq 2 \\ 2^{1-\frac{1}{p}}, & \textit{for} \quad 2 < p < \infty \end{cases}.$$

In the next result we show some properties of the above MNCs (see [1, 3]).

Proposition 1.6. *The MNC $\mu \in \{\chi, \kappa, \beta\}$ satisfies the following properties for all $B, B_1, B_2 \in \mathcal{B}(X)$:*

1. *Regularity:* $\mu(B) = 0$ *if, and only if, B is precompact.*
2. *Invariance under closure and convex hull:* $\mu(B) = \mu(\overline{B}) = \mu(\mathrm{Conv}(B))$.
3. *Semi-additivity:* $\mu(B_1 \cup B_2) = \max\{\mu(B_1), \mu(B_2)\}$.
4. *Algebraic semi-additivity:* $\mu(B_1 + B_2) \leq \mu(B_1) + \mu(B_2)$.
5. *Semi-homogeneity:* $\mu(\lambda B) = |\lambda|\mu(B)$, *for all* $\lambda \in \mathbb{R}$.
6. *Invariance under translations:* $\mu(x_0 + B) = \mu(B)$, *for all* $x_0 \in X$.

In what follows, $\mathbf{0}$ is the null vector of X and for a given $B \in \mathcal{B}(X)$, the distance from $\mathbf{0}$ to B is denoted by $d(\mathbf{0}, B) := \inf\{\|x\| : x \in B\}$. Also, $\mathcal{B}_{\overline{\mathrm{Conv}}}(X)$ stands for the class of nonempty, bounded, closed and convex subsets of X. With this notation, we can give the following definition (see, for instance, [3]):

Definition 1.7. *The modulus of noncompact convexity associated to the MNC μ is the function $\Delta_{X,\mu} : [0, \mu(U_X)] \longrightarrow I$ defined by*

$$\Delta_{X,\mu}(\varepsilon) := \inf\left\{1 - d(\mathbf{0}, B) : B \in \mathcal{B}_{\overline{\mathrm{Conv}}}(X) \cap U_X, \mu(B) \geq \varepsilon\right\} \quad \textit{for all } \varepsilon \in [0, \mu(U_X)],$$

and the nearly uniform convexity characteristic of X associated to μ, μ-NUC-characteristic, is

$$\varepsilon_\mu(X) := \sup\left\{\varepsilon \in [0, \mu(U_X)] : \Delta_{X,\mu}(\varepsilon) = 0\right\}.$$

Remark 1.8. *The name of $\varepsilon_\mu(X)$ is justified by the following fact (see [3, Remark V.1.1.4]): X is UNC if, and only if, $\varepsilon_\mu(X) = 0$ for $\mu \in \{\chi, \kappa, \beta\}$.*

The function $\Delta_{X,\kappa}(\varepsilon)$ was introduced and analyzed in [18], $\Delta_{X,\chi}(\varepsilon)$ in [5] and $\Delta_{X,\beta}(\varepsilon)$ in [9]. It is not very hard to prove the following inequalities (see also [3, Remark 1.4, p. 86]).

$$\delta_X(\varepsilon) \leq \Delta_{X,\kappa}(\varepsilon) \leq \Delta_{X,\beta}(\varepsilon) \leq \Delta_{X,\chi}(\varepsilon), \tag{2}$$

for all $\varepsilon \in [0, 1]$, and consequently

$$\varepsilon_0(X) \geq \varepsilon_\kappa(X) \geq \varepsilon_\beta(X) \geq \varepsilon_\chi(X). \tag{3}$$

For some Banach spaces, we know an explicit formula for $\Delta_{X,\mu}(\varepsilon)$, $\mu \in \{\chi, \kappa, \beta\}$.

Example 1.9. *Fixed $1 < p < \infty$, let ℓ_p be as in Example 1.5. Then, we have (see [3, Chapter V])*

$$\Delta_{\ell_p,\kappa}(\varepsilon) = 1 - \left(1 - \left(\tfrac{\varepsilon}{2}\right)^p\right)^{\frac{1}{p}} \quad , \Delta_{\ell_p,\beta}(\varepsilon) = 1 - \left(1 - \tfrac{\varepsilon^p}{2}\right)^{\frac{1}{p}},$$

$$\Delta_{\ell_p,\chi}(\varepsilon) = 1 - \left(1 - \varepsilon^p\right)^{\frac{1}{p}},$$

where, in each case, ε is in the interval $[0, \mu(U_X)]$, with $\mu \in \{\chi, \kappa, \beta\}$.

However, at the present day, the value of $\Delta_{X,\mu}(\varepsilon)$ is unknown for an arbitrary Banach space X and a MNC μ. For instance, the exact values of $\Delta_{L^p,\kappa}$ are unknown if $1 < p < 2$, although they are different from those of $\Delta_{\ell_p,\kappa}$ (see, for instance, [29]). Anyway, the analysis of the geometry of X based on the modulus $\Delta_{X,\mu}$, as well as $\varepsilon_\mu(X)$ for a given MNC μ, is based in inequalities. We will recall in Section 3 some important results, related with the geometry of X, which are given in terms of certain inequalities that involve $\Delta_{X,\mu}$ and/or $\varepsilon_\mu(X)$, for $\mu \in \{\chi, \kappa, \beta\}$.

On the other hand by using the so called *degree of nondensifiablity*, denoted by ϕ and explained in detail in Section 2, our main goal is to introduce and analyze *the modulus of nondensifiable convexity* $\Delta_X^\phi(\varepsilon)$ and the *ϕ-NUC-characteristic of* X, $\varepsilon^\phi(X)$. Such concepts, as well as its main properties, are given in Section 3. We show that, although ϕ is not a MNC, $\Delta_X^\phi(\varepsilon)$ and $\varepsilon^\phi(X)$ share some properties with $\Delta_{X,\mu}$ and $\varepsilon_\mu(X)$, respectively, for a given MNC μ. Also, we prove in Theorem 3.8 that Δ_X^ϕ (resp. $\varepsilon_\mu(X)$) is an upper (resp. lower) for $\Delta_{X,\mu}$ (resp. $\varepsilon_\mu(X)$), for an arbitrary MNC μ.

To conclude our exposition, in Section 4 we characterize the normal structure of X in terms of ϕ. Also, by using $\varepsilon^\phi(X)$, we give a sufficient condition for X has the so called *weak fixed point property*.

2. The degree of nondensifiability and some relationships with the MNCs

In what follows, (E, d) will be a metric space and $\mathcal{B}(E)$ the class of non-empty and bounded subsets of E. As in Sectino 1, $(X, \|\cdot\|)$ is a Banach space. For a better comprehension of the manuscript, we start this section by recalling the following (see, for instance, [25]).

Definition 2.1. *Let $\alpha \geq 0$ and $B \in \mathcal{B}(E)$. A continuous mapping $\gamma : I \longrightarrow (E, d)$ is said to be an α-dense curve in B if it satisfies:*

(i) $\gamma(I) \subset B$.

(ii) For any $x \in B$ there is $t \in I$ such that $d(x, \gamma(t)) \leq \alpha$.

If for every $\alpha > 0$ there exists an α-dense curve in B then B is said to be densifiable.

For a detailed exposition of the above concepts, we refer to [7, 24–27]. From the concept of α-dense curve, we can give the following (see [16, 26]):

Definition 2.2. *For each $\alpha \geq 0$ we denote by $\Gamma_{\alpha,B}$ the class of α-dense curves in $B \in \mathcal{B}(E)$. The degree of nondensifiability, DND, is the mapping $\phi : \mathcal{B}(E) \longrightarrow \mathbb{R}_+$ defined as*

$$\phi(B) := \inf\left\{\alpha \geq 0 : \Gamma_{\alpha,B} \neq \emptyset\right\}, \quad \textit{for all } B \in \mathcal{B}(E).$$

Let us note that ϕ is well defined. Indeed, given $B \in \mathcal{B}(E)$ take $x_0 \in B$ and define $\gamma(t) := x_0$ for all $t \in I$. Thus, γ is an α-dense curve in B for any $\alpha \geq \mathrm{Diam}(B)$ (the diameter of B). So, $\phi(B) \leq \mathrm{Diam}(B)$.

Example 2.3. *(see [26]) We have:*

$$\phi(U_X) = \begin{cases} 0, & \textit{if X is finite dimensional} \\ 1, & \textit{if X is infinite dimensional} \end{cases}.$$

From now on, $\mathcal{B}_{\mathrm{arc}}(E)$ will be the class of nonempty, bounded and arc-wise connected subsets of E, and the some notation for X. We state some basic properties of the DND in the following result (see [15, 16]):

Proposition 2.4. *The DND ϕ satisfies the following properties:*

(M1) Regularity on $\mathcal{B}_{\mathrm{arc}}(E)$: $\phi(B) = 0$ if, and only if, B is precompact, for all $B \in \mathcal{B}_{\mathrm{arc}}(E)$.

(M2) Invariant under closure: $\phi(B) = \phi(\overline{B})$, for all $B \in \mathcal{B}(E)$.

In particular, for $E := X$ we hav:

(B1) Semi-homogeneity: $\phi(\lambda B) = |\lambda|\phi(B)$, for any $\lambda \in \mathbb{R}$ and $B \in \mathcal{B}(X)$.

(B2) Invariant under translations: $\phi(x_0 + B) = \phi(B)$, for all $x_0 \in X$ and $B \in \mathcal{B}(X)$.

(B3) $\phi(\mathrm{Conv}(B_1 \cup B_2)) \leq \max\{\phi(\mathrm{Conv}(B_1)), \phi(\mathrm{Conv}(B_2))\}$, for all $B_1, B_2 \in \mathcal{B}(X)$.

(B4) Algebraic semi-additivity: $\phi(B_1 + B_2) \leq \phi(B_1) + \phi(B_2)$, for all $B_1, B_2 \in \mathcal{B}(X)$.

Despite the above properties, the DND ϕ is not a MNC:

Example 2.5. *In the space L^1 (see Example 1.5), consider the set*

$$B := \Big\{ f \in L^1 : f \geq 0 \text{ and } \int_0^1 f(x)dx = 1 \Big\}.$$

Then, $\phi(B) = 2$ (see [16]) and noticing Example 2.3

$$1 = \phi(U_{L^1}) = \phi(B \cup U_{L^1}) < 2 = \max \Big\{ \phi(B), \phi(U_{L^1}) \Big\}.$$

Other examples that evidence the differences between the DND ϕ and the MNCs are given in [12–15]. However, the DND ϕ and the MNCs are related by the following result (see [16]):

Theorem 2.6. *For every $B \in \mathcal{B}(X)$, the inequality $\mu(B) \leq \mu(U_X)\phi(B)$ holds.*

Furthermore, for the MNCs given in Section 1 we can provide these others inequalities:

Proposition 2.7. *For $\mu \in \{\chi, \kappa, \beta\}$ and $B \in \mathcal{B}_{\mathrm{arc}}(X)$, we have:*

1. $\chi(B) \leq \phi(B) \leq 2\chi(B)$.
2. $\frac{1}{2}\kappa(B) \leq \phi(B) \leq \kappa(B)$.
3. $\frac{1}{\beta(U_X)}\beta(B) \leq \phi(B) \leq 2\beta(B)$.

Moreover, these inequalities are the best possible in infinite dimensional Banach spaces.

Proof. The inequalities in (1) were proved in [16], and they are the best possible. The left hand-side inequalities in (2) and (3) follow from Theorem 2.6, and the right hand-side inequality of (3) is a consequence of the inequalities (1) and (1). So, we prove the right hand-side inequality of (2).

Given $B \in \mathcal{B}_{\mathrm{arc}}(X)$ taking any $\delta > \kappa(B)$, there are $B_1, \ldots, B_n \in \mathcal{B}(X)$, with $\mathrm{Diam}(B_i) \leq \delta$ such that $B \subset \cup_{i=1}^n B_i$. Take $x_i \in B \cap B_i$, for $i = 1, \ldots, n$, and let $\gamma : I \longrightarrow B$ be a continuous mapping joining the vectors $x_1, \ldots, x_n$. We can define such γ because of B is arc-wise connected. Then, for a given $x \in B$, taking $t_i \in I$ such that $\gamma(t_i) = x_i$ and $\|x - x_i\| \leq \mathrm{Diam}(B_i) \leq \delta$, we have

$$\|x - \gamma(t_i)\| = \|x - x_i\| \leq \delta,$$

and therefore γ is a δ-dense curve in B and so $\phi(B) \leq \delta$. By letting $\delta \to \kappa(B)$, we infer that $\phi(B) \leq \kappa(B)$.

On the other hand, from the properties of ϕ and the MNCs κ, β, if $B \in \mathcal{B}_{\text{arc}}(X)$ is precompact, all the inequalities in (2) and (3) become (trivially) into equalities. Noticing Examples 1.4 and 2.3, we have

$$1 = \frac{1}{2}\kappa(U_X) = \phi(U_X) < \kappa(U_x) = 2,$$

and the inequalities in (2) can be strict. Let c_0 be the sequence space of the null sequences, endowed its usual supremum norm, and $\{e_n : n \geq 1\}$ its standard basis. Let $B := \overline{\text{Conv}}(\{e_n : n \geq 1\})$. Then, $\kappa(B) = 1$ and as was proved in [16], $\phi(B) = 1$. Therefore, $\frac{1}{2}\kappa(B) < \phi(B) = \kappa(B)$.

For the inequalities in (3), from Examples 1.5 and 2.3, we find

$$\frac{\beta(U_{\ell_p})}{\beta(U_{\ell_p})} = 1 = \phi(U_{\ell_p}) < 2\beta(U_{\ell_1}) = 2^{1+\frac{1}{p}}, \quad \text{for all } p \geq 1.$$

Let $\{e_n : n \geq 1\}$ be the standard basis of ℓ_1 and $B := \overline{\text{Conv}}(\{e_n : n \geq 1\})$. If γ is an α-dense curve in B, for some $\alpha > \phi(B)$, as $\gamma(I)$ is compact for a given $0 < \varepsilon < 1$ there exists $n_0 \geq 1$ such that (see [3, Theorem 2.4.1])

$$\sum_{n \geq n_0+1} y_n \leq \varepsilon, \quad \text{for all } y := (y_n)_{n\geq 1} \in \gamma(I).$$

Thus, for $x := e_{n_0+1} \in B$, there is $y \in \gamma(I)$ such that $\|x - y\| \leq \alpha$ and from the above inequality

$$\alpha \geq \|x - y\| = \sum_{n=1}^{n_0} y_n + |1 - y_{n_0+1}| + \sum_{n \geq n_0+2} y_n \geq \sum_{n=1}^{n_0} y_n + 1 - \varepsilon,$$

and by the arbitrariness of $\varepsilon > 0$, $\alpha > \phi(B)$ and $n_0 > 1$, we infer that $\phi(B) \geq 2$. But, from the right-hand side inequality (1) of the statement, $\phi(B) \leq 2\chi(B) \leq 2$ and consequently $\phi(B) = 2$. So, again noticing Example 1.5, we have

$$\frac{\beta(B)}{\beta(U_{\ell_1})} = \frac{2}{2} = 1 < \phi(B) < 4 = 2\phi(B).$$

Finally, noticing inequalities (1), for the set $B \subset L^1$ of Example (2.5), we have $\beta(B) \geq 1$. Fixed $\varepsilon > 0$, for each $n > 1$, if $\mathbf{1}_A$ denotes the characteristic function of the set $A \subset I$

$$\beta(B) + \varepsilon \geq \|2n\mathbf{1}_{[0,\frac{1}{2n}]} - n\mathbf{1}_{[0,\frac{1}{n}]}\| = \int_0^{\frac{1}{2n}} n dx + \int_{\frac{1}{2n}}^{\frac{1}{n}} n dx = 1,$$

and therefore, by the arbitrariness of ε, $\beta(B) = 1$. Then, we conclude that $\phi(B) = 2\beta(B)$ and this completes the proof. □

3. Main results

Let us note that given $B \in \mathcal{B}_{\overline{\text{Conv}}}(X) \cap U_X$ and an α-dense curve in B, put γ, for some $\alpha \geq 0$, for every $x \in B$ and $y \in \gamma(I)$ we have $\|x - y\| \leq \|x\| + \|y\| \leq 2$. Thus, $\phi(B) \leq 2$ and therefore

$$\varsigma(U_X) := \sup\left\{\phi(B) : B \in \mathcal{B}_{\overline{\text{Conv}}}(X) \cap U_X\right\} \leq 2.$$

Also, this bound is attained, for instance, in L^1 (see Example 2.5). However, at the present day, we do not know if $\varsigma(U_X) = 2$ for every infinite dimensional Banach space X. Without loss of generality, in what follows we assume that $\varsigma(U_X)$ is achieved for some $B \in \mathcal{B}_{\overline{\text{Conv}}}(X) \cap U_X$, otherwise in the below definitions and results we replace the closed interval $[0, \varsigma(U_X)]$ by the semi-open interval $[0, \varsigma(U_X))$.

The main concepts of this paper are given in the following definition:

Definition 3.1. *The modulus of nondensifiable convexity is the function* $\Delta_X^{\phi} : [0, \varsigma(U_X)] \longrightarrow I$ *defined by*

$$\Delta_X^{\phi}(\varepsilon) := \inf\Big\{1 - d(\mathbf{0}, B) : B \in \mathcal{B}_{\overline{\mathrm{Conv}}}(X) \cap U_X, \phi(B) \geq \varepsilon\Big\} \quad \text{for all } \varepsilon \in [0, \varsigma(U_X)],$$

and the nearly uniform convexity characteristic of X *associated to* ϕ, ϕ*-NUC-characteristic, is*

$$\varepsilon^{\phi}(X) := \sup\Big\{\varepsilon \in [0, \varsigma(U_X)] : \Delta_X^{\phi}(\varepsilon) = 0\Big\}.$$

Remark 3.2. *The name of* $\varepsilon^{\phi}(X)$ *will be justified in Corollary 3.14.*

For clarity, we divide the remainder of this section into two parts.

3.1. Some properties of the modulus of nondensifiable convexity

The results of this section are devoted to prove some properties of the modulus of nondensifiable convexity Δ_X^{ϕ}.

Related with the continuity of Δ_X^{ϕ}, we have the following:

Proposition 3.3. *The function* Δ_X^{ϕ} *is continuous on* $[0, 1)$.

Proof. Fixed $\varepsilon_1 \in [0, 1)$ take $\varepsilon_2 \in [\varepsilon_1, 1)$ and $r > 0$ such that $\phi(B_1) > \varepsilon_1$, for some $B_1 \in \mathcal{B}_{\overline{\mathrm{Conv}}}(X) \cap U_X$, and

$$1 - d(\mathbf{0}, B_1) \leq \Delta_X^{\phi}(\varepsilon_1) + r. \tag{4}$$

Define $B_2 := kB_1$, with $k := \frac{1-\varepsilon_2}{1-\varepsilon_1}$. Then, $\|x\| \leq k$ for all $x \in B_2$, $d(\mathbf{0}, B_2) = kd(\mathbf{0}, B_1)$ and from (3) of Proposition 2.4 $\phi(B_2) = k\phi(B_1)$. Putting $C := \cup_{x \in B_2} \bar{B}(x, 1-k)$ (the closed ball centered at x and radius $1 - k$), we have

$$d(\mathbf{0}, C) = kd(\mathbf{0}, B_1) - 1 + k,$$

and noticing [3, Theorem II.2.10], $\chi(C) = k\chi(B_1) + 1 - k > \varepsilon_2$. Consequently, by Proposition 2.7, $\phi(C) \geq \chi(C) > \varepsilon_2$. Then, from (4), we have

$$\Delta_X^{\phi}(\varepsilon_2) \leq 1 - d(\mathbf{0}, C) = 1 - kd(\mathbf{0}, B_1) + 1 - k = k\Big(1 - d(\mathbf{0}, B_1)\Big) + 2(1-k) \leq$$

$$k\Big(\Delta_X^{\phi}(\varepsilon_1) + r\Big) + 2(1-k),$$

and letting $r \to 0$,

$$\Delta_X^{\phi}(\varepsilon_2) \leq k\Delta_X^{\phi}(\varepsilon_1) + 2(1-k).$$

So,

$$\Delta_X^{\phi}(\varepsilon_2) - \Delta_X^{\phi}(\varepsilon_1) \leq (1-k)\Big(2 - \Delta_X^{\phi}(\varepsilon_1)\Big) \leq 2(1-k) = 2\frac{\varepsilon_2 - \varepsilon_1}{1 - \varepsilon_1} \longrightarrow 0,$$

as $\varepsilon_2 \to \varepsilon_1$, and the result holds.
□

The above result was proved in [5, Theoem 3] for $\Delta_{X,\chi}$, and in [31] for $\Delta_{X,\mu}$, μ being a MNC satisfying certain properties.

When X is reflexive, $\Delta_{X,\chi}(\varepsilon)$ is a subhomogeneous function (see [5, Theorem 4]). The same result holds for $\Delta_X^{\phi}(\varepsilon)$:

Theorem 3.4. *Let X be a reflexive Banach space. Then, for every $k \in (0,1)$ we have*

$$\Delta_X^{\phi}(k\varepsilon) \leq k\Delta_X^{\phi}(\varepsilon), \quad \text{for all } \varepsilon \in [0, \varsigma(U_X)].$$

Proof. Fixed $k \in (0,1)$ and $\varepsilon \in [0, \varsigma(U_X)]$, let any $r > 0$. Then, there exists $B \in \mathcal{B}_{\overline{\mathrm{Conv}}}(X) \cap U_X$ such that

$$1 - d(\mathbf{0}, B) \leq \Delta_X^{\phi}(\varepsilon) + r \quad \text{and} \quad \phi(B) \geq \varepsilon. \tag{5}$$

As X is reflexive, there is $x_0 \in X$ such that $d(\mathbf{0}, B) = \|x_0\|$ (see, for instance, [22]). Define the set

$$B_1 := kB + \frac{1-k}{\|x_0\|}x_0 \in \mathcal{B}_{\overline{\mathrm{Conv}}}(X) \cap U_X.$$

Thus, $\phi(B_1) = k\phi(B) \geq k\varepsilon$ (see Proposition 2.4) and $d(\mathbf{0}, B_1) = kd(\mathbf{0}, B) + 1 - k$. Therefore,

$$d(\mathbf{0}, B) = \frac{1}{k}\Big[d(\mathbf{0}, B_1) + k - 1\Big]. \tag{6}$$

So, noticing (5) and (6), we have

$$1 - \Delta_X^{\phi}(\varepsilon) \leq d(\mathbf{0}, B) + r = \frac{1}{k}\Big[d(\mathbf{0}, B_1) + k - 1\Big] + r \leq$$

$$\frac{1}{k}\Big[1 - \Delta_X^{\phi}(k\varepsilon) + k - 1\Big] + r \leq 1 - \frac{\Delta_X^{\phi}(k\varepsilon)}{k} + \frac{r}{k},$$

and as $r > 0$ can be arbitrarily small, the result follows.
□

Remark 3.5. *The above result is trivial for $k = 0, 1$ and the reflexivity of X is not required. Indeed, for $k = 1$ the equality $\Delta_X^{\phi}(k\varepsilon) = k\Delta_X^{\phi}(\varepsilon)$ is trivial, while for $k = 0$ we have $\Delta_X^{\phi}(0) = 0$ (taking, for instance, any $x_1 \in U_X$ with $\|x_1\| = 1$, for $B := \{x_1\}$ we obtain the desired result).*

Now, we give some consequences of the above theorem.

Corollary 3.6. *If X is reflexive, then Δ_X^{ϕ} is strictly increasing on the interval $[\varepsilon_X^{\phi}, \varsigma(U_X)]$.*

Proof. Let $\varepsilon_1, \varepsilon_2 \in [\varepsilon_X^{\phi}, \varsigma(U_X)]$, $\varepsilon_1 < \varepsilon_2$ and define $k := \varepsilon_1/\varepsilon_2 < 1$. Then, by Theorem 3.4

$$\Delta_X^{\phi}(\varepsilon_1) = \Delta_X^{\phi}(k\varepsilon_2) \leq k\Delta_X^{\phi}(\varepsilon_2) < \Delta_X^{\phi}(\varepsilon_2),$$

and the result holds. □

Corollary 3.7. *If X is reflexive, then $\Delta_X^{\phi}(\varepsilon) \leq \varepsilon$ for all $\varepsilon \in [0, \varsigma(U_X)]$.*

Proof. As we have noted in Remark 3.5, $\Delta_X^{\phi}(0) = 0$ and the inequality $\Delta_X^{\phi}(1) \leq 1$ holds trivially from the definition of the modulus of nondensifiable convexity Δ_X^{ϕ}.

If $\varepsilon \in (0,1)$ then by Theorem 3.4

$$\Delta_X^{\phi}(\varepsilon) \leq \varepsilon\Delta_X^{\phi}(1) \leq \varepsilon.$$

If $\varepsilon \in (1, \varsigma(U_X)]$, by Corollary 3.6, we have

$$\Delta_X^{\phi}(\varepsilon) \leq \Delta_X^{\phi}\big(\varsigma(U_X)\big) \leq 1 < \varepsilon,$$

and this completes the proof. □

3.2. Some relationships between the moduli Δ_X^{ϕ} and $\Delta_{X,\mu}$

Taking into account Theorem 2.6, we can provide a lower bound for the function $\Delta_{X,\mu}$:

Theorem 3.8. *Let μ be MNC defined on $\mathcal{B}(X)$. Then, we have*

$$\Delta_{X,\mu}(\varepsilon) \geq \Delta_X^{\phi}\Big(\frac{\varepsilon}{\mu(U_X)}\Big), \quad \text{for all } \varepsilon \in [0, \mu(U_X)].$$

Consequently, $\varepsilon_\mu(X) \leq \mu(U_X)\varepsilon^{\phi}(X)$.

Proof. We only prove the second inequality, because of the first can be obtained directly from Theorem 2.6 and the definitions of $\Delta_{X,\mu}$ and Δ_X^{ϕ}. As

$$\Big\{\varepsilon \in [0, \mu(U_X)] : \Delta_{X,\mu}(\varepsilon) = 0\Big\} \subset \Big\{\varepsilon \in [0, \mu(U_X)] : \Delta_X^{\phi}\Big(\frac{\varepsilon}{\mu(U_X)}\Big) = 0\Big\},$$

noticing the definition of $\varepsilon_\mu(X)$ and the properties of the supremum, we have

$$\varepsilon_\mu(X) \leq \sup\Big\{\varepsilon \in [0, \mu(U_X)] : \Delta_X^{\phi}\Big(\frac{\varepsilon}{\mu(U_X)}\Big) = 0\Big\}.$$

Then, by putting $\tilde{\varepsilon} := \varepsilon/\mu(U_X)$ we find

$$\varepsilon_\mu(X) \leq \mu(U_X) \sup\Big\{\tilde{\varepsilon} \in I : \Delta_X^{\phi}(\tilde{\varepsilon}) = 0\Big\} \leq$$

$$\mu(U_X) \sup\Big\{\tilde{\varepsilon} \in [0, \varsigma(U_X)] : \Delta_X^{\phi}(\tilde{\varepsilon}) = 0\Big\} = \mu(U_X)\varepsilon^{\phi}(X).$$

□

As consequence of the above result, we obtain the next:

Corollary 3.9. *If $\varepsilon^{\phi}(X) < 1$, then X is reflexive.*

Proof. Assume that X is not reflexive. Then, following the proof of [3, Theorem V.1.7], we find that $\varepsilon_\chi(X) \geq 1$ and therefore, noticing Theorem 3.8, $\varepsilon^{\phi}(X) \geq 1$. □

On the other hand, bearing in mind Proposition 2.7 and Theorem 3.8, we can relate the moduli Δ_X^{ϕ} and $\Delta_{X,\mu}$ in the following way:

Proposition 3.10. *For all ε where the below functions are well defined we have:*

(1) $\Delta_X^{\phi}(\varepsilon) \leq \Delta_{X,\chi}(\varepsilon) \leq \Delta_X^{\phi}(2\varepsilon)$.

(2) $\Delta_X^{\phi}\big(\frac{\varepsilon}{2}\big) \leq \Delta_{X,\kappa}(\varepsilon) \leq \Delta_X^{\phi}(\varepsilon)$.

(3) $\Delta_X^{\phi}\big(\frac{\varepsilon}{\beta(U_X)}\big) \leq \Delta_{X,\beta}(\varepsilon) \leq \Delta_X^{\phi}(2\varepsilon)$.

From the above result we can derive a lower bound for the modulus of nondensifiable convexity:

Corollary 3.11. *For every $\varepsilon \in [0, \varsigma(U_X)]$, $\Delta_X^{\phi}(\varepsilon) \geq \delta_X(\varepsilon)$. In particular, if X is a Hilbert space then we have*

$$\Delta_X^{\phi}(\varepsilon) \geq 1 - \sqrt{1 - \frac{\varepsilon^2}{4}}.$$

Proof. Noticing (2) and (2) of Proposition 3.10, the inequality $\Delta_X^{\phi}(\varepsilon) \geq \delta_X(\varepsilon)$, for all $\varepsilon \in [0, \varsigma(U_X)] \subset [0, 2]$, follows. When X is a Hilbert space, it is known that $\delta_X(\varepsilon) = 1 - \sqrt{1 - \frac{\varepsilon^2}{4}}$ (see, for instance, [3]). $\square$

In Remark 1.8 has been justified the name of the function $\varepsilon_\mu(X)$: X is UNC if, and only if, $\varepsilon_\mu(X) = 0$ for $\mu \in \{\chi, \kappa, \beta\}$. To conclude this section, we will prove the same result for the ϕ-NUC-characteristic of X, $\varepsilon^{\phi}(X)$. Before, we show two examples of not NUC Banach spaces for which $\varepsilon^{\phi}(X) > 0$.

Example 3.12. *Let L^1 be the Banach space of Example 1.5. Then, $\Delta_{L^1}^{\phi}(\varepsilon) = 0$ for every $\varepsilon \in [0, 2]$. Indeed, given $\varepsilon \in [0, 2]$, let $B \subset L^1$ be the set of Example 2.5. So, as $\phi(B) = 2 \geq \varepsilon$ and $d(\mathbf{0}, B) = 1$ we infer that $\Delta_{L^1}^{\phi}(\varepsilon) = 0$ for all $\varepsilon \in [0, 2]$. Consequently, $\varepsilon^{\phi}(L^1) = 2$.*

Example 3.13. *Let $C(I)$ the Banach space of the continuous functions defined on I, endowed its usual supremum norm, and consider*

$$B := \Big\{x \in C(I) : 0 = x(0) \leq x(t) \leq x(1) = 1, \text{ for all } t \in I\Big\} \in \mathcal{B}_{\overline{\mathrm{Conv}}}(X) \cap U_{C(I)}.$$

Then, in [12, Example 3.4] we proved that $\phi(B) = 1$ and so, $\Delta_{C(I)}^{\phi}(\varepsilon) = 0$ for all $\varepsilon \in [1, \varsigma(U_X)]$. Therefore, we have $\varepsilon^{\phi}(C(I)) \geq 1$.

Corollary 3.14. *X is NUC if, and only if, $\varepsilon^{\phi}(X) = 0$.*

Proof. If $\varepsilon^{\phi}(X) = 0$, then by Theorem 3.8 $\varepsilon_\chi(X) = 0$ and, noticing Remark 1.8, X is NUC.

If X is NUC then, again by Remark 1.8, $\varepsilon_\kappa(X) = 0$. This means that $\Delta_{X,\kappa}(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$. Therefore, from (2) of Proposition 3.10, $\Delta_X^{\phi}(\varepsilon) > 0$ for each $\varepsilon \in (0, 2] \supset (0, \varsigma(U_X)]$. So, $\varepsilon^{\phi}(X) = 0$ as $\Delta_X^{\phi}(0) = 0$ (see Remark 3.5) and the result holds.

$\square$

4. Normal structure

Other important geometric property of a Banach space is that of *normal structure*, introduced by Brodskiĭ and Mil'man [28] in 1948. We recall such concept in the following lines.

For a given $B \in \mathcal{B}(X)$ which contains more than one point, the *Chebyshev radius of B* (see, for instance, [3]) is the number

$$r(B) := \inf\Big\{\sup\{\|x - y\| : x \in B\} : y \in B\Big\}.$$

In what follows, as the Chebyshev radius will be used in the below definitions and results, the closed, bounded and convex sets considered will have more than one point. That is, $\mathcal{B}_{\overline{\mathrm{Conv}}}(X)$ will be the class of bounded, closed and convex subset of X with more than one point. Likewise, we denote $\mathcal{B}^{w}_{\overline{\mathrm{Conv}}}(X)$ will be the class of bounded, closed, weakly compact and convex subset of X with more than one point.

Definition 4.1. *A set $B \in \mathcal{B}_{\overline{\mathrm{Conv}}}(X)$ is said to have normal structure if $r(B) < \mathrm{Diam}(B)$. If each $B \in \mathcal{B}_{\overline{\mathrm{Conv}}}(X)$ (resp. $B \in \mathcal{B}^{w}_{\overline{\mathrm{Conv}}}(X)$) has normal structure, X is said to have normal structure (resp. weak normal structure).*

Clearly, the above definition is equivalent to the following one: $B \in \mathcal{B}_{\overline{\mathrm{Conv}}}(X)$ has normal structure if, and only if, there exists some $x_0 \in B$ such that

$$\sup\Big\{\|x_0 - x\| : x \in B\Big\} < \mathrm{Diam}(B).$$

Often, such x_0 is called a *non-diametral point* of B.

In our next result, we prove that the concepts of normal structure of a set $B \in \mathcal{B}_{\overline{\mathrm{Conv}}}(X)$ and the DND of B, $\phi(B)$, are closely related.

Proposition 4.2. *A set $B \in \mathcal{B}_{\overline{\mathrm{Conv}}}(X)$ has normal structure if, and only if,* $\phi(B) < \mathrm{Diam}(B)$.

Proof. Let $B \in \mathcal{B}_{\overline{\mathrm{Conv}}}(X)$. If B has normal structure, take any non-diametral point of B, put x_0, and define $\alpha := \sup\{\|x_0 - x\| : x \in B\} < \mathrm{Diam}(B)$. Let $\gamma : I \longrightarrow X$ be the mapping defined as $\gamma(t) := x_0$ for all $t \in I$. Then, is clear that γ is an α-dense curve in B, and therefore $\phi(B) \leq \alpha < \mathrm{Diam}(B)$.

Now, assume $\phi(B) < \mathrm{Diam}(B)$ and take $\phi(B) < \alpha < \mathrm{Diam}(B)$. If γ is an α-dense curve in B, by the compactness of $\gamma(I)$, for $0 < \varepsilon < \mathrm{Diam}(B) - \alpha$ there exists $\{y_1, \ldots, y_n\} \subset \gamma(I)$ such that $\gamma(I) \subset \{y_1, \ldots, y_n\} + \varepsilon U_X$. Therefore, we have

$$B \subset \gamma(I) + \alpha U_X \subset \{y_1, \ldots, y_n\} + (\alpha + \varepsilon) U_X. \tag{7}$$

Now, for each $x \in B$ noticing (7) there is $1 \leq j \leq n$ such that $\|x - y_j\| \leq \alpha + \varepsilon$. Thus, if $x_0 := \frac{1}{n}\sum_{i=1}^{n} y_i \in B$, we find

$$\|x - x_0\| \leq \frac{1}{n}\Big(\sum_{i=1}^{n} \|x - y_i\|\Big) \leq \frac{1}{n}\Big(\sum_{i=1}^{n} \|x - y_j\| + \sum_{i=1}^{n} \|y_j - y_i\|\Big) \leq$$

$$\frac{1}{n}\Big(\alpha + \varepsilon + (n-1)\mathrm{Diam}(B)\Big),$$

and so, taking into account the choice of ε

$$\sup\Big\{\|x - x_0\| : x \in B\Big\} \leq \frac{1}{n}\Big(\alpha + \varepsilon + (n-1)\mathrm{Diam}(B)\Big) <$$

$$\frac{1}{n}\Big(\mathrm{Diam}(B) + (n-1)\mathrm{Diam}(B)\Big) = \mathrm{Diam}(B),$$

or, in other words, x_0 is a non-diametral point of B and consequently B has normal structure.
□

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On the other hand, the concept of normal (and weak normal) structure is specially important in the metric fixed point theory and, in particular, to state the existence of fixed points of nonexpansive mappings, i.e. of those mappings $T : C \subseteq X \longrightarrow X$ such that $\|T(X) - T(y)\| \leq \|x - y\|$ for all $x, y \in C$. The fixed point theory for such mappings is very rich; for concrete references and results, see the recent papers [20, 32].

At this point, it is convenient to recall the following:

Definition 4.3. *X is said to have the fixed point property, if every nonexpansive mapping $T : C \longrightarrow C$, with $B \in \mathcal{B}_{\overline{\mathrm{Conv}}}(X)$, has some fixed point. If this condition holds for each weakly compact $C \in \mathcal{B}_{\overline{\mathrm{Conv}}}(X)$, X is said to have the weak fixed point property.*

In 1965 Kirk [21] published his germinal paper which proved the existence of fixed points for nonexpansive mappings in reflexive Banach spaces with normal structure:

Theorem 4.4. *If X has normal structure, then has the weak fixed point property.*

To conclude our exposition, we provide a sufficient condition, in terms of the ϕ-NUC-characteristic, for X has the weak fixed point property:

Theorem 4.5. *If $\varepsilon^{\phi}(X) < 1/\beta(U_X)$, then X has the weak fixed point property.*

Proof. Assume $\varepsilon^{\phi}(X) < 1/\beta(U_X)$. Then, in view of Theorem 3.8, we have $\varepsilon_{\beta}(X) < 1$ and therefore X has normal structure (see [3, Corollary VI.4.7]). So, by virtue of Corollary 3.9, as X is reflexive the result follows noticing Theorem 4.4.
□

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