



Extended Rakočević's property

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Abstract. The purpose of this paper is to introduce and study new extension of Rakočević's property (w) and property (b) introduced by Berkani–Zariouh in [8], in connection with other Weyl type theorems and recent properties. We prove in particular, the two following results:

1. A bounded linear operator T satisfies property $(w_{\pi_{00}})$ if and only if T satisfies property (w) and $\sigma_{uf}(T) = \sigma_{uw}(T)$.
 2. T satisfies property $(gw_{\pi_{00}})$ if and only if T satisfies property $(w_{\pi_{00}})$ and $\pi_0(T) = p_0^a(T)$.
- Furthermore, some classes of operators are considered as illustrating examples.

1. Introduction and basic definitions

This paper is a continuation of [4], where we introduced and studied the new approach to a-Weyl's theorem. Our purpose is to investigate new spectral properties named (bz_1) , (gbz_1) , $(w_{\pi_{00}})$ and $(gw_{\pi_{00}})$ (see Definitions 3.1 and 4.1) for bounded linear operators as new versions of properties (bz) , $(w_{\pi_{00}^a})$ and $(gw_{\pi_{00}^a})$, which have been introduced and studied in [4]. The results obtained are summarized in the diagram presented at the end of this paper. In order to simplify, we use the same symbols and notations used in [4]. For more details on several classes and spectra originating from Fredholm theory or B-Fredholm theory, we refer the reader to [4, 8]. The following list, summarizes the same notations and symbols used in [4], which will be needed in the sequel.

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$\sigma_b(T)$: Browder spectrum of T	$\Delta^g(T) := \sigma(T) \setminus \sigma_{bw}(T)$
$\sigma_{uf}(T)$: upper semi-Fredholm spectrum of T	$\Delta_{uf}^g(T) := \sigma_a(T) \setminus \sigma_{ubf}(T)$
$\sigma_{ubf}(T)$: upper semi-B-Fredholm spectrum of T	$\Delta_{uf}(T) := \sigma_a(T) \setminus \sigma_{uf}(T)$
$\sigma_{ub}(T)$: upper semi-Browder spectrum of T	$\Delta_a^g(T) := \sigma_a(T) \setminus \sigma_{ubw}(T)$
$\sigma_w(T)$: Weyl spectrum of T	$\Delta(T) := \sigma(T) \setminus \sigma_w(T)$
$\sigma_{bw}(T)$: B-Weyl spectrum of T	$p_{00}(T)$: poles of T of finite rank
$\sigma_{uw}(T)$: upper semi-Weyl spectrum of T	$p_0^a(T)$: left poles of T
$\sigma_{ubw}(T)$: upper semi-B-Weyl spectrum of T	$p_{00}^a(T)$: left poles of T of finite rank
$\sigma_{ld}(T)$: left Drazin spectrum of T	$p_0(T)$: poles of T
$\sigma_d(T)$: Drazin spectrum of T ,	$\pi_0^a(T) := \text{iso } \sigma_a(T) \cap \sigma_p(T)$
$\sigma_p(T)$: eigenvalues of T	$\pi_0(T) := \text{iso } \sigma(T) \cap \sigma_p(T)$
$\sigma_p^f(T)$: eigenvalues of T of finite multiplicity	$\pi_{00}(T) := \text{iso } \sigma(T) \cap \sigma_p^f(T)$
$\Delta_a(T) := \sigma_a(T) \setminus \sigma_{uw}(T)$	$\pi_{00}^a(T) := \text{iso } \sigma_a(T) \cap \sigma_p^f(T)$

Definition 1.1. [2, 4, 8, 9] Let $T \in L(X)$. We say that

- (i) T satisfies property (w) if $\Delta_a(T) = \pi_{00}(T)$.
- (ii) T satisfies property (gw) if $\Delta_a^g(T) = \pi_0(T)$.
- (iii) T satisfies property (b) if $\Delta_a(T) = p_{00}(T)$.
- (iv) T satisfies property (gb) if $\Delta_a^g(T) = p_0(T)$.
- (v) T satisfies property (bz) if $\Delta_{uf}(T) = p_{00}^a(T)$.
- (vi) T satisfies property (gbz) if $\Delta_{uf}^g(T) = p_0^a(T)$.

To give the reader a good overview of the subject, we summarize in the following theorem some known results which are useful in this paper.

Theorem 1.2. [2, 4, 8] The following statements hold for every $T \in L(X)$.

- (i) Property (gb) holds for T if and only if property (b) holds for T and $p_0(T) = p_0^a(T)$.
- (ii) Property (gw) holds for T if and only if property (w) holds for T and $p_0^a(T) = \pi_0(T)$.
- (iii) $\sigma_{ubf}(T) = \sigma_{ubw}(T)$ if and only if $\sigma_{uf}(T) = \sigma_{uw}(T)$.
- (iv) Property (gbz) holds for T if and only if property (bz) holds for T .

2. Remarks and improvement

The following property has relevant role in local spectral theory: a bounded linear operator $T \in L(X)$ is said to have the *single-valued extension property* (SVEP for short) at $\lambda \in \mathbb{C}$ if for every open neighborhood U_λ of λ , the function $f \equiv 0$ is the only analytic solution of the equation $(T - \mu I)f(\mu) = 0 \quad \forall \mu \in U_\lambda$. We denote by $\mathcal{S}(T) = \{\lambda \in \mathbb{C} : T \text{ does not have the SVEP at } \lambda\}$ and we say that T has SVEP if $\mathcal{S}(T) = \emptyset$. We say that T has the SVEP on $A \subset \mathbb{C}$, if T has the SVEP at every $\lambda \in A$. (For more details about this property, we refer the reader to [1]).

From [4, Theorem 2.7], we have $T \in L(X)$ has the SVEP on $(\sigma_{uf}(T))^c$ if and only if T has the SVEP on $(\sigma_{ubf}(T))^c$, where $(\sigma_{uf}(T))^c$ and $(\sigma_{ubf}(T))^c$ means respectively, the complementary of upper semi-Fredholm spectrum and the complementary of upper semi-B-Fredholm spectrum of an operator T .

Lemma 2.1. For every $T \in L(X)$, we have the following equivalence

$$\sigma_{uw}(T) = \sigma_w(T) \iff \sigma_{ubw}(T) = \sigma_{bw}(T).$$

Proof. \Leftarrow) Let $\lambda \notin \sigma_{uw}(T)$ be arbitrary and without loss of generality we can assume that $\lambda = 0$. So T is an upper semi-Weyl and in particular, it is an upper semi-B-Weyl. Since $\sigma_{ubw}(T) = \sigma_{bw}(T)$, it follows that $0 \notin \sigma_w(T)$ and hence $\sigma_{uw}(T) = \sigma_w(T)$.

\Rightarrow) Suppose that $0 \notin \sigma_{ubw}(T)$. Then from the punctured neighborhood theorem for semi-B-Fredholm

operators [6, Corollary 3.2], there exists $\epsilon > 0$ such that $T - \mu I$ is an upper semi-Fredholm and $\text{ind}(T - \mu I) = \text{ind}(T)$ for every $0 < |\mu| < \epsilon$. This implies that $\mu \notin \sigma_{uw}(T) = \sigma_w(T)$ and so $\text{ind}(T - \mu I) = \text{ind}(T) = 0$. Hence $\sigma_{ubw}(T) = \sigma_{bw}(T)$. \square

Now, we give an improvement of some results of [7]. But to give the reader a global and precise view, we recall here these results in the same order of the paper [7].

Lemma I: [7, Lemma 2.1] Let $T \in L(X)$. If T^* has the SVEP on $(\sigma_{uw}(T))^C$, then $\sigma(T) = \sigma_a(T)$, $\sigma_{uw}(T) = \sigma_w(T)$.

Theorem I: [7, Theorem 2.2] Let $T \in L(X)$. Then the following assertions are equivalent.

- (a) T^* has the SVEP on $(\sigma_{uw}(T))^C$;
- (b) T satisfies property (b) and $\sigma_{uw}(T) = \sigma_w(T)$.

Lemma II: [7, Lemma 2.4] Let $T \in L(X)$. If T^* has the SVEP on $(\sigma_{ubw}(T))^C$, then $\sigma(T) = \sigma_a(T)$, $\sigma_{ubw}(T) = \sigma_{bw}(T) = \sigma_d(T)$.

Theorem II: [7, Theorem 2.5] Let $T \in L(X)$. Then the following assertions are equivalent.

- (c) T^* has the SVEP on $(\sigma_{ubw}(T))^C$;
- (d) T satisfies property (gb) and $\sigma_{ubw}(T) = \sigma_{bw}(T)$.

Using Lemma 2.1, we remark that Lemma II is an immediate consequence [since $(\sigma_{uw}(T))^C \subset (\sigma_{ubw}(T))^C$] of Lemma I and can be combined in a single as follows:

Lemma 2.2. Let $T \in L(X)$. If T^* has the SVEP on $(\sigma_{uw}(T))^C$, then we have

- (i) $\sigma(T) = \sigma_a(T)$, $\sigma_{ub}(T) = \sigma_{uw}(T) = \sigma_w(T) = \sigma_b(T)$.
- (ii) $\sigma_{ld}(T) = \sigma_{ubw}(T) = \sigma_{bw}(T) = \sigma_d(T)$, $p_0(T) = p_0^a(T)$ and $p_{00}(T) = p_{00}^a(T)$.

Proof. (i) See the proof of [7, Lemma 2.1].

(ii) From Lemma 2.1, we have $\sigma_{ubw}(T) = \sigma_{bw}(T)$. It is well known that $\sigma_w(T) = \sigma_b(T)$ is equivalent to say that $\sigma_{bw}(T) = \sigma_d(T)$, and $\sigma_{uw}(T) = \sigma_{ub}(T)$ is equivalent to say that $\sigma_{ubw}(T) = \sigma_{ld}(T)$. Thus $\sigma_{ld}(T) = \sigma_{ubw}(T) = \sigma_{bw}(T) = \sigma_d(T)$. So $p_0(T) = \sigma(T) \setminus \sigma_d(T) = \sigma_a(T) \setminus \sigma_d(T) = p_0^a(T)$, and this implies that $p_{00}(T) = p_{00}^a(T)$. \square

We prove in the following corollary, that the statements (a), (b), (c) and (d) of Theorem I and Theorem II are equivalent.

Corollary 2.3. For every $T \in L(X)$, the following statements are equivalent.

- (i) T^* has the SVEP on $(\sigma_{uw}(T))^C$;
- (ii) T satisfies property (b) and $\sigma_{uw}(T) = \sigma_w(T)$;
- (iii) T^* has the SVEP on $(\sigma_{ubw}(T))^C$;
- (iv) T satisfies property (gb) and $\sigma_{ubw}(T) = \sigma_{bw}(T)$.

Proof. (i) \iff (ii) See [7, Theorem 2.2].

(iii) \iff (iv) See [7, Theorem 2.5].

(iii) \implies (i) Obvious.

(i) \implies (iv) If T^* has the SVEP on $(\sigma_{uw}(T))^C$, then $\sigma_{uw}(T) = \sigma_w(T)$. From Lemma 2.2, $\Delta_a^g(T) = p_0(T)$ and $\sigma_{ubw}(T) = \sigma_{bw}(T)$. \square

3. On the extension of property (b)

We begin this section by the following definition, in which we introduce new spectral properties named (bz_1) and (gbz_1) which are extensions (see Theorem 3.3 bellow) of properties (b) and (gb), respectively.

Definition 3.1. A bounded linear operator $T \in L(X)$ is said to satisfy:

- (i) Property (bz_1) if $\Delta_{uf}(T) = p_{00}(T)$, or equivalently $\sigma_a(T) = \sigma_{uf}(T) \sqcup p_{00}(T)$.
- (ii) Property (gbz_1) if $\Delta_{uf}^g(T) = p_0(T)$, or equivalently $\sigma_a(T) = \sigma_{ubf}(T) \sqcup p_0(T)$.

Example 3.2. Here and elsewhere the operators R and L are defined on the Hilbert space $l^2(\mathbb{N})$ (which is usually denoted by l^2) by

$$R(x_1, x_2, x_3, \dots) = (0, x_1, x_2, x_3, \dots) \text{ and } L(x_1, x_2, x_3, \dots) = (x_2, x_3, x_4, \dots).$$

1. It is well known that $\sigma_a(R) = C(0, 1)$, where $C(0, 1)$ is the unit circle of \mathbb{C} and so $p_{00}(R) = p_0(R) = \emptyset$. From [4, Example 2.2], we have $\sigma_{ubf}(R) = \sigma_{uf}(R) = C(0, 1)$. On the Banach space $l^2 \oplus l^2$, we define the operator T by $T = R \oplus 0$. Then $\sigma_a(T) = \sigma_{uf}(T) = C(0, 1) \cup \{0\}$ and $p_{00}(T) = p_0(T) = \emptyset$. So $\Delta_{uf}(T) = p_{00}(T)$ and this means that T satisfies property (bz_1) . On the other hand, it is easily seen that $\sigma_{ubf}(T) = C(0, 1)$. So $\Delta_{uf}^g(T) = \{0\} \neq p_0(T)$, and this means that T does not satisfy property (gbz_1) .
2. Let T be the operator defined on l^2 by $T(x_1, x_2, x_3, \dots) = (\frac{x_2}{2}, \frac{x_3}{3}, \frac{x_4}{4}, \dots)$. It is easily seen that $\sigma_a(T) = \sigma_{uf}(T) = \sigma_{ubf}(T) = \{0\}$. Since $\text{asc}(A) = \infty$, it follows that $p_0(T) = p_{00}(T) = \emptyset$. So $\Delta_{uf}(T) = p_{00}(T)$ and $\Delta_{uf}^g(T) = p_0(T)$; this means that T satisfies properties (gbz_1) and (bz_1) .
3. Let T be the operator defined on the Banach space $l^2 \oplus l^2$ by $T = L \oplus R$. We have $\sigma_a(T) = \sigma_a(L) \cup \sigma_a(R) = D(0, 1)$, where $D(0, 1)$ is the closed unit disc of \mathbb{C} and so $p_{00}(T) = p_0(T) = \emptyset$. It is (see also [4]) easily seen that $\sigma_{uf}(T) = C(0, 1)$. So T does not satisfy neither property (bz_1) nor property (gbz_1) .

As we have showed in Example 3.2, there exist in nature operators who do not satisfy neither property (bz_1) nor property (gbz_1) . Nonetheless, we explore some conditions which guarantee these properties for a bounded linear operator. We begin by giving, a relationship between properties (bz_1) and (b) , and between properties (gbz_1) and (gb) .

Theorem 3.3. The following statements hold for every $T \in L(X)$.

- (i) T satisfies property (bz_1) if and only if T satisfies property (b) and $\sigma_{uf}(T) = \sigma_{uw}(T)$.
- (ii) T satisfies property (gbz_1) if and only if T satisfies property (gb) and $\sigma_{ubf}(T) = \sigma_{ubw}(T)$.

Proof. (i) Suppose that T satisfies property (bz_1) , that's $\Delta_{uf}(T) = p_{00}(T)$. As $\Delta_a(T) \subset \Delta_{uf}(T)$, then $\Delta_a(T) \subset p_{00}(T)$ and since the inclusion $\Delta_a(T) \supset p_{00}(T)$ is always true, it follows that $\Delta_a(T) = p_{00}(T)$. Moreover, we have $\sigma_{uf}(T) = \sigma_a(T) \setminus p_{00}(T) = \sigma_{uw}(T)$. The converse is obvious.

(ii) Assume that T satisfies property (gbz_1) , that's $\Delta_{uf}^g(T) = p_0(T)$. As $\Delta_a^g(T) \subset \Delta_{uf}^g(T)$, then $\Delta_a^g(T) \subset p_0(T)$ and since the inclusion $\Delta_a^g(T) \supset p_0(T)$ is always true, it follows that $\Delta_a^g(T) = p_0(T)$. Furthermore, we have $\sigma_{ubf}(T) = \sigma_a(T) \setminus p_0(T) = \sigma_{ubw}(T)$. The converse is also clear. \square

Remark 3.4. Generally, property (b) and property (gb) do not imply property (bz_1) and property (gbz_1) , respectively. For this, we consider the left shift operator L defined on l^2 , we have $\sigma_a(L) = \sigma_{ubw}(L) = \sigma_{uw}(L) = D(0, 1)$ and $p_0(L) = \emptyset$. So L satisfies property (gb) and then property (b) . It is already showed in [4, Example 2.3] that $\sigma_{uf}(L) = C(0, 1)$. Thus $\Delta_{uf}(L) \neq \emptyset = p_{00}(L)$ and L does not satisfy property (bz_1) . Hence it does not satisfy property (gbz_1) . Note also that from [4, Remark 6], we have $\sigma_{ubf}(L) = C(0, 1)$.

In the next corollary, we show that an operator satisfying property (gbz_1) , satisfies property (bz_1) . The operator $T = R \oplus 0$ given in Example 3.2, proves that the converse is not generally true. Furthermore, we give a condition of equivalence between them.

Corollary 3.5. Let $T \in L(X)$. The following statements are equivalent.

- (i) T satisfies property (gbz_1) ;
- (ii) T satisfies property (bz_1) and $p_0(T) = p_0^a(T)$.

Proof. (i) \implies (ii) The inclusion $\Delta_{uf}(T) \subset \Delta_{uf}^g(T)$ is always true. As T satisfies property (gbz_1) then $\Delta_{uf}(T) \subset p_0(T)$. Hence $\Delta_{uf}(T) = p_{00}(T)$. Moreover,

$$p_0(T) = \text{iso } \sigma_a(T) \cap (\sigma_{ubf}(T))^C = \text{iso } \sigma_a(T) \cap (\sigma_{ubw}(T))^C = p_0^a(T).$$

(i) \longleftarrow (ii) Is an immediate consequence of Theorems 3.3 and 1.2. \square

Corollary 3.6. *For every $T \in L(X)$, we have the following statements.*

- (i) T satisfies property (bz_1) if and only if T satisfies (bz) and $p_{00}(T) = p_{00}^a(T)$.
- (ii) T satisfies property (gbz_1) if and only if T satisfies (gbz) and $p_0(T) = p_0^a(T)$.

Proof. (i) Since T satisfies property (bz_1) , then

$p_{00}(T) = \text{iso } \sigma_a(T) \cap (\sigma_{uf}(T))^C = \text{iso } \sigma_a(T) \cap (\sigma_{uw}(T))^C = p_{00}^a(T)$. So T satisfies property (bz) . The converse is clear.

(ii) Follows directly from Corollary 3.5, the first assertion (i) and Theorem 1.2. \square

Remark 3.7. *From Corollary 3.6, if $T \in L(X)$ satisfies property (bz_1) or property (gbz_1) , then it satisfies property (bz) . But the converse is not true, as the following example shows. We consider the operator T defined on the Banach space $l^2 \oplus l^2$ by $T = R \oplus P$, where P is the operator defined on l^2 by $P(x_1, x_2, x_3, \dots) = (0, x_2, x_3, x_4, \dots)$. Then $\sigma_a(T) = C(0, 1) \cup \{0\}$, $\sigma_{uf}(T) = \sigma_{uw}(T) = C(0, 1)$, $p_0^a(T) = p_{00}^a(T) = \{0\}$. So $\Delta_{uf}(T) = p_{00}^a(T)$. This means that T satisfies property (bz) [or equivalently (gbz)]. But T does not satisfy neither property (bz_1) nor property (gbz_1) , since $\sigma_{ubf}(T) = \sigma_{ubw}(T) = C(0, 1)$ and $p_0(T) = p_{00}(T) = \emptyset$.*

Corollary 3.8. *Let $T \in L(X)$. If T and T^* have the SVEP on $(\sigma_{uf}(T))^C$, then T satisfies properties (gbz_1) and (bz_1) .*

Proof. As T have the SVEP on $(\sigma_{uf}(T))^C$ then from [4, Theorem 2.7], T satisfies property (gbz) . By Lemma 2.2, the SVEP for T^* on $(\sigma_{uf}(T))^C$ implies that $p_0(T) = p_0^a(T)$. Thus by Corollary 3.6, property (gbz_1) holds for T and then property (bz_1) holds also for T . \square

Remark 3.9. *We cannot guarantee the property (gbz_1) , for an operator T , if we restrict only to the SVEP of T on $(\sigma_{uf}(T))^C$ or to the SVEP of T^* on $(\sigma_{uf}(T))^C$. The operator $T = R \oplus P$ defined above has the SVEP, but it does not satisfy properties (gbz_1) and (bz_1) . Here T has the SVEP, but $\mathcal{S}(T^*) = \mathcal{S}(L) = \{\lambda \in \mathbb{C} : 0 \leq |\lambda| < 1\}$ and $\sigma_{uf}(T) = C(0, 1)$. On the other hand, It is already mentioned that the left shift operator L does not satisfy the properties (bz_1) and (gbz_1) , even if $L^* = R$ has the SVEP.*

4. On the extended Rakočević's property

Similarly to the Definition 3.1, we introduce in the next definition a new extension of Rakočević's property (w) , and a new extension of property (gw) .

Definition 4.1. *A bounded linear operator $T \in L(X)$ is said to satisfy:*

- i) Property $(w_{\pi_{00}})$ if $\Delta_{uf}(T) = \pi_{00}(T)$, or equivalently $\sigma_a(T) = \sigma_{uf}(T) \sqcup \pi_{00}(T)$.
- ii) Property $(gw_{\pi_{00}})$ if $\Delta_{uf}^g(T) = \pi_0(T)$, or equivalently $\sigma_a(T) = \sigma_{ubf}(T) \sqcup \pi_0(T)$.

Example 4.2.

1. The operator $T = R \oplus 0$ given in Example 3.2, satisfies property $(w_{\pi_{00}})$, since $\Delta_{uf}(T) = \pi_{00}(T) = \emptyset$. But it does not satisfy property $(gw_{\pi_{00}})$, since $\Delta_{uf}^g(T) = \{0\} \neq \pi_0(T) = \emptyset$. Observe that $\sigma(T) = D(0, 1)$.

2. Let I the identity operator on an infinite dimensional Banach space X . It is clear that $\sigma_a(I) = \sigma_{uf}(I) = \pi_0(I) = \{1\}$ and $\sigma_{ubf}(I) = \pi_{00}(I) = \emptyset$. So $\Delta_{uf}(I) = \emptyset = \pi_{00}(I)$ and $\Delta_{uf}^g(I) = \{1\} = \pi_0(I)$. So the identity operator satisfies the properties $(w_{\pi_{00}})$ and $(gw_{\pi_{00}})$.

3. Let L be the left shift operator defined on l^2 . It is already mentioned above that $\sigma_{uf}(L) = \sigma_{ubf}(L) = C(0, 1)$. As $\sigma_a(L) = \sigma(L) = D(0, 1)$, then $\pi_0(L) = \pi_{00}(L) = \emptyset$. So the left shift operator does not satisfy neither property $(w_{\pi_{00}})$ nor property $(gw_{\pi_{00}})$.

We prove in the next theorem that properties $(w_{\pi_{00}})$ and $(gw_{\pi_{00}})$ are respectively, extensions of properties (w) and (gw) .

Theorem 4.3. *The following statements hold for every $T \in L(X)$.*

- (i) *T satisfies property $(w_{\pi_{00}})$ if and only if T satisfies property (w) and $\sigma_{uf}(T) = \sigma_{uw}(T)$.*
- (ii) *T satisfies property $(gw_{\pi_{00}})$ if and only if T satisfies property (gw) and $\sigma_{ubf}(T) = \sigma_{ubw}(T)$.*

Proof. (i) Suppose that T satisfies property $(w_{\pi_{00}})$, that's $\Delta_{uf}(T) = \pi_{00}(T)$. Then $\Delta_a(T) \subset \pi_{00}(T)$. If $\lambda \in \pi_{00}(T)$, then $\lambda \in \text{iso } \sigma(T)$ and so T has the SVEP at λ . Since $T - \lambda I$ is upper semi-Fredholm, then $\text{asc}(T - \lambda I)$ is finite and then $\lambda \notin \sigma_{uw}(T)$. Hence $\Delta_a(T) = \pi_{00}(T)$, and consequently, $\sigma_{uf}(T) = \sigma_a(T) \setminus \pi_{00}(T) = \sigma_{uw}(T)$. The converse is clear.

(ii) $\Delta_a^g(T) \subset \pi_0(T)$, since T satisfies property $(gw_{\pi_{00}})$. Moreover, $\pi_0(T) = \text{iso } \sigma_a(T) \cap (\sigma_{ubf}(T))^C$. By the proof of [5, Theorem 2.8], $\pi_0(T) = \text{iso } \sigma_a(T) \cap (\sigma_{ubw}(T))^C \subset \Delta_a^g(T)$. Hence $\Delta_a^g(T) = \pi_0(T)$, and this means that T satisfies property (gw) . Moreover, $\sigma_{ubf}(T) = \sigma_a(T) \setminus \pi_0(T) = \sigma_{ubw}(T)$. The converse is also clear. \square

Remark 4.4. *Note that the sufficient condition “ $\sigma_{uf}(T) = \sigma_{uw}(T)$ ” [which is equivalent to the condition “ $\sigma_{ubf}(T) = \sigma_{ubw}(T)$ ”] assumed in the two statements of Theorem 4.3 is crucial, as we can see in the following example: The left shift operator L satisfies properties (gw) and (w) , since $\Delta_a^g(L) = \pi_0(L) = \emptyset$ and $\Delta_a(L) = \pi_{00}(L) = \emptyset$. But it does not satisfy neither $(w_{\pi_{00}})$ nor $(gw_{\pi_{00}})$, since $\Delta_{uf}(L) \neq \emptyset = \pi_{00}(L)$ and $\Delta_{uf}^g(L) \neq \emptyset = \pi_0(L)$.*

In the next corollary, we show that an operator satisfying property $(gw_{\pi_{00}})$, satisfies property $(w_{\pi_{00}})$. The operator $T = S \oplus 0$ defined on $l^2 \oplus l^2$, where S is defined on l^2 by $S(x_1, x_2, \dots) = (0, \frac{x_1}{2}, \frac{x_2}{3}, \dots)$ proves that the converse is not generally true. Indeed, we have $\sigma_a(T) = \sigma_{uf}(T) = \{0\}$ and $\pi_{00}(T) = \emptyset$. So T satisfies property $(w_{\pi_{00}})$. But it does not satisfy property $(gw_{\pi_{00}})$, since $\pi_0(T) = \{0\}$.

Furthermore, we give a condition of equivalence between them.

Corollary 4.5. *Let $T \in L(X)$. The following statements are equivalent.*

- (i) *T satisfies property $(gw_{\pi_{00}})$;*
- (ii) *T satisfies property $(w_{\pi_{00}})$ and $\pi_0(T) = p_0^a(T)$.*

Proof. (i) \implies (ii) Since $\Delta_{uf}(T) \subset \Delta_{uf}^g(T)$ and T satisfies property $(gw_{\pi_{00}})$, then $\Delta_{uf}(T) \subset \pi_{00}(T)$. On the other hand, $\pi_{00}(T) \subset \pi_0(T) = \Delta_{uf}^g(T)$. Hence $\Delta_{uf}(T) = \pi_{00}(T)$. Moreover,

$$\pi_0(T) = \text{iso } \sigma_a(T) \cap (\sigma_{ubf}(T))^C = \text{iso } \sigma_a(T) \cap (\sigma_{ubw}(T))^C = p_0^a(T).$$

(i) \longleftarrow (ii) Is an immediate consequence of Theorems 4.3 and 1.2. \square

Remark 4.6. *It is easy to get that if $T \in L(X)$ satisfies property $(w_{\pi_{00}})$, then*

$$p_{00}(T) = \pi_{00}(T) = p_{00}^a(T).$$

In addition, if T satisfies property $(gw_{\pi_{00}})$, then $p_0(T) = \pi_0(T) = p_0^a(T)$, and this implies that $p_{00}(T) = \pi_{00}(T) = p_{00}^a(T)$.

According to [4], an operator $T \in L(X)$ is said to satisfy property $(w_{\pi_{00}^a})$ if $\Delta_{uf}(T) = \pi_{00}^a(T)$ and is said to satisfy property $(gw_{\pi_{00}^a})$ if $\Delta_{uf}^g(T) = \pi_0^a(T)$.

Corollary 4.7. *For every $T \in L(X)$, we have the following statements.*

- (i) *T satisfies property $(w_{\pi_{00}})$ if and only if T satisfies (bz_1) and $\pi_{00}(T) = p_{00}(T)$.*
- (ii) *T satisfies property $(gw_{\pi_{00}})$ if and only if T satisfies (gbz_1) and $\pi_0(T) = p_0(T)$.*

Remark 4.8. *There exist operators satisfying property (bz_1) [resp., property (gbz_1)] which do not satisfy property $(w_{\pi_{00}})$ [resp., property $(gw_{\pi_{00}})$].*

1. Let $T \in L(l^2)$ be defined by $T(x_1, x_2, \dots) = (\frac{x_2}{2}, \frac{x_2}{3}, \dots)$. Property (bz_1) holds for T , since $\sigma_a(T) = \sigma_{uf}(T) = \{0\}$ and $p_{00}(T) = \emptyset$. While property $(w_{\pi_{00}})$ does not hold for T , since $\pi_{00}(T) = \{0\}$. Note that here $p_{00}(T) \neq \pi_{00}(T)$.

2. Let Q be defined on $l^1(\mathbb{N})$ by $Q(x_1, x_2, \dots) = (0, \alpha_1 x_1, \alpha_2 x_2, \dots, \alpha_k x_k, \dots)$, where (α_i) is a sequence of complex numbers such that $0 < |\alpha_i| \leq 1$ and $\sum_{i=1}^{\infty} |\alpha_i| < \infty$.

And we define the operator T on $l^1(\mathbb{N}) \oplus l^1(\mathbb{N})$ by $T = Q \oplus 0$. We have $\sigma(T) = \sigma_a(T) = \{0\}$ and $\pi_0(T) = \{0\}$. It is easily seen that the range $\mathcal{R}(T^n)$ of T^n is not closed for any $n \in \mathbb{N}$, so that $\sigma_{ubf}(T) = \{0\}$ and $p_0(T) = \emptyset$. Thus $\Delta_{uf}^g(T) = p_0(T)$ and $\Delta_{uf}^g(T) \neq \pi_0(T)$, as desired. Note also that here $p_0(T) \neq \pi_0(T)$.

Remark 4.9.

1. The properties $(w_{\pi_{00}})$ and $(w_{\pi_{00}^a})$ are independent.

We consider the operator $S = R \oplus 0$ defined on the Banach space $l^2 \oplus \mathbb{C}^n$. We have $\sigma_a(S) = C(0, 1) \cup \{0\}$, $\sigma_{uf}(S) = C(0, 1)$, $\pi_{00}^a(S) = \{0\}$ and $\pi_{00}(S) = \emptyset$. So $\Delta_{uf}(S) = \pi_{00}^a(S)$, but $\Delta_{uf}(S) \neq \pi_{00}(S)$.

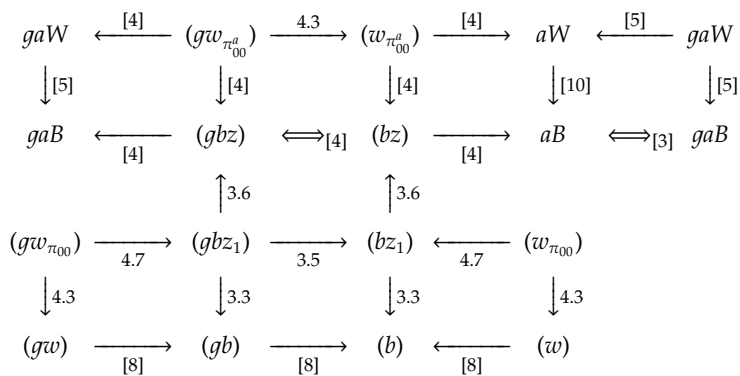
Now, we consider the operator $U = R \oplus T$ defined on the $l^2 \oplus l^2$, where T is defined l^2 by $T(x_1, x_2, \dots) = (\frac{x_2}{2}, \frac{x_2}{3}, \dots)$. Then $\sigma_a(U) = C(0, 1) \cup \{0\}$, $\sigma_{uf}(U) = C(0, 1) \cup \{0\}$, $\pi_0^a(U) = \pi_{00}^a(U) = \{0\}$ and $\pi_0(U) = \pi_{00}(U) = \emptyset$. So $\Delta_{uf}(U) = \pi_{00}(U)$, but $\Delta_{uf}(U) \neq \pi_{00}^a(U)$.

2. The properties $(gw_{\pi_{00}})$ and $(gw_{\pi_{00}^a})$ are independent.

We take the operator $V = R \oplus A$ defined $l^2 \oplus l^2$, where A is defined by $A(x_1, x_2, \dots) = (0, \frac{-x_1}{2}, 0, \dots)$. V satisfies property $(gw_{\pi_{00}^a})$, since $\sigma_a(V) = C(0, 1) \cup \{0\}$, $\sigma_{ubf}(V) = C(0, 1)$ and $\pi_0^a(V) = \{0\}$. But, it does not satisfy property $(gw_{\pi_{00}})$, since $\pi_0(V) = \emptyset$. On the other hand, the operator U considered in the first point, does not satisfy property $(w_{\pi_{00}^a})$, and this implies that it does not satisfy also property $(gw_{\pi_{00}^a})$. But, it satisfies property $(gw_{\pi_{00}})$, since $\sigma_{ubf}(U) = C(0, 1) \cup \{0\}$.

Conclusion:

As conclusion, we give a summary of the results obtained in this paper. In the following diagram, which extends the one that has been presented in [4], arrows signify implications between the properties introduced in this paper and those that have been introduced in [4], and other Weyl type theorems (generalized or not). The numbers near the arrows are references to the results in the present paper (numbers without brackets) or to the bibliography therein (numbers in square brackets).



Moreover, counterexamples were given to show that the reverse of each implication (presented with number without bracket) in the diagram is not true. Nonetheless, it was proved that under some extra assumptions, these implications are equivalences.

Remark: This paper will be followed by a second one (in preparation), in which we will consider a “Weyl-type” version of the results obtained, by introducing and studying the following properties.

Definition 4.10. [article in preparation] An operator $T \in L(X)$ is said to satisfy:

1. property (bz_2) if $\Delta_{uf}(T) = p_0(T)$.
2. property (bz_3) if $\Delta_{uf}(T) = p_0^a(T)$.
3. property (w_{π_0}) if $\Delta_{uf}(T) = \pi_0(T)$.
4. property $(w_{\pi_0^a})$ if $\Delta_{uf}(T) = \pi_0^a(T)$.

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