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A Note on c-almost Periodic Ultradistributions and c-almost Periodic Hyperfunctions

Marko Kostić[a]

[a]*Faculty of Technical Sciences, University of Novi Sad, Trg D. Obradovića 6, 21125 Novi Sad, Serbia*

Abstract. The classes of c-almost periodic functions and c-almost periodic distributions have recently been introduced and analyzed. In this note, we consider the classes of c-almost periodic ultradistributions and c-almost periodic hyperfunctions.

1. Introduction and preliminaries

In this paper, we will always assume that $(X, \|\cdot\|)$ is a complex Banach space as well as that $I = \mathbb{R}$ or $I = [0, \infty)$, $c \in \mathbb{C}$ and $|c| = 1$. Set $S_1 \equiv \{z \in \mathbb{C} : |z| = 1\}$.

Let $f : I \to X$ be a continuous function and let a number $\epsilon > 0$ be given. We call a number $\tau > 0$ an (ϵ, c)-period for $f(\cdot)$ if $\|f(t + \tau) - cf(t)\| \leq \epsilon$ for all $t \in I$. By $\vartheta_c(f, \epsilon)$ we denote the set consisting of all (ϵ, c)-periods for $f(\cdot)$. In a joint research study [12] with M. T. Khalladi, A. Rahmani, M. Pinto and D. Velinov, we have recently introduced the following notion (cf. also [13]-[14]):

Definition 1.1. It is said that $f(\cdot)$ is c-almost periodic if and only if for each $\epsilon > 0$ the set $\vartheta_c(f, \epsilon)$ is relatively dense in $[0, \infty)$, which means that there exists $l = l(\epsilon) > 0$ such that any subinterval of $[0, \infty)$ of length l meets $\vartheta_c(f, \epsilon)$ The space consisting of all c-almost periodic functions from the interval I into X will be denoted by $AP_c(I : X)$.

If $c = 1$, resp. $c = -1$, then we recover the notion of almost periodicity, resp. almost anti-periodicity; if $c = 1$, then we also denote $AP(I : X) \equiv AP_c(I : X)$. There is an enormous literature about almost periodic functions and their applications to ordinary differential equations and partial differential equations; let us recall that this important class of functions was introduced by the Danish mathematician H. Bohr around 1925. For more details about the subject, we refer the reader to the forthcoming monograph [19] and references cited therein.

The classes of scalar-valued bounded distributions and scalar-valued almost periodic distributions have been introduced by L. Schwartz [26] and later extended to the vector-valued case by I. Cioranescu in [6];

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Email address: marco.s@verat.net (Marko Kostić)

the class of scalar-valued almost periodic ultradistributions was introduced by I. Cioranescu in [7] and later extended to the vector- valued case in [20]; see also the recent research studies [21] by M. Kostić, [2] by C. Bouzar, F. Z. Tchouar and the list of references given in the monograph [18]. In a recent joint work [11] with V. Fedorov, S. Pilipović and D. Velinov, we have introduced and analyzed various classes of c-almost periodic distributions.

The main aim of this note is to consider c-almost periodic ultradistributions and c-almost periodic hyperfunctions with values in complex Banach spaces. Before beginning our work, the author would like to express his sincere thanks to Prof. M. Hasler and M. T. Khalladi for many stimulating discussions during the preparation of this manuscript.

2. c-Almost periodic ultradistributions and c-almost periodic hyperfunctions

In this note, we analyze c-almost periodic ultradistributions and c-almost periodic hyperfunctions; we will skip all related details concerning c-uniformly recurrent ultradistributions (hyperfunctions) and semi-c-periodic ultradistributions (hyperfunctions).

Assume that (M_p) is a sequence of positive real numbers satisfying $M_0 = 1$ and the following conditions:

(M.1): $M_p^2 \leq M_{p+1}M_{p-1},\ \ p \in \mathbb{N},$

(M.2): $M_p \leq AH^p \sup_{0\leq i\leq p} M_iM_{p-i},\ \ p \in \mathbb{N},$ for some $A,\ H > 1,$

We will occasionally use conditions

(M.3'): $\sum_{p=1}^{\infty} \frac{M_{p-1}}{M_p} < \infty,$

(M.3): $\sup_{p\in\mathbb{N}} \sum_{q=p+1}^{\infty} \frac{M_{q-1}M_{p+1}}{pM_pM_q} < \infty,$

(C): The sequence (M_p^2) satisfies (M.3).

Let us recall that conditions (M.3') and (C) are substantially weaker than (M.3) as well that condition (C) has been essentially employed in the analysis of almost periodic hyperfunctions [3] carried out by J. Chung, S.-Y. Chung, D. Kim, H. J. Kim and the analysis of representations of quasianalytic ultradistributions carried out by S.-Y. Chung, D. Kim [4] (it is well known that (M_p) satisfies (C) if and only if there exists a positive integer $k \in \mathbb{N}$ such that $\liminf_{p\to+\infty}(m_{kp}/m_p)^2 > k$, where $m_p := M_p/M_{p-1}$ for all $p \in \mathbb{N}$ as well as that H. Petzche has proved, in [25], that (M_p) satisfies (M.3) if and only if there exists a positive integer $k \in \mathbb{N}$ such that $\liminf_{p\to+\infty} m_{kp}/m_p > k$). If $s > 1$, then the Gevrey sequence $(p!^s)$ satisfies the above conditions, while the sequence $(p!^s)$ satisfies (M.1), (M.2) and (C) for $s > 1/2$.

The space of Beurling, resp., Roumieu ultradifferentiable functions, is defined by $\mathcal{D}^{(M_p)} := \mathrm{indlim}_{K\Subset\Subset\mathbb{R}}\mathcal{D}_K^{(M_p)}$, resp., $\mathcal{D}^{\{M_p\}} := \mathrm{indlim}_{K\Subset\Subset\mathbb{R}}\mathcal{D}_K^{\{M_p\}}$, where $\mathcal{D}_K^{(M_p)} := \mathrm{projlim}_{h\to\infty}\mathcal{D}_K^{M_p,h}$, resp., $\mathcal{D}_K^{\{M_p\}} := \mathrm{indlim}_{h\to 0}\mathcal{D}_K^{M_p,h}$, $\mathcal{D}_K^{M_p,h} := \{\phi \in C^\infty(\mathbb{R}) : \mathrm{supp}\phi \subseteq K,\ \|\phi\|_{M_p,h,K} < \infty\}$ and

$$\|\phi\|_{M_p,h,K} := \sup\left\{\frac{h^p|\phi^{(p)}(t)|}{M_p} : t \in K,\ p \in \mathbb{N}_0\right\}.$$

The asterisk $*$ is used to designate both, the Beurling case (M_p) or the Roumieu case $\{M_p\}$. The space consisting of all linear continuous functions from $\mathcal{D}^*$ into X, denoted by $\mathcal{D}'^*(X) := L(\mathcal{D}^* : X)$, is said to be the space of all X-valued ultradistributions of $*$-class.

Let us recall (see [15]-[17] for the basic introduction to the theory of ultradistributions) that an entire function of the form $P(\lambda) = \sum_{p=0}^{\infty} a_p\lambda^p$, $\lambda \in \mathbb{C}$, is of class (M_p), resp., of class $\{M_p\}$, if there exist $l > 0$ and $C > 0$, resp., for every $l > 0$ there exists a constant $C > 0$, such that $|a_p| \leq Cl^p/M_p$, $p \in \mathbb{N}$. The corresponding ultradifferential operator $P(D) = \sum_{p=0}^{\infty} a_pD^p$ is of class (M_p), resp., of class $\{M_p\}$. For more details about convolution of scalar-valued ultradistributions (ultradifferentiable functions), see [15]. The convolution of Banach space valued ultradistributions and scalar-valued ultradifferentiable functions will be taken in the sense of considerations given on page 685 of [17]. As in the distributional case, we define $\langle T_h, \phi \rangle := \langle T, \phi(\cdot - h) \rangle$, $T \in \mathcal{D}'^*(X)$, $h > 0$, $\phi \in \mathcal{D}^*$.

The Sato space $\mathcal{F}_H$ consists of all infinitely differentiable functions $\phi : \mathbb{R} \to \mathbb{C}$ satisfying that there exist $h > 0$ and $k > 0$ such that

$$\|\phi\|_{p,k} := \sup_{x \in \mathbb{R}, p \in \mathbb{N}_0} \frac{h^p \left|\phi^{(p)}(x)\right| e^{k|x|}}{p!} < +\infty.$$

Let $\mathcal{F}_H$ be topologized by the corresponding inductive limit topology induced by these seminorms. The space of all X-valued Fourier hyperfunctions, denoted by $\mathcal{F}'_H(X)$, is defined as the space of all linear continuous mappings $T : \mathcal{F}_H \to X$, equipped with the strong topology.

Now we will consider bounded ultradistributions and bounded hyperfunctions with values in complex Banach spaces. First of all, for every $h > 0$, we define

$$\mathcal{D}_{L^1}\big((M_p), h\big) := \left\{ f \in \mathcal{D}_{L^1} \ ; \ \|f\|_{1,h} := \sup_{p \in \mathbb{N}_0} \frac{h^p \|f^{(p)}\|_1}{M_p} < \infty \right\}.$$

Then $(\mathcal{D}_{L^1}((M_p), h), \|\cdot\|_{1,h})$ is a Banach space and the space of all X-valued bounded Beurling ultradistributions of class (M_p), resp., X-valued bounded Roumieu ultradistributions of class $\{M_p\}$, is defined as the space consisting of all linear continuous mappings from $\mathcal{D}_{L^1}((M_p))$, resp., $\mathcal{D}_{L^1}(\{M_p\})$, into X, where

$$\mathcal{D}_{L^1}\big((M_p)\big) := \text{projlim}_{h \to +\infty} \mathcal{D}_{L^1}\big((M_p), h\big),$$

resp.,

$$\mathcal{D}_{L^1}\big(\{M_p\}\big) := \text{indlim}_{h \to 0+} \mathcal{D}_{L^1}\big((M_p), h\big).$$

These spaces, carrying the strong topologies, will be shortly denoted by $\mathcal{D}'_{L^1}((M_p) : X)$, resp., $\mathcal{D}'_{L^1}(\{M_p\} : X)$. It is well known that $\mathcal{D}^{(M_p)}$, resp. $\mathcal{D}^{\{M_p\}}$, is a dense subspace of $\mathcal{D}_{L^1}((M_p))$, resp., $\mathcal{D}_{L^1}(\{M_p\})$, as well as that $\mathcal{D}_{L^1}((M_p)) \subseteq \mathcal{D}_{L^1}(\{M_p\})$.

In particular case $M_p := p!$, the space $\mathcal{D}'_{L^1}(\{p!\} : X)$ is said to be the space of bounded hyperfunctions. As in the scalar-valued case, this space is contained in the space $\mathcal{F}'_H(X)$ of all X-valued Fourier hyperfunctions (see also [5, Definition 3.1] for the multi-dimensional analogue).

Recall that the heat kernel $E(x,t)$ is defined by $E(x,t) := (4\pi t)^{-1/2} e^{-x^2/4t}$, $x \in \mathbb{R}$, $t > 0$ and $E(x,t) := 0$, $x \in \mathbb{R}$, $t \leq 0$. It can be simply shown that the function $E(\cdot, t)$ belongs to the Sato space for every fixed real number $t > 0$ as well as that for each $x \in \mathbb{R}$ and $t > 0$ the function $E(x - \cdot, t)$ belongs to the space $\mathcal{D}_{L^1}(\{p!\} : X)$. Hence, for each Fourier hyperfunction $T \in \mathcal{F}'_H(X)$, its Gauss transform $u(x,t) := \langle T, E(x - \cdot, t) \rangle$ is infinitely differentiable in $\mathbb{R} \times (0, \infty)$.

We would like to note that the statements of [5, Theorem 3.4, Theorem 3.5] continue to hold in the vector-valued case. In connection with this observation, it should be only observed that the existence of functions $g(x)$ and $h(x)$, established on [5, p. 2425, l. -3] (see also [3, p. 735, l. -1; l. -5]), follows from the facts (see [1, Example 3.7.6, Example 3.7.8] for more details) that the Laplacian Δ with maximal distributional domain ($\equiv A$) generates a strongly continuous Gaussian semigroup on $L^p(\mathbb{R}^n : X)$, the operator A generates a polynomially bounded once integrated Gaussian semigroup on $L^\infty(\mathbb{R}^n : X)$, the basic results about the existence and uniqueness of mild solutions of the abstract (ill-posed) Cauchy problems of the first order and the conclusion established on [5, p. 2425, l. -4]. In particular, the statement of [3, Theorem 3.1] can be extended to the vector-valued case:

Theorem 2.1. *Suppose that $T \in \mathcal{F}'_H(X)$. Then the following statements are equivalent:*

(i) We have $T \in \mathcal{D}'_{L^1}(\{p!\} : X)$.

(ii) $T * \varphi \in L^\infty(\mathbb{R} : X)$ for all $\varphi \in \mathcal{F}_H$.

(iii) There exist two bounded continuous functions $f : \mathbb{R} \to X$, $g : \mathbb{R} \to X$ and an ultradifferential operator P of class $\{p!^2\}$ such that $T = P(-\Delta) f + g$.

(iv) *The Gauss transform $u(x,t)$ of T is infinitely differentiable in $(0,\infty)^2$ and solves the heat equation in $(0,\infty)^2$, as well as for every $\epsilon > 0$ there exists a constant $c > 0$ such that*

$$\|u(x,t)\| \leq ce^{\epsilon/t}, \quad x \in \mathbb{R},\ t > 0$$

and

$$\langle T, \varphi \rangle = \lim_{t \to 0+} \int_{-\infty}^{+\infty} u(x,t)\varphi(x)\,dx, \quad \varphi \in \mathcal{D}_{L^1}(\{p!\} : X).$$

Concerning bounded quasianalytic ultradistributions, we would like to note that the statement of [3, Lemma 4.2] also holds in the vector-valued case.

Concerning c-almost periodic ultradistributions, we will use the function space

$$\mathcal{E}^*_{AP_c}(X) := \left\{ \phi \in \mathcal{E}^*(X) : \phi^{(i)} \in AP_c(\mathbb{R} : X) \text{ for all } i \in \mathbb{N}_0 \right\},$$

which is a slight generalization of the space $\mathcal{E}^*_{AP}(X)$ used in [20], with $c = 1$.

In [7] and [20], a bounded X-valued ultradistribution $T \in \mathcal{D}'_{L^1}((M_p) : X)$, resp., $T \in \mathcal{D}'_{L^1}(\{M_p\} : X)$, is said to be almost periodic of Beurling class (M_p), resp., almost periodic of Roumeiu class $\{M_p\}$, if and only if there exists a sequence of X-valued trigonometric polynomials converging to T in $\mathcal{D}'_{L^1}((M_p) : X)$, resp., $\mathcal{D}'_{L^1}(\{M_p\} : X)$. If the sequence (M_p) satisfies (M.3), then $T \in \mathcal{D}'_{L^1}((M_p) : X)$ is almost periodic if and only if $T * \varphi \in AP(\mathbb{R} : X)$ for all $\varphi \in \mathcal{D}^{(M_p)}$.

Concerning [20, Theorem 2], the following result should be stated for c-almost periodicity:

Theorem 2.2. *Let (M_p) satisfy the conditions* (M.1), (M.2) *and* (M.3'), *and let $T \in \mathcal{D}'_{L^1}((M_p) : X)$, resp., $T \in \mathcal{D}'_{L^1}(\{M_p\} : X)$. Consider the following assertions:*

(i) *There exists an ultradifferential operator $P(D) = \sum_{p=0}^{\infty} a_p D^p$ of class (M_p), resp., of class $\{M_p\}$, and functions $f,\ g \in AP_c(\mathbb{R} : X)$ such that the function $t \mapsto (f(t), g(t))$, $t \in \mathbb{R}$ is c-almost periodic and $T = P(D)f + g$ for all $\varphi \in \mathcal{D}_{L^1}((M_p))$, resp., $\varphi \in \mathcal{D}_{L^1}(\{M_p\})$.*

(ii) *For every $\varphi \in \mathcal{D}^*$, we have $T * \varphi \in AP_c(\mathbb{R} : X)$.*

(iii) *$T \in \mathcal{D}'^*_{L^1}((M_p) : X)$, resp. $T \in \mathcal{D}'^*_{L^1}(\{M_p\} : X)$, and there exists a sequence (ϕ_n) in $\mathcal{E}^*_{AP_c}(X)$ such that $\lim_{n\to\infty} \phi_n = T$ for the topology of $\mathcal{D}'_{L^1}((M_p) : X)$, resp. $\mathcal{D}'_{L^1}(\{M_p\} : X)$.*

(iv) *There exists $h > 0$ such that for each compact set $K \subseteq \mathbb{R}$, in the Beurling case, resp., for each compact set $K \subseteq \mathbb{R}$ and for each $h > 0$, in the Roumieu case, the following holds $T * \varphi \in AP_c(\mathbb{R} : X)$, $\varphi \in \mathcal{D}^{M_p,h}_K$.*

Then we have (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv).

Unfortunately, if (M_p) additionally satisfies (M.3), then the equivalence of the above assertions cannot be so simply clarified in the Beurling case (see e.g., [7, Lemma 2] and the proofs of [7, Theorem 1, Theorem 2]); more precisely, it is not clear how one can prove that (iv) implies (i) for c-almost periodicity; we can only prove that (iv) implies that there exists an ultradifferential operator $P(D) = \sum_{p=0}^{\infty} a_p D^p$ of class (M_p), resp., of class $\{M_p\}$, and functions $f,\ g \in AP_c(\mathbb{R} : X)$ such that $T = P(D)f + g$ for all $\varphi \in \mathcal{D}_{L^1}((M_p))$, resp., $\varphi \in \mathcal{D}_{L^1}(\{M_p\})$.

Concerning almost periodic quasianalytic ultradistributions, we would like to note that the statement of [3, Theorem 4.3] continues to hold in the vector-valued case. Concerning c-almost periodic quasianalytic ultradistribution and asymptotically c-almost periodic ultradistributions of $*$-class, let us only mention that the notion introduced in [20, Definition 1, Definition 2] as well as the notion of space $B'_0(X)$ can be straightforwardly extended to the ultradistributional case (cf. also the recent article [8] by A. Debrouwere, L. Neyt and J. Vindas). Also, it could be very interesting to reconsider [20, Theorem 3] for asymptotical c-almost periodicity.

Now we will consider the class of c-almost periodic hyperfunctions. We will follow the approach of J. Chung, S.-Y. Chung, D. Kim and H. J. Kim obeyed in [3]. In this paper, the authors use the operation calculus approach to hyperfunctions developed by T. Matsuzawa in [22]-[24], which is based on the use of Gauss kernels.

First of all, we introduce the vector-valued analogue of [3, Definition 3.2]:

Definition 2.3. A hyperfunction $T \in \mathcal{D}'_{L^1}(\{p!\} : X)$ is said to be almost periodic if and only if there exists a sequence of trigonometric polynomials in X which converges to T in $\mathcal{D}'_{L^1}(\{p!\} : X)$.

Further on, we want to emphasize that the statement of [3, Theorem 3.5] can be extended to the vector-valued case:

Theorem 2.4. *Suppose that $T \in \mathcal{D}'_{L^1}(\{p!\} : X)$. Then the following statements are equivalent:*

(i) T is almost periodic.

(ii) $T * \varphi \in AP(\mathbb{R} : X)$ for all $\varphi \in \mathcal{F}_H$.

(iii) There exist two almost periodic functions $f : \mathbb{R} \to X$, $g : \mathbb{R} \to X$ and an ultradifferential operator P of class $\{p!^2\}$ such that $T = P(-\Delta)f + g$.

(iv) The Gauss transform $u(x, t)$ of T is almost periodic.

Now we would like to introduce the notion of a c-almost periodic hyperfunction, which extends the notion of an almost periodic hyperfunction ($c = 1$) due to Theorem 2.4(ii):

Definition 2.5. Suppose that $c \in S_1$ and $T \in \mathcal{D}'_{L^1}(\{p!\} : X)$. Then T is said to be c-almost periodic if and only if $T * \varphi \in AP_c(\mathbb{R} : X)$ for all $\varphi \in \mathcal{F}_H$.

Immediately from definition, it follows that any c-almost periodic hyperfunction is almost periodic, bounded and belongs to the Fourier class of hyperfunctions as well that the space of c-almost periodic functions is closed under differentiation. Many structural properties of c-almost periodic hyperfunctions can be obtained by using the corresponding structural properties of space $AP_c(\mathbb{R} : X)$ given in [12]; for example, any almost anti-periodic hyperfunction (obtained by plugging $c = -1$ in the above definition) is almost periodic and any c-almost periodic hyperfunction is almost anti-periodic, provided that $|c| = 1$, $p \in \mathbb{Z}\backslash\{0\}$, $q \in \mathbb{N}$, $(p, q) = 1$ and $\arg(c) = (p/q)\pi$. Furthermore, many structural properties of c-almost periodic hyperfunctions can be obtained analogously as for c-almost periodic distributions; for example, the statements of [11, Proposition 2.5, Proposition 2.6] continue to hold for c-almost periodic hyperfunctions.

Concerning c-almost periodic hyperfunctions, we have the following analogue of Theorem 2.4:

Theorem 2.6. *Suppose that $T \in \mathcal{D}'_{L^1}(\{p!\} : X)$. Consider the following statements:*

(i) There exists a X^2-valued c-almost periodic function $x \mapsto (f(x), g(x))$, $x \in \mathbb{R}$ and an ultradifferential operator P of class $\{p!^2\}$ such that $T = P(-\Delta)f + g$.

(ii) For every φ, $\psi \in \mathcal{F}_H$, the function $x \mapsto ((T * \varphi)(x), (T * \psi)(x))$, $x \in \mathbb{R}$ is c-almost periodic.

(iii) T is c-almost periodic.

(iv) The Gauss transform $u(x, t)$ of T is c-almost periodic.

Then we have (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) $\Leftrightarrow$ (iv).

Proof. The proofs of the equivalence (iii) $\Leftrightarrow$ (iv), the implication (ii) $\Rightarrow$ (i) and the implication (ii) $\Rightarrow$ (iii) can be given similarly as in the proof of [3, Theorem 3.5]. In order to see that (i) implies (ii), we can argue as in the proof of [11, Theorem 2.8]. Speaking-matter-of-factly, let φ, $\psi \in \mathcal{F}_H$. Let $\epsilon > 0$ be given, and let τ be a common (c,ϵ)-almost period of functions $f(\cdot)$ and $g(\cdot)$. If we assume that $\|\varphi\|_{p,k} < \infty$ for some $h,\ k > 0$, then for each $t \in \mathbb{R}$ we have

$$(T*\varphi)(t) = \langle T, \varphi(t-\cdot)\rangle = \sum_{p=0}^{\infty}(-1)^p a_p \int_{-\infty}^{+\infty} \varphi^{(2p)}(v) f(t-v)\, dv + \int_{-\infty}^{+\infty} \varphi(v) g(t-v)\, dv$$

and therefore

$$\begin{aligned}
&\big\|(T*\varphi)(t+\tau) - c(T*\varphi)(t)\big\| \\
&\leq \Bigg\| \sum_{p=0}^{\infty}(-1)^p a_p \int_{-\infty}^{+\infty} \varphi^{(2p)}(v)\big[f(t+\tau-v) - cf(t-v)\big]\, dv \\
&+ \int_{-\infty}^{+\infty} \varphi(v)\big[g(t+\tau-v) - cg(t-v)\big]\, dv \Bigg\| \\
&\leq \epsilon\left[\sum_{p=0}^{\infty}|a_p| \int_{-\infty}^{+\infty}\left|\varphi^{(2p)}(v)\right| dv + \int_{-\infty}^{+\infty}|\varphi(v)|\, dv\right].
\end{aligned}$$

We have the existence of a finite real number $M \geq 1$ such that $|\varphi^{(2p)}(v)| \leq Mh^{-p}e^{-k|v|}(2p)!$ for all $p \in \mathbb{N}_0$ and $v \in \mathbb{R}$. Moreover, for any $l \in (0, h/4)$, we have the existence of a finite real number $c > 0$ such that $|a_p| \leq cl^p p!^2$ for all $p \in \mathbb{N}_0$, so that we can continue the calculus as follows:

$$\begin{aligned}
&\leq \epsilon\left[\sum_{p=0}^{\infty} cl^p p!^2 Mh^{-p}(2p)! \int_{-\infty}^{+\infty} e^{-k|v|}\, dv + \int_{-\infty}^{+\infty} e^{-k|v|}\, dv\right] \\
&\leq \epsilon\left[\sum_{p=0}^{\infty} cl^p 2^{2p} Mh^{-p} \int_{-\infty}^{+\infty} e^{-k|v|}\, dv + \int_{-\infty}^{+\infty} e^{-k|v|}\, dv\right].
\end{aligned}$$

A similar estimate holds with the function $\psi(\cdot)$ considered, with the same number τ. This simply completes the proof. $\square$

Remark 2.7. Consider the following condition:

(i)' There exist two c-almost periodic functions $x \mapsto f(x)$, $x \in \mathbb{R}$, $x \mapsto g(x)$, $x \in \mathbb{R}$ and an ultradifferential operator P of class $\{p!^2\}$ such that $T = P(-\Delta)f + g$.

Then we clearly have that (i) implies (i)' but it is not clear whether (i)' implies (ii).

We close the paper with the observation that (Q,T) affine-periodic solutions and pseudo (Q,T) affine-periodic solutions for various classes of systems of ordinary differential equations have recently been analyzed by many mathematicians. The notion of (Q,T) affine-periodicity is a special case of the notion of $(w,\mathbb{T})$-periodicity, which has recently been introduced and analyzed in the infinite-dimensional setting by M. Fečkan, K. Liu and J.-R. Wang in [10]. We can extend this notion to the almost periodic setting in the following way: Let $\mathbb{T} : X \to X$ be a linear isomorphism. For a given $\varepsilon > 0$, a real number $\tau > 0$ is called $(\varepsilon,\mathbb{T})$-almost period of a continuous function $f : I \to X$ if and only if

$$\|f(t+\tau) - \mathbb{T}f(t)\| < \varepsilon,\ t \in I.$$

Denote by $\vartheta_{\mathbb{T}}(f,\varepsilon)$ the set of all $(\varepsilon,\mathbb{T})$-almost periods of $f(\cdot)$, i.e.,

$$\vartheta_{\mathbb{T}}(f,\varepsilon) := \left\{\tau \in I : \sup_{t\in I}\|f(t+\tau) - \mathbb{T}f(t)\| < \varepsilon\right\}.$$

A continuous function $f : I \to X$ is called $\mathbb{T}$-almost periodic if and only if for any $\varepsilon > 0$ the set $\vartheta_{\mathbb{T}}(f, \varepsilon)$ is relatively dense in $[0, \infty)$.

In the case that there exists an integer $k \in \mathbb{N}_0$ such that $\mathbb{T}^k = I$, the notion of $(w, \mathbb{T})$-periodicity is a special case of the notion of $\mathbb{T}$-almost periodicity; the converse statement does not true in general. In the case that $\mathbb{T} = cI$, where $c \in \mathbb{C} \setminus \{0\}$ and I denotes the identity operator on X, the notion of $\mathbb{T}$-almost periodicity reduces to the notion of c-almost periodicity. For more details, we refer the reader to the forthcoming paper [9].

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