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Modified Inertial Subgradient Extragradient Method for a Variational Inequality Problem in Hilbert Space

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Abstract. In this paper, we introduce and study strong convergence of modified inertial-type subgradient extragradient method for finding solution of variational inequality problem and common fixed point problem of an infinite family of demimetric mappings in a Hilbert space. Our results substantially improve and generalize some well-known results in the literature.

1. Introduction

Let C be a nonempty, closed and convex subset of a real Hilbert space H. A mapping $G : D(G) \subset H \to H$ is said to be: (i) *monotone* if for all $x, y \in D(G)$,

$$\langle Gx - Gy, x - y\rangle \geq 0, \tag{1}$$

holds, where $D(G)$ denotes the domain of G. (ii) $L-$*Lipschitz* if there exists $L > 0$ and for all $x, y \in D(G)$, we have

$$\|Gx - Gy\| \leq L\|x - y\|.$$

The set of fixed points of G is given by $F(G) = \{x \in D(G) : G(x) = x\}$. Let $G : C \to H$ be a nonlinear mapping. The variational inequality problem introduced and studied by Stampacchia [31] is to:

$$\text{find } u \in C \text{ such that } \langle Gu, v - u\rangle \geq 0,\ \forall v \in C. \tag{2}$$

The set of solution of variational inequality problem is denoted by $VI(C, G)$. The problem of solving a variational inequality of the form (2) has been intensively studied by numerous authors due to its various applications in several physical problems, such as in operational research, economics and engineering

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design (see [4–6]). The ideas and techniques of variational inequalities are being applied in a variety of diverse areas of sciences and proving to be productive and innovative. It has been shown that this theory provides a simple, natural, and unified framework for a general treatment of many unrelated problems. In recent years, considerable interest has been shown in developing various extensions and generalizations of variational inequalities, both for mathematical theories and for their applications. Iterative methods for solving these problems have been proposed and analyzed by many authors (see, for example, [12, 13, 20] and references therein). These methods include extragradient method introduced by Korpelevič [24]. Censor et al.[14] modified the method proposed by Korpelevič [24] by replacing one of the projections with a projection onto a half-space. The modified method of Censor et al. [14] is known as subgradient extragradient method and is studied in [15, 16]. Under some appropriate condtions, the subgradient extragradient method in [14] converges weakly to a solution of a variational inequality. In 2014, Kraikaew and Saejung [25] introduced a subgradient extragradient method which converges strongly to a solution of the variational inequality as follows:

Theorem 1.1. (Kraikaew and Saejung [25]) *Let $S : H \to H$ be a quasi-nonexpansive mapping such that $I - S$ is demiclosed at zero and $G : C \to H$ be a monotone and $L-$Lipschitz mapping. Let τ be a positive real number such that $\tau L < 1$. Suppose that $VI(C, G) \cap F(S) \neq \emptyset$. Let $\{x_n\} \subset H$ be a sequence generated by*

$$\begin{cases} x_0 \in H; \\ y_n = P_C(x_n - \tau G(x_n)), \\ T_n = \{w \in H : \langle x_n - \tau G(x_n) - y_n, w - y_n\rangle \leq 0\}, \\ z_n = \alpha_n x_0 + (1 - \alpha_n)P_{T_n}(x_n - \tau G(y_n)), \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)Sz_n, \end{cases} \tag{3}$$

where $\beta_n \subset [a, b] \subset]0, 1[$ for some $a, b \in (0, 1)$, $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$ and P_C denotes the metric projection of C onto H. Then $\{x_n\}$ converges strongly to $P_{VI(C,G)\cap F(S)}x_0$.

In order to speed up rate of convergence of an iterative method (to reduce computational cost), Polyak [29] studied the heavy ball method, an inertial extrapolation to accelerate process for minimizing a smooth convex function h. The inertial algorithm is the following two-step iterative method:

$$\begin{cases} x_0, x_1 \in H, \\ w_n = x_n + \theta_n(x_n - x_{n-1}) \\ x_{n+1} = w_n + \lambda_n \nabla h(x_n), \ n \geq 1 \end{cases} \tag{4}$$

where $\theta_n \in [0, 1)$ and λ_n is a step-size parameter to be chosen sufficiently small. The main difference compared to a standard gradient method is that in each iteration, the extrapolated term w_n is used instead of x_n. It is remarkable that this minor change significantly improves the performance of the method. The term $\theta_n(x_n - x_{n-1})$ is called inertial; hence method (4) is called inertial method. In view of great potential, study of inertial-type method has attracted the attention of several researchers, see for example [2, 3, 11, 17, 37]. In 2015, Bot and Csetnek [7] introduced the so-called inertial hybrid proximal extragradient method, which combines inertial type method and hybrid proximal extragradient method for a maximal monotone operator. Inspired by this method, Dong et al. [8] proposed an algorithm by including inertial terms in the extragradient method as follows:

$$\begin{cases} x_0, x_1 \in H, \\ w_n = x_n + \theta_n(x_n - x_{n-1}) \\ y_n = P_C(w_n - \tau G w_n), \\ x_{n+1} = (1 - \beta_n)w_n + \beta_n P_C(w_n - \tau G y_n). \end{cases} \tag{5}$$

Under appropriate conditions, they proved that the sequence $\{x_n\}$ converges weakly to an element of $VI(C, G)$. Recently, Thong and Hieu [36] combined the inertial technique with the subgradient extragradient method and proposed a method, called inertial subgradient extragradient method, for solving variational inequality problem (2) in a Hilbert spaces as follows:

$$\begin{cases} x_0, x_1 \in H, \\ w_n = x_n + \theta_n(x_n - x_{n-1}) \\ y_n = P_C(w_n - \tau G w_n), \\ T_n = \{x \in H : \langle w_n - \tau G w_n - y_n, x - y_n \rangle \leq 0\}, \\ x_{n+1} = P_{T_n}(w_n - \tau G y_n) \end{cases} \tag{6}$$

where $\tau > 0$, $\theta_n \geq 0$ are suitable parameters. Under some appropriate conditions imposed on these parameters, they proved weak convergence of the sequence $\{x_n\}$ generated by (6) to some point in V I(C,G). Motivated and inspired by the work of Kraikaew and Saejund [25] and Thong and Hiew [36], in this paper, we study common fixed point problem of a new modified inertial subgradient extragradient method for an infinite family of demimetric mappings $\{T_i\}_{i=1}^{\infty}$ introduced by Takahash [34] and a monotone L-Lipschitz mapping G on a Hilbert space. We prove a strong convergence theorem for modified inertial method to find a solution of variational inequality problem of G and a common fixed point of the family $\{T_i\}_{i=1}^{\infty}$.

2. Preliminaries

The metric projection from H onto C is the mapping: $P_C : H \to C$, for each $x \in H$, there exists a unique point $z = P_C(x)$ such that

$$\|x - z\| = \inf_{y \in C} \|x - y\|.$$

Lemma 2.1. [18]*Let $x \in H$ and $z \in C$ be an arbitrary point. Then we have*

(i) *$z = P_C(x)$ if and only if the following relation holds*

$$\langle x - z, y - z \rangle \leq 0, \quad \forall y \in C. \tag{7}$$

(ii) *There holds the relation*

$$\langle P_C(x) - P_C(y), x - y \rangle \geq \|P_C(x) - P_C(y)\|^2, \ \forall x, y \in H.$$

(iii) *For $x \in H$ and $y \in C$*

$$\|y - P_C(x)\|^2 + \|x - P_C(x)\|^2 \leq \|x - y\|^2.$$

Lemma 2.2. **[27]** *Let H be a real Hilbert space. Then for all $x, y, z \in H$ and $\alpha, \beta, \gamma \in [0, 1]$ with $\alpha + \beta + \gamma = 1$, we have*

$$\|\alpha x + \beta y + \gamma z\|^2 = \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x - y\|^2 - \alpha\gamma\|x - z\|^2 - \beta\gamma\|y - z\|^2.$$

Lemma 2.3. ([33]) *For any $x, y, u, v \in H$, we have the following statements:*

(i) $|\langle x, y \rangle| \leq \|x\|\|y\|$;

(ii) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle$ (the subdifferential inequality).

(iii) $2\langle x - y, u - v \rangle = \|x - v\|^2 + \|y - u\|^2 - \|x - u\|^2 - \|y - v\|^2$.

Recently, Takahashi [34] introduced the notion of a new nonlinear mapping in a smooth, strictly convex and reflexive Banach space as follows:

Definition 2.4. *Let E be a smooth, strictly convex and reflexive Banach space, C be a nonempty, closed and convex subset of E and let η be a real number in $(-\infty, 1)$. Then a mapping $T : C \to E$ with $F(T) \neq \emptyset$, is called $\eta -$ demimetric [34] if,*

$$\langle x - q, J(x - Tx)\rangle \geq \frac{1-\eta}{2}\|x - Tx\|^2,$$

for any $x \in C$ and $q \in F(T)$, where J is the duality mapping on E.
In a Hilbert space H, the above definition becomes: A mapping $T : C \to H$ with $F(T) \neq \emptyset$, is called $\eta -$ demimetric if

$$\langle x - q, x - Tx\rangle \geq \frac{1-\eta}{2}\|x - Tx\|^2, \tag{8}$$

for any $x \in C$ and $q \in F(T)$.
A mapping $T : C \to H$ is a generalized hybrid if there exist $\alpha, \beta \in \mathbb{R}$ such that, for all $x, y \in C$

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2.$$

It is clear that if T is generalized hybrid and $F(T) \neq \emptyset$, then T is $0-$ *demimetric*. Note that the class of generalized hybrid mappings covers several well-known mappings (see [23] for details). For example, if T is a $(1, 0)-$generalized hybrid mapping, then T is nonexpansive, that is, $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$ and quasi-nonexpansive if $F(T) \neq \emptyset$. Also, T is nonspreading [21, 22] if $\alpha = 2$ and $\beta = 1$, that is,

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2, \ \ \forall\, x, y \in C.$$

Furthermore, T is hybrid [32] for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$, that is,

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2, \ \ \forall\, x, y \in C.$$

Osilike and Isiogugu [28] introduced a class of nonlinear mapping more general than the nonspreading mapping in Hilbert space, namely k-strictly pseudononspreading mapping as follows: A mapping $T : D(T) \to H$ is called k-strictly pseudononspreading if there exists $k \in [0, 1)$ such that

$$\begin{aligned} \|Tx - Ty\|^2 \ \ \leq \ \ & \|x - y\|^2 + k\|x - Tx - (y - Ty)\|^2 \\ & +2\langle x - Tx, y - Ty\rangle \ \ \forall\, x, y \in D(T). \end{aligned} \tag{9}$$

But, if T is k-strictly pseudononspreading with $F(T) \neq \emptyset$, then for $y \in F(T)$, (9) becomes

$$\|Tx - y\|^2 \leq \|x - y\|^2 + k\|x - Tx\|^2 \ \ \forall\, x \in D(T). \tag{10}$$

By (iii) in Lemma 2.3 and (10), we obtain

$$\|x - Tx\|^2 + \|y - x\|^2 - 2\langle x - y, x - Tx\rangle \leq \|x - y\|^2 + k\|x - Tx\|^2 \ \ \forall\, x \in D(T),$$

hence

$$\langle x - y, x - Tx\rangle \geq \frac{1-k}{2}\|x - Tx\|^2.$$

Therefore, T is a k-demimetric mapping with $k \in [0, 1)$.

Lemma 2.5. ([1, 34, 35]) *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $k \in (-\infty, 0)$ and let T be a $k-$demimetric mapping of C into H such that $F(T) \neq \emptyset$. Let λ be a real number with $0 < \lambda \leq 1 - k$ and defined $S = (1-\lambda) + \lambda T$. Then*

(i) $F(T) = F(S)$,

(ii) *$F(T)$ is closed and convex,*

(iii) *S is a quasi-nonexpansive mapping of C into H.*

Lemma 2.6. (Song [30]) *Let H be a Hilbert space and C be a nonempty convex subset of H. Assume that $\{T_i\}_{i=1}^{\infty} : C \to H$ is an infinite family of k_i-demimetric mappings with $\sup\{k_1 : i \in \mathbb{N}\} < 1$ such that $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Assume that $\{\eta_i\}_{i=1}^{\infty}$ is a positive sequence such that $\sum_{i=1}^{\infty} \eta_i = 1$. Then $\sum_{i=1}^{\infty} \eta_i T_i : C \to H$ is a k-demimetric mapping with $k = \sup\{k_i : i \in \mathbb{N}\}$ and $F(\sum_{i=1}^{\infty} \eta_i T_i) = \bigcap_{i=1}^{\infty} F(T_i)$.*

Lemma 2.7. *[25] Let H be a Hilbert space and C be nonempty convex subset of H. Let $G : H \to H$ be a monotone and $L-$Lipschitz on C. Let $U = P_C(I - \mu G)$, where $\mu > 0$. If $\{x_n\}$ is a sequence in H satisfying $x_n \rightharpoonup q$ and $x_n - Ux_n \to 0$, then $q \in VI(C, G) = F(U)$.*

Lemma 2.8. *[25] Let H be a Hilbert space and C be nonempty convex subset of H. Let $G : H \to H$ be a monotone and $L-$Lipschitz on C, τ a positive number and suppose that $VI(C, G) \neq \emptyset$. Let $\{x_n\}$ be a sequence in H defined by*

$$\begin{cases} x_0 \in H; \\ y_n = P_C(x_n - \tau G(x_n)), \\ T_n = \{w \in H : \langle x_n - \tau G(x_n) - y_n, w - y_n\rangle \leq 0\}, \\ z_n = P_{T_n}(x_n - \tau G(y_n)), \end{cases}$$

then for all $u \in VI(C, G)$, we have

$$\|z_n - u\|^2 \leq \|x_n - u\|^2 - (1 - \tau L)\|y_n - x_n\|^2 - (1 - \tau L)\|z_n - y_n\|^2. \tag{11}$$

In particular, if $\tau L \leq 1$, we have $\|z_n - u\| \leq \|x_n - u\|$.

Lemma 2.9. ([26]) *Let $\{a_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$.*

$$a_{m_k} \leq a_{m_k+1} \text{ and } a_k \leq a_{m_k+1}.$$

In fact, $m_k = \max\{j \leq k : a_j < a_{j+1}\}$.

Lemma 2.10. ([9]) *Let $\{a_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$, and $\{t_n\}$ be sequences of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1 - t_n - \gamma_n)a_n + \gamma_n a_{n-1} + t_n s_n + \delta_n,$$

where $\sum_{n=n_0}^{\infty} t_n = +\infty$, $\sum_{n=n_0}^{\infty} \delta_n < +\infty$ for each $n \geq n_0$ (where n_0 is a positive integer) and $\{\gamma_n\} \subset [0, \frac{1}{2}]$, $\limsup_{n\to\infty} s_n \leq 0$. Then, the sequence $\{a_n\}$ converges strongly to zero.

The demiclosedness principle for mappings plays an important role in our proof in the subsequent section.

Definition 2.11. (Chidume and Maruster [10]) *A self-mapping T on a Banach space is said to be demiclosed at y, if for any sequence $\{x_n\}$ which converges weakly to x, and if the sequence $\{Tx_n\}$ converges strongly to y, then $T(x) = y$. In particular, if $y = 0$, then T is demiclosed at 0.*

3. Main Results

In this section, we introduce a modified subgradient extragradient method, by using the term $\theta_n(x_{n-1}-x_n)$ which is called *modified inertial*. Inertial can be viewed as method of speeding up the convergence properties (see [2, 3, 7] for more details). Note that modified inertial term $\theta_n(x_{n-1} - x_n)$ not only speed up the rate of convergence of algorithms, the computational cost of algorithms with modified inertial term are less expensive compared to the normal inertial term $\theta_n(x_n - x_{n-1})$, see [19] (for example, w_n in (3.1) can be written as a convex combination).

For this reason, the following method is different from the methods studied in [8, 25, 36, 37].

Theorem 3.1. *Let H be a real Hilbert space and C be a nonempty closed and convex subset of H. Let $\{T_i\}_{i=1}^{\infty} : H \to H$ be an infinite family of k_i-demimetric mapping and demiclosed at zero with $k_i \in (-\infty, 1)$ for each $i \geq 1$ and $k = \max\{k_i : i \geq 1\} \leq 1$. Let $G : H \to H$ be a monotone and $L-$Lipschits mapping with $L > 0$. Let τ be a positive real number such that $\tau L < 1$. Assume $\Gamma := \bigcap_{i=1}^{\infty} F(T_i) \cap VI(C, G) \neq \emptyset$. For any fixed $u \in H$, let $\{x_n\}_{n=1}^{\infty}$ be a sequence defined iteratively by arbitrary chosen $x_0, x_1 \in H$ as:*

$$\begin{cases} w_n = x_n + \theta_n(x_{n-1} - x_n), \\ y_n = P_C(w_n - \tau G w_n), \\ T_n = \{x \in H : \langle w_n - \tau G w_n - y_n, x - y_n\rangle \leq 0\}, \\ z_n = P_{T_n}(w_n - \tau G y_n), \\ v_n = (1 - \lambda_n) z_n + \lambda_n \sum_{i=1}^{\infty} \gamma_i T_i z_n, \\ x_{n+1} = \alpha_n x_n + \beta_n v_n + (1 - \alpha_n - \beta_n) u \ n \geq 1, \end{cases} \tag{12}$$

where $\{\theta_n\} \subset [0, \frac{1}{2}]$, $\{\alpha_n\}$, $\{\beta_n\}, \{\gamma_i\} \subset (0, 1)$ and $\{\lambda_n\} \subset (0, +\infty)$ satisfy the following conditions

(C1) $0 < a \leq \theta_n < \beta_n \leq \frac{1}{2}$, *for all $n \geq 1$ and for some a,*

(C2) $\lim_{n\to\infty}(1 - \alpha_n - \beta_n) = 0$ *and* $\sum_{n=1}^{\infty}(1 - \alpha_n - \beta_n) = +\infty$,

(C3) $\sum_{i=1}^{\infty} \gamma_i = 1$,

(C4) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$.

Then $\{x_n\}$ converges strongly to $p := P_\Gamma u$.

Proof. Let $p \in \Gamma := \bigcap_{i=1}^{\infty} F(T_i) \cap VI(C, G)$. By (12) and Lemma 2.8, we obtain

$$\|z_n - p\|^2 \leq \|w_n - p\|^2 - (1 - \tau L)\|y_n - w_n\|^2 - (1 - \tau L)\|z_n - y_n\|^2. \tag{13}$$

Also, from modified inertial part of (12) and convexity of the norm, we obtain

$$\begin{aligned} \|w_n - p\|^2 &= \|x_n + \theta_n(x_{n-1} - x_n) - p\|^2 \\ &= \|(1 - \theta_n)(x_n - p) + \theta_n(x_{n-1} - p)\|^2 \\ &\leq (1 - \theta_n)\|x_n - p\|^2 + \theta_n\|x_{n-1} - p\|^2. \end{aligned} \tag{14}$$

Furthermore, letting $D := \sum_{i=1}^{\infty} \gamma_i T_i$, by Lemma 2.6, D is $k-$demimetric mapping. Let $S_n := (1 - \lambda_n)I + \lambda_n D$. Then by Lemma 2.5, S_n is a quasi-nonexpansive mapping and $F(S_n) = F(D) = \bigcap_{i=1}^{\infty} F(T_i)$. Therefore from (v_n) in (12), we obtain

$$\|v_n - p\| = \|S_n z_n - p\| \leq \|z_n - p\|. \tag{15}$$

Hence, from (12) − (15) and Lemma 2.2, letting ($\delta_n := 1 - \alpha_n - \beta_n$ which implies $\alpha_n + \beta_n = 1 - \delta_n$), we obtain

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\alpha_n x_n + \beta_n v_n + \delta_n u - p\|^2 \\
&= \|\alpha_n(x_n - p) + \beta_n(v_n - p) + \delta_n(u - p)\|^2 \\
&\leq \alpha_n\|x_n - p\|^2 + \beta_n\|v_n - p\|^2 + \delta_n\|u - p\|^2 - \alpha_n\beta_n\|x_n - v_n\|^2 \\
&\leq \alpha_n\|x_n - p\|^2 + \beta_n\|z_n - p\|^2 + \delta_n\|u - p\|^2 - \alpha_n\beta_n\|x_n - v_n\|^2 \\
&\leq \alpha_n\|x_n - p\|^2 + \beta_n\Big(\|w_n - p\|^2 - (1 - \tau L)\|y_n - w_n\|^2 \\
&\quad -(1 - \tau L)\|z_n - y_n\|^2\Big) + \delta_n\|u - p\|^2 - \alpha_n\beta_n\|x_n - v_n\|^2 \\
&= \alpha_n\|x_n - p\|^2 + \beta_n\|w_n - p\|^2 - (1 - \tau L)\beta_n\|y_n - w_n\|^2 \\
&\quad -(1 - \tau L)\beta_n\|z_n - y_n\|^2 + \delta_n\|u - p\|^2 - \alpha_n\beta_n\|x_n - v_n\|^2 \\
&\leq \alpha_n\|x_n - p\|^2 + \beta_n[(1 - \theta_n)\|x_n - p\|^2 + \theta_n\|x_{n-1} - p\|^2] \\
&\quad -(1 - \tau L)\beta_n\|y_n - w_n\|^2 - (1 - \tau L)\beta_n\|z_n - y_n\|^2 \\
&\quad +\delta_n\|u - p\|^2 - \alpha_n\beta_n\|x_n - v_n\|^2 \\
&= (1 - \delta_n - \beta_n\theta_n) + \|x_n - p\|^2 + \beta_n\theta_n\|x_{n-1} - p\|^2 + \delta_n\|u - p\|^2 \\
&\quad -\beta_n[(1 - \tau L)\|y_n - w_n\|^2 + (1 - \tau L)\|z_n - y_n\|^2 + \delta_n\|x_n - v_n\|^2] \qquad (16) \\
&\leq (1 - \delta_n - \beta_n\theta_n)\|x_n - p\|^2 + \beta_n\theta_n\|x_{n-1} - p\|^2 + \delta_n\|u - p\|^2 \\
&\leq \max\{\|x_n - p\|^2, \|x_{n-1} - p\|^2, \|u - p\|^2\}.
\end{aligned}$$

By induction

$$\|x_n - p\| \leq \max\{\|x_1 - p\|^2, \|x_0 - p\|^2, \|u - p\|^2\}.$$

Hence, $\{x_n\}$ is bounded. With this $\{v_n\}$, $\{z_n\}$, $\{y_n\}$ and $\{w_n\}$ are also bounded. Therefore, from (16), we obtain

$$\begin{aligned}
&\beta_n[(1 - \tau L)\|y_n - w_n\|^2 + (1 - \tau L)\|z_n - y_n\|^2 + \delta_n\|x_n - v_n\|^2] \\
&\quad \leq (1 - \delta_n - \beta_n\theta_n)\|x_n - p\|^2 + \beta_n\theta_n\|x_{n-1} - p\|^2 \\
&\quad\quad +\delta_n\|u - p\|^2 - \|x_{n+1} - p\|^2 \\
&\quad \leq \Big(\|x_n - p\|^2 - \|x_{n+1} - p\|^2\Big) + \delta_n\|u - p\|^2 \\
&\quad\quad +\beta_n\theta_n\Big(\|x_{n-1} - p\|^2 - \|x_n - p\|^2\Big). \qquad (17)
\end{aligned}$$

We divide the remaining proof in two cases.

Case 1. Assume that $\{\|x_n - p\|^2\}_{n=1}^{\infty}$ is non-increasing sequence of real numbers. Since $\{\|x_n - p\|^2\}_{n=1}^{\infty}$ is bounded, therefore its limit exists. It follows that $\lim_{n\to\infty}(\|x_n - p\|^2 - \|x_{n+1} - p\|^2) = \lim_{n\to\infty}(\|x_{n-1} - p\|^2 - \|x_n - p\|^2) = 0$. From the fact that $\delta_n \to 0$ as $n \to \infty$ and (17), we obtain

$$\lim_{n\to\infty}\|y_n - w_n\| = \lim_{n\to\infty}\|z_n - y_n\| = \lim_{n\to\infty}\|x_n - v_n\| = 0. \qquad (18)$$

Now, from recursion formula (x_{n+1}) in (12), we get

$$\|x_{n+1} - x_n\| \leq \beta_n\|v_n - x_n\| + \delta_n\|u - x_n\|,$$

it follows from (18) and the fact that $\delta_n \to 0$ that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \qquad (19)$$

Also, from recursion formula (w_n) in (12) and (19), we obtain

$$\lim_{n\to\infty}\|w_n - x_n\| = \lim_{n\to\infty}\theta_n\|x_{n-1} - x_n\| = 0. \qquad (20)$$

In view of

$$\|y_n - x_n\| \leq \|y_n - w_n\| + \|w_n - x_n\|,$$

it follows from (18) and (20), that

$$\lim_{n\to\infty}\|y_n - x_n\| = 0. \tag{21}$$

As

$$\|z_n - v_n\| \le \|z_n - y_n\| + \|y_n - x_n\| + \|x_n - v_n\|$$

so from (17) and (21), we obtain

$$\lim_{n\to\infty}\|z_n - v_n\| = 0. \tag{22}$$

Furthermore, since

$$\|x_n - z_n\| \le \|x_n - v_n\| + \|u_n - z_n\|,$$

therefore it follows from (17) and (22) that

$$\lim_{n\to\infty}\|x_n - z_n\| = 0. \tag{23}$$

Since $\{x_n\}$ is bounded, there exist a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup x^* \in C$. On the other hand, since T_i are k_i−demimetric for each $i \ge 1$, and $p \in \bigcap_{i=1}^{\infty} F(T_i)$, therefore from (8), we obtain

$$\begin{aligned}
\langle z_n - p, z_n - v_n\rangle &= \langle z_n - p, z_n - S_n z_n\rangle \\
&= \lambda_n \sum_{i=1}^{\infty} \gamma_i \langle z_n - p, z_n - T_i z_n\rangle \\
&\ge \lambda_n \sum_{i=1}^{\infty} \gamma_i \frac{1-k_i}{2}\|z_n - T_i z_n\|^2 \\
&\ge \frac{1-k}{2}\lambda_n \sum_{i=1}^{\infty} \gamma_i \|z_n - T_i z_n\|^2.
\end{aligned}$$

From (22) and the fact that $k \le 1$, it follows for each $i \ge 1$ that

$$\lim_{n\to\infty}\|z_n - T_i z_n\| = 0. \tag{24}$$

Furthermore, since T_i is demiclosed at zero for each $i \ge 1$ and $x_n \rightharpoonup x^*$ as $n \to \infty$ therefore by (23), we get $z_n \rightharpoonup x^*$. Hence by (24), we conclude that $x^* \in \bigcap_{i=1}^{\infty} F(T_i)$. Also, as $x_n \rightharpoonup x^*$, so by (20), $w_n \rightharpoonup x^*$. Therefore by (18) and Lemma 2.7, we obtain $x^* \in VI(C, G)$. Hence $x^* \in \Gamma = \bigcap_{i=1}^{\infty} F(T_i) \bigcap VI(C, G)$. Also

$$\limsup_{n\to\infty}\langle u - p, x_n - p\rangle = \lim_{j\to\infty}\langle u - p, x_{n_j} - p\rangle = \langle u - p, x^* - p\rangle.$$

Since $p = P_\Gamma u$, by (7) in Lemma 2.1, it follows that $\langle u - p, x^* - p\rangle \le 0$. Hence

$$\limsup_{n\to\infty}\langle u - p, x_n - p\rangle = \langle u - p, x^* - p\rangle \le 0. \tag{25}$$

Also since, $\langle u - p, x_{n+1} - p\rangle = \langle u - p, x_{n+1} - x_n\rangle + \langle u - p, x_n - p\rangle$, therefore it follows from (19) and (25), that

$$\limsup_{n\to\infty}\langle u - p, x_{n+1} - p\rangle \le 0. \tag{26}$$

Finally, we show that $x_n \to p$ as $n \to \infty$. From (12), (13) and $\tau L < 1$, we get $\|z_n - p\| \leq \|w_n - p\|$. Therefore by Lemma 2.3, (14) and (15), we obtain

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \langle \alpha_n x_n + \beta_n v_n + \delta_n u - p, x_{n+1} - p\rangle \\
&= \langle \alpha_n(x_n - p) + \beta_n(v_n - p), x_{n+1} - p\rangle + \delta_n\langle u - p, x_{n+1} - p\rangle \\
&\leq \|\alpha_n(x_n - p) + \beta_n(v_n - p)\|\|x_{n+1} - p\| + \delta_n\langle u - p, x_{n+1} - p\rangle \\
&\leq \alpha_n\|x_n - p\|\|x_{n+1} - p\| + \beta_n\|v_n - p\|\|x_{n+1} - p\| \\
&\quad +\delta_n\langle u - p, x_{n+1} - p\rangle \\
&\leq \alpha_n\|x_n - p\|\|x_{n+1} - p\| + \beta_n\|z_n - p\|\|x_{n+1} - p\| \\
&\quad +\delta_n\langle u - p, x_{n+1} - p\rangle \\
&\leq \alpha_n\|x_n - p\|\|x_{n+1} - p\| + \beta_n\|w_n - p\|\|x_{n+1} - p\| \\
&\quad +\delta_n\langle u - p, x_{n+1} - p\rangle \\
&\leq \frac{\alpha_n}{2}[\|x_n - p\|^2 + \|x_{n+1} - p\|^2] + \frac{\beta_n}{2}[\|w_n - p\|^2 + \|x_{n+1} - p\|^2] \\
&\quad +\delta_n\langle u - p, x_{n+1} - p\rangle \\
&= \frac{\alpha_n}{2}\|x_n - p\|^2 + \frac{\beta_n}{2}\|w_n - p\|^2 + \frac{\alpha_n + \beta_n}{2}\|x_{n+1} - p\|^2 \\
&\quad +\delta_n\langle u - p, x_{n+1} - p\rangle \\
&\leq \frac{\alpha_n}{2}\|x_n - p\|^2 + \frac{\beta_n}{2}\|w_n - p\|^2 + \frac{1}{2}\|x_{n+1} - p\|^2 \\
&\quad +\delta_n\langle u - p, x_{n+1} - p\rangle \\
&\leq \frac{\alpha_n}{2}\|x_n - p\|^2 + \frac{\beta_n}{2}[(1 - \theta_n)\|x_n - p\|^2 + \theta_n\|x_{n-1} - p\|^2] \\
&\quad +\frac{1}{2}\|x_{n+1} - p\|^2 + \delta_n\langle u - p, x_{n+1} - p\rangle \\
&\leq \frac{1 - \delta_n - \beta_n\theta_n}{2}\|x_n - p\|^2 + \frac{\beta_n\theta_n}{2}\|x_{n-1} - p\|^2 \\
&\quad +\frac{1}{2}\|x_{n+1} - p\|^2 + \delta_n\langle u - p, x_{n+1} - p\rangle.
\end{aligned}$$

Therefore

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leq (1 - \delta_n - \beta_n\theta_n)\|x_n - p\|^2 + \beta_n\theta_n\|x_{n-1} - p\|^2 \\
&\quad +2\delta_n\langle u - p, x_{n+1} - p\rangle. \qquad (27)
\end{aligned}$$

Hence, applying Lemma 2.10 and (26) to (27), we obtain $x_n \to p$.

Case 2. Assume that $\{\|x_n - p\|^2\}_{n=1}^{\infty}$ is non-decreasing sequence of real numbers. Set $\Phi_n := \|x_n - p\|^2$ then there exists a subsequence $\{\Phi_{n_i}\}$ of $\{\Phi_n\}$ such that $\Phi_{n_i} < \Phi_{n_i+1}$ for all $i \in \mathbb{N}$. Thus by Lemma 2.9, let $s : \mathbb{N} \to \mathbb{N}$ be a mapping for all $n \geq n_0$ (for some n_0 large enough) defined by

$$s(n) := \max\{k \in \mathbb{N} : k \leq n, \Phi_k \leq \Phi_{k+1}\}.$$

Clearly, s is a non-decreasing sequence, $s(n) \to \infty$ as $n \to \infty$ and

$$0 \leq \Phi_{s(n)} \leq \Phi_{s(n)+1}, \ \forall\, n \geq n_0.$$

This implies that $\|x_{s(n)} - p\|^2 \leq \|x_{s(n)+1} - p\|^2, \ \forall\, n \geq n_0$. Thus $\lim_{n\to\infty}\|x_{s(n)} - p\|$ exists. Following the argument as in Case 1, we can show that

$$\lim_{n\to\infty}\|y_{s(n)} - w_{s(n)}\| = \lim_{n\to\infty}\|z_{s(n)} - y_{s(n)}\| = \lim_{n\to\infty}\|x_{s(n)} - v_{s(n)}\| = 0$$

and $\lim_{n\to\infty}\|x_{s(n)+1} - x_{s(n)}\| = 0$. As $\{x_{s(n)}\}$ is bounded, so there exists a subsequence of $\{x_{s(n)}\}$, still denoted by $\{x_{s(n)}\}$ which converges weakly to x^*. By an argument similar to the one in Case 1, we can show that $x^* \in \Gamma$ and

$$\limsup_{n\to\infty}\langle u - p, x_{s(n)+1} - p\rangle \leq 0.$$

Furthermore, from (27), we obtain

$$\begin{aligned}\|x_{s(n)+1} - p\|^2 \quad &\leq \quad (1 - \delta_{s(n)} - \beta_{s(n)}\theta_{s(n)})\|x_{s(n)} - p\|^2 + \beta_{s(n)}\theta_{s(n)}\|x_{s(n)-1} - p\|^2 \\ &\quad +2\delta_{s(n)}\langle u - p, x_{s(n)+1} - p\rangle. \end{aligned} \tag{28}$$

Note that $\Phi_{s(n)} \leq \Phi_{s(n)+1}\ \ \forall\ n \geq n_0$, which means that $\Phi_{s(n)-1} \leq \Phi_{s(n)}$ and $\beta_{s(n)}\theta_{s(n)} > 0$. Now it follows from (28) that

$$\|x_{s(n)} - p\|^2 \leq \langle u - p, x_{s(n)+1} - p\rangle.$$

This implies that

$$\limsup_{n\to\infty}\|x_{s(n)} - p\| \leq 0.$$

Thus

$$\lim_{n\to\infty}\|x_{s(n)} - p\| = 0$$

and

$$\|x_{s(n)+1} - p\| \leq \|x_{s(n)+1} - x_{s(n)}\| + \|x_{s(n)} - p\| \to 0 \text{ as } \ n \to \infty.$$

Therefore

$$\lim_{n\to\infty}\Phi_{s(n)} = \Phi_{s(n)+1} = 0.$$

Furthermore, for $n \geq n_0$. it is easy to see that $\Phi_{s(n)} \leq \Phi_{s(n)+1}$ if $n \neq s(n)$ (that is $s(n) < n$), because $\Phi_k \geq \Phi_{k+1}$ for $s(n) + 1 \leq k \leq n$. As a consequence, we obtain for all $n \geq n_0$.

$$0 \leq \Phi_n \leq \max\{\Phi_{s(n)}, \Phi_{s(n)+1}\} = \Phi_{s(n)+1}.$$

Hence, $\lim_{n\to\infty}\Phi_n = 0$ gives $\lim_{n\to\infty}\|x_n - p\| = 0$. Thus, $x_n \to p$ as $n \to \infty$. □

Remark 3.2.

(i) *A generalized hybrid mapping T with $F(T) \neq \emptyset$ is a 0-demimetric mapping. So Theorem 3.1 holds good for an infinite family of generalized hybrid mappings.*

(ii) *A nonspreading mapping is a $(2, 1)$-generalized hybrid mapping. Therefore a nonspreading mapping T with $F(T) \neq \emptyset$ is a 0-demimetric mapping. So Theorem 3.1 holds for an infinite family of nonspreading mappings.*

(iii) *A hybrid mapping is a $(\frac{3}{2}, \frac{1}{2})$-generalized hybrid mapping. Therefore a hybrid mapping T with $F(T) \neq \emptyset$ is a 0-demimetric mapping. So Theorem 3.1 holds for an infinite family of hybrid mappings.*

(iv) *For $k \in [0, 1)$, a k-strictly pseudononspreading mapping T with $F(T) \neq \emptyset$ is a k-demimetric mapping. So Theorem 3.1 holds good for an infinite family of k-strictly pseudononspreading mappings.*

4. Numerical Example

Let $\mathbb{R}^3$ be the three-dimensional Euclidean space with the usual inner product

$$\langle x, y\rangle = x_1y_1 + x_2y_2 + x_3y_3,$$

where $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3) \in \mathbb{R}^3$ and and usual norm

$$\|x\|^2 = x_1^2 + x_2^2 + x_3^2, \forall x = (x_1, x_2, x_3) \in \mathbb{R}^3.$$

Table 1: Convergence behavior of iterative algorithm (3.1) for the initial values $x_0 = (0.5, 0.25, 0.2)$, $x_1 = (0.125, 0.16666, 0.066666)$.

# of iter.	Algorithm (3.1)
0	$(0.5, 0.25, 0.2)$
1	$(0.125, 0.16666, 0.066666)$
2	$(0.171483, 0.216903, 0.132913)$
3	$(0.175577, 0.210647, 0.146857)$
$\vdots$	$\vdots$
50	$(0.005741, 0.005741, 0.005741)$
$\vdots$	$\vdots$
100	$(0.002761, 0.002761, 0.002761)$
$\vdots$	$\vdots$
150	$(0.001809, 0.001809, 0.001809)$
$\vdots$	$\vdots$

Given a half-space $C = \left\{z \in \mathbb{R}^3 : \langle u, z - w_0\rangle \leq 0\right\}$ of a real Hilbert space, where $u\,(\neq 0)$ and w_0 are two xed element in $\mathbb{R}^3$, then for any $x_0 \in H$, we have

$$P_C x_0 = \begin{cases} x_0 - \frac{\langle u, x_0 - w_0\rangle}{\|u\|^2} u, \ \langle u, x_0 - w_0\rangle > 0; \\ x, \ \langle u, x_0 - w_0\rangle \leq 0. \end{cases}$$

Let $Gx = \frac{1}{3}x$, $\tau = \frac{1}{2}$, $T_1 x = -\frac{2}{3}x$, then T_1 is $-\frac{1}{5}-$demimetric mapping and $T_2 x = -\frac{3}{4}x$, then T_2 is $-\frac{1}{7}-$demimetric mapping. Take $\lambda_n = \frac{2n-1}{6n}$, $\gamma_1 = \gamma_2 = \frac{1}{2}$, $\alpha_n = \frac{3n+1}{6n}$, $\beta_n = \frac{2n-1}{4n}$, $\theta_n = \frac{1}{4}$ $\forall n \geq 1$ and $u = w_0 = (1, 1, 1)$, $x_0 = (0.5, 0.25, 0.2)$, $x_1 = (0.125, 0.16666, 0.066666)$. Then the following table shows convergence behavior of algorithm (3.1) and the following figure shows convergence behavior of $\|x_n - (0, 0, 0)\|$ for the initial values $x_0 = (0.5, 0.25, 0.2)$, $x_1 = (0.125, 0.16666, 0.066666)$.

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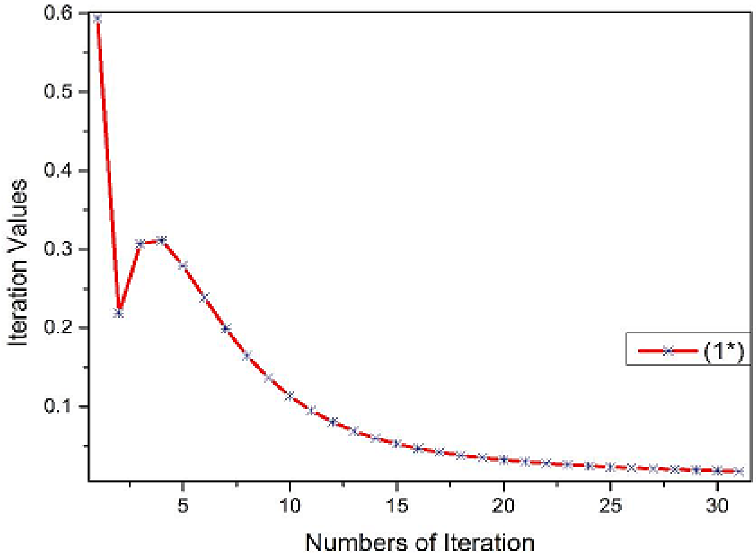

Figure 1: (1*) represents convergence behavior of $\|x_n - (0, 0, 0)\|$ for the initial values $x_0 = (0.5, 0.25, 0.2)$, $x_1 = (0.125, 0.16666, 0.066666)$.

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