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Adjugates on linear algebras

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Abstract. We present an abstract version of the "adjugate" or classical adjoint of a square matrix.

Introduction

The *Adjugate*, or "classical adjoint", of a square matrix, is constructed [1] by a curiously elaborate routine: striking out the row and column intersecting in an arbitrary entry of a $(k+1)\times(k+1)$ matrix, replacing the original matrix entry by the *determinant* of the $k\times k$ matrix thus revealed, then moving the result to its mirror image in the diagonal, and finally either changing its sign or not. Since the "determinant" itself is derived from another adjugate, the stage would appear to be set for an induction. In this note we try to ride two horses: on the one hand to provide this induction, and the other to embed it in a more abstract, axiomatic, environment.

1. Cofactors

By a *cofactor*, on a real or complex linear algebra A, with identity 1, we shall understand a mapping

1.1 $$\mathrm{adj} : a \mapsto a^\sim$$

satisfying the following three conditions: for arbitrary $a, b \in A$

1.2 $$1^\sim = 1\ ;$$

1.3 $$(ba)^\sim = a^\sim b^\sim\ ;$$

1.4 $$a^\sim a = a a^\sim = |a| 1 \in \mathbb{K}1\ .$$

Here $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ is either the real or the complex field, or possibly the rationals $\mathbb{Q}$. The number $|a|$ will be referred to as the *determinant* of a:

1.5 $$\mathrm{adj}(a)a = a\ \mathrm{adj}(a) = \det(a)1 \in \mathbb{K}1\ .$$

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The archetype of a cofactor is defined on the 2×2 matrices $A = \mathbb{K}^{2\times 2}$:

1.6 $$\mathrm{adj}\begin{pmatrix} a & m \\ n & b \end{pmatrix} = \begin{pmatrix} a & m \\ n & b \end{pmatrix}^{\sim} = \begin{pmatrix} b & -m \\ -n & a \end{pmatrix} .$$

Evidently this is a very simple construction, easily remembered. For this very special case, there are additional properties of the cofactor: for arbitrary $a, b \in A$ and $\alpha, \beta \in \mathbb{K}$

1.7 $$(a^{\sim})^{\sim} = a \ ;$$

1.8 $$(\alpha a + \beta b)^{\sim} = \alpha a^{\sim} + \beta b^{\sim} \ ;$$

1.9 $$a + a^{\sim} = \mathrm{tr}(a)1 \in \mathbb{K}1 \ .$$

The reader will recall that

1.10 $$\det \begin{pmatrix} a & m \\ n & b \end{pmatrix} = ab - mn = ba - mn = ba - nm \ ;$$

for the *trace* of (1.9) we have

1.11 $$\mathrm{tr}\begin{pmatrix} a & m \\ n & b \end{pmatrix} = a + b \ .$$

Part of our objective is to see that the adjugates of the Introduction are indeed “cofactors”, to characterize them as cofactors which are in some sense “minimal”, and then to show that these abstract matrix cofactors are accessible inductively for the square matrices $\mathbb{K}^{k\times k}$ of all orders $k \in \mathbb{N}$.

2. Invertibility and uniqueness

Our first observation is that the determinant of (1.5) satisfies conditions analagous to (1.2)-(1.4):

Lemma 2.1. *For arbitrary $a, b \in A$*

2.1 $$|1| = 1 \ ;$$

2.2 $$|ba| = |b| \ |a| = |a| \ |b| \ ;$$

2.3 $$|a|b = b|a| \ .$$

In particular the scalar element $|a|1 \in \mathbb{K}1 \subseteq A$ “commutes” with every $b \in A$.

Proof. Observe
$$|ba|1 = (ba)^{\sim}(ba) = a^{\sim}(b^{\sim}b)a = a^{\sim}|b|a = |a| \ |b|1.$$

□

The condition (1.3) in a sense nearly follows from (1.4): if $a, b \in A$ there is implication

2.4 $$c = a^{\sim}b^{\sim} \Longrightarrow (ba)c = c(ba) = |a| \ |b|1 \ .$$

We next claim that the determinant “determines” whether or not an element is invertible:

Lemma 2.2. *For arbitrary $a, b \in A$ there is implication*

$$ba = 1 \Longrightarrow |a| \neq 0 \ .$$

Conversely if $|a| \neq 0$ then

2.5 $$b = (1/|a|)a^{\sim} \Longrightarrow ba = 1 = ab \ .$$

Proof. Obviously

$$ba = 1 \Longrightarrow |b| \ |a| = |ba| = |1| = 1.$$

□

Observe that we have shown more than we advertised; if $a \in A$ has even half an inverse then it is fully invertible:

2.6 $$a \in A^{-1}_{left} \cup A^{-1}_{right} \Longrightarrow a \in A^{-1} \equiv A^{-1}_{left} \cap A^{-1}_{right} \ .$$

The implication (2.5) is sometimes known as *Cramer's Rule*.

We notice that the conditions (1.2)-(1.4) do not uniquely determine the "cofactor": indeed if the mapping $a \mapsto a^{\sim}$ satisfies these three conditions, and if $k \in \mathbb{N}$, then so do the mappings

2.7 $$a \mapsto |a|^k a^{\sim} \ , \ a \mapsto -a^{\sim} \ ,$$

with associated "determinants" $|a|^{k+1}$, $-|a|$. The special case (1.6) is in a sense "minimal" among all solutions of the system of conditions (1.2)-(1.4). We are not really troubled by this non-uniqueness; each such cofactor does its job of providing an effective determinant of invertibility. If however we replace the real or complex numbers $\mathbb{K}$ by the integers $\mathbb{Z}$, or the continuous functions $C(\Omega)$ on a topological space Ω, we need to be a little more careful:

2.8 $$a \in A^{-1} \Longrightarrow |a| \in \mathbb{K}^{-1} \ .$$

This is the same as the condition $|a| \neq 0$ for the real and the complex numbers, but when $\mathbb{K} = \mathbb{Z}$ it requires that

2.9 $$|a| \in \{1, -1\} \ ,$$

while for $C(\Omega)$ it means

2.10 $$|a|^{-1}(0) = \emptyset \subseteq \Omega \ .$$

3. Block triangles

To inductively extend a cofactor from 2×2 to $k\times k$ matrices we look at "block triangles". Miraculously, we can in a sense work with 2×2 matrices throughout. We start by looking at linear algebras of the form

3.1 $$G = \begin{pmatrix} A & M \\ N & B \end{pmatrix} = \left\{ \begin{pmatrix} a & m \\ n & b \end{pmatrix} : (a,m,n,b) \in A\times M\times N\times B \right\} ,$$

where A and B are linear algebras with identity, and M and N are bimodules over A and B. Formally M is a (left A, right B) bimodule and N a (left B, right A) bimodule; in addition there are bilinear mappings

3.2 $$(m,n)\mapsto m\cdot n : M\times N\to A\ ;\ (m,n)\mapsto n\cdot m : M\times N\to B\ .$$

Formal multiplication of 2×2 matrices lays the structure bare. There is an idempotent

3.3 $$P = P^2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in G\ ,$$

for which (to within isomorphism)

3.4 $$A = PGP\ ,\ M = PG(I-P)\ ,\ N = (I-P)GP\ ,\ B = (I-P)G(I-P)\ .$$

The "block triangles" occur when either $M = O$ or $N = O$; we now show that cofactors on A and B combine to generate cofactors on both kinds of block triangle.

Theorem 3.1. *Cofactors $a\mapsto a^\sim$ and $b\mapsto b^\sim$ on A and B generate cofactors*

3.5 $$\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \mapsto \begin{pmatrix} |b|a^\sim & -a^\sim m b^\sim \\ 0 & |a|b^\sim \end{pmatrix}$$

and

3.6 $$\begin{pmatrix} a & 0 \\ n & b \end{pmatrix} \mapsto \begin{pmatrix} |b|a^\sim & 0 \\ -b^\sim n a^\sim & |a|b^\sim \end{pmatrix}\ .$$

Proof. Perform the obvious multiplications. □

If in particular A and B are the matrix algebras $\mathbb{K}^{k\times k}$ and $\mathbb{K}^{\ell\times\ell}$, and the cofactors $a\mapsto a^\sim$ and $b\mapsto b^\sim$ are generated by the traditional recipe, deleting rows and columns, then it is not hard to see that the cofactors generated on the triangles by Theorem 3.1 are also given by the traditional recipe.

More generally if $T = RS$ is the product of an upper and a lower block triangle we define $T^\sim = S^\sim R^\sim$, and more generally still for a word in upper and lower block triangles

3.7 $$T = S_k \dots S_2 S_1 \Longrightarrow T^\sim = S_1^\sim S_2^\sim \dots S_k^\sim\ .$$

This gives a "partial generalization" of Theorem 3.1:

Lemma 3.2. *Generally*

3.8 $$a \in A^{-1} \Longrightarrow \begin{pmatrix} a & m \\ n & b \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ na^{-1} & 1 \end{pmatrix} \begin{pmatrix} a & m \\ 0 & b - na^{-1}m \end{pmatrix} \ ,$$

and

3.9 $$b \in B^{-1} \Longrightarrow \begin{pmatrix} a & m \\ n & b \end{pmatrix} = \begin{pmatrix} 1 & mb^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a - mb^{-1}n & 0 \\ n & b \end{pmatrix} \ ,$$

To produce the cofactors, and hence also the determinant, combine (3.5), (3.6) and (3.7).
Hence for example ([7] Problem 7.1 ; cf [5]) if $a \in A^{-1}$

3.10 $$\begin{pmatrix} a & m \\ n & b \end{pmatrix} \in G^{-1} \Longleftrightarrow b - na^{-1}m \in B^{-1} \ .$$

To extend our cofactors from $k \times k$ to $(k+1) \times (k+1)$ matrices apply Lemma 3.2 with $A = \mathbb{K}$ and $B = \mathbb{K}^{k \times k}$: if $a \neq 0$ then $a \in A^{-1}$ and (3.8) applies. If $a = 0$ then we may have to perform a row interchange, premultiplying by another invertible matrix, which will have determinant ± 1. While this seems very simple, such a multiplier may not be accessible as a product of block triangles:

Theorem 3.3. $T = \begin{pmatrix} a & m \\ n & b \end{pmatrix} \in \mathbb{K}^{2\times 2}$ *is triangular if and only if* $mn = 0$*, and is the product of two triangles if and only if*

3.11 $$ab = 0 \Longrightarrow mn = 0 \ .$$

Proof. Observe

$$\begin{pmatrix} a'' & m'' \\ 0 & b'' \end{pmatrix} \begin{pmatrix} a' & 0 \\ n' & b' \end{pmatrix} = \begin{pmatrix} a & m \\ n & b \end{pmatrix} \Longrightarrow mn = m''bn' \ ;$$

$$\begin{pmatrix} a' & 0 \\ n' & b' \end{pmatrix} \begin{pmatrix} a'' & m'' \\ 0 & b'' \end{pmatrix} = \begin{pmatrix} a & m \\ n & b \end{pmatrix} \Longrightarrow nm = n'am''$$

□

For a fundamental example

3.12 $$\omega = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is not the product of an upper and a lower triangle. To derive the formula (1.6) for the adjugate of a 2×2 matrix we combine Theorem 3.1 with two declarations:

Definition 3.4.

3.13 $$a \in \mathbb{K} = \mathbb{K}^{1\times 1} \Longrightarrow a^{\sim} = 1 \ ;$$

3.14 $$a = \omega \in \mathbb{K}^{2\times 2} \Longrightarrow a^{\sim} = -\omega \ .$$

It is evident that arbitrary $a \in \mathbb{K}^{2\times 2}$ can be written in the form

3.15 $$a = \omega^{\nu_0} u^{\nu_1} v^{\nu_{-1}} \ ,$$

where u and v are an upper and a lower triangle and each ν_j is in $\{0, 1\}$. Thus (3.14), together with Lemma 3.2, determines the cofactor for all 2×2 matrices.

4. Interchanges

It turns out that the two assertions of Definition 3.4, together with Theorem 3.1, also determine the cofactor for arbitrary 3×3 matrices, and indeed enable us to perform the induction which determines the adjugate for an arbitrary square matrix:

Theorem 4.1. *If*

4.1 $$T = \begin{pmatrix} a & m \\ n & b \end{pmatrix} \in G = \begin{pmatrix} A & M \\ N & B \end{pmatrix}$$

with

4.2 $$A = \mathbb{K} = \mathbb{K}^{1\times 1} \ , \ B = \mathbb{K}^{k\times k} \ ,$$

then the cofactor $T^{\sim} \in G$ is determined by the cofactor $b^{\sim} \in B$.

Proof. If $0 \neq a \in \mathbb{K}$ then $b^{\sim} \in B$ determines $T^{\sim} \in G$ by (3.8) from Lemma 3.2, while if $a = 0$ and also $n = 0$ then (3.5) from Theorem 3.1 does the job. Writing

4.3 $$T = \begin{pmatrix} a & m \\ n & b \end{pmatrix} = \begin{pmatrix} a & m \\ n_1 & b_1 \\ n_1' & b_1' \end{pmatrix}$$

gives

4,4 $$S_1 = \begin{pmatrix} \omega & 0 \\ 0 & 1 \end{pmatrix} \Longrightarrow S_1 T = \begin{pmatrix} n_1 & b_1 \\ a & m \\ n_1' & b_1' \end{pmatrix} \ .$$

We have just interchanged the first two rows of the matrix T, replacing $a = 0$ by n_1. More generally,

4.5 $$T = \begin{pmatrix} a & m \\ n_j'' & b_j'' \\ n_{j-1} & b_{j-1} \\ n_j & b_j \\ n_j''' & b_j''' \end{pmatrix} \ , \ S_j = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & 1 \end{pmatrix} \Longrightarrow S_j T = \begin{pmatrix} a & m \\ n_j'' & b_j'' \\ n_j & b_j \\ n_{j-1} & b_{j-1} \\ n_j''' & b_j''' \end{pmatrix} \ :$$

we can interchange row $j-1$ and row j, bringing n_j one step closer to $a=0$. Repeating the process,

4.6 $$R_j = S_1 S_2 \dots S_j \Longrightarrow R_j T = \begin{pmatrix} n_j & b_j \\ a & m \\ n_j^{iv} & b_j^{iv} \end{pmatrix} \Longrightarrow T = S_j \dots S_2 S_1 \begin{pmatrix} n_j & b_j \\ a & m \\ n_j^{iv} & b_j^{iv} \end{pmatrix} .$$

It follows that if $n_j \neq 0$ then the cofactor of the product $R_j T$ is again given by (3.8). Now each factor $S_\ell = S_\ell^{-1}$ can be treated as a block diagonal, is its own inverse, and has determinant -1; the induction is complete. $\square$

It would of course be economical to take n_j to be the first non-zero entry in the column n; it is not however necessary for the argument. For example the 2×2 interchange ω is the tip of an iceberg, none of the rest of which is needed to make the induction; for each $k \in \mathbb{N}$:

4.7 $$\varpi_1 = \omega \ ; \ \varpi_{k+1} = \begin{pmatrix} 0 & \varpi_k \\ 1 & 0 \end{pmatrix}$$

For example, to write out the adjugate of a general 3×3 matrix, with

4.8 $$m = \begin{pmatrix} m_1 & m_2 \end{pmatrix} \in \mathbb{K}^{1\times 2} \ ; \ n = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \in \mathbb{K}^{2\times 1} \ ,$$

write

4.9 $$m^\sim = \begin{pmatrix} m_2 \\ -m_1 \end{pmatrix} \ ; \ n^\sim = \begin{pmatrix} n_2 & -n_1 \end{pmatrix} \ ,$$

and observe

4.10 $$mm^\sim = 0 = n^\sim n \ ; \ (nm)^\sim = m^\sim n^\sim \ .$$

Then with $a \in \mathbb{K}$ and $b \in \mathbb{K}^{2\times 2}$, we find, for general $T \in \mathbb{K}^{3\times 3}$,

4.11 $$\begin{pmatrix} a & m \\ n & b \end{pmatrix}^\sim = \begin{pmatrix} |b| & -mb^\sim \\ -b^\sim n & ab^\sim - (nm)^\sim \end{pmatrix} \ .$$

Notice that the entries in the row $-m^\sim b$, the column $-bn^\sim$ and the 2×2 matrix $-(nm)^\sim$ are the familiar 2×2 determinants from the classical recipe for the left hand side of (4.11). Checking the conditions (1.4),

4.12 $$\begin{pmatrix} |b| & -mb^\sim \\ -b^\sim n & ab^\sim - (nm)^\sim \end{pmatrix} \begin{pmatrix} a & m \\ n & b \end{pmatrix} = \begin{pmatrix} |b|a - mb^\sim n & |b|m - mb^\sim b \\ -b^\sim na + ab^\sim n - (nm)^\sim n & -(nm)^\sim b - b^\sim nm + ab^\sim b \end{pmatrix}$$

and

4.13 $$\begin{pmatrix} a & m \\ n & b \end{pmatrix} \begin{pmatrix} |b| & -mb^\sim \\ -b^\sim n & ab^\sim - (nm)^\sim \end{pmatrix} = \begin{pmatrix} a|b| - mb^\sim n & -amb^\sim + mab^\sim - m(nm)^\sim \\ n|b| - bb^\sim n & -nmb^\sim + bab^\sim - b(nm)^\sim \end{pmatrix} \ .$$

On the right hand sides of (4.12) and (4.13) it is easy to see that the off-diagonals are all zero; to see that all four diagonal entries coincide observe that

4.14 $$\mathrm{tr}((nm)^\sim b) = \mathrm{tr}(b(nm)^\sim) = mb^\sim n \ .$$

This is a familiar observation about the trace of a product of 2×2 matrices, which apparently also extends to products of rows and columns.

For example, from (4.7) with $k = 2$,

4.15 $$\varpi_2^\sim = -\varpi_2 = -\varpi_2^{-1} \ .$$

Theorem 4.1 says that a cofactor mapping $T \mapsto T^\sim$, defined on the disjoint union of all square matrices $\mathbb{K}^{k\times k}$, and subject to (1.2)-(1.4), together with (3.5), (3.6), (3.13) and (3.14), is uniquely determined, and therefore given by the classical recipe. Theorem 4.1 therefore also shows that the classical recipe for the adjugate of a square matrix can be proved by induction. It also closes a tiny gap in our discussion [13] of the extension of the adjugate to the "socle" of a Banach algebra, making it now self-contained.

We can also observe that the classical recipe $a \mapsto \mathrm{adj}(a)$ generates a cofactor which is in a sense minimal among adjugate mappings $a \mapsto a^\sim$ satisfying the conditions (1.2)-(1.4):

4.16 $$a^\sim \in \{\det(a)^k \mathrm{adj}(a) : k \in \mathbb{Z}^+\} \cup \{-\mathrm{adj}(a)\} \ .$$

The condition (1.4) of course restricts a "cofactor" $a \mapsto a^\sim$ to square matrices, making our little foray into the realm of rows and columns (cf [10]) something of an *abus de notation*.

5. Operator matrices

The algebra of adjugates and determinants extends not just to the real and complex fields $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ but also to more general rings. If D is a ring, not necessarily commutative, with identity 1, then we can consider the ring

5.1 $$D^{2\times 2} = \left\{ T = \begin{pmatrix} a & m \\ n & b \end{pmatrix} : (a, m, n, b) \in D^4 \right\} \ ,$$

and write

5.2 $$T = \begin{pmatrix} a & m \\ n & b \end{pmatrix} \Longrightarrow [T] = \{a, m, n, b\} \subseteq D \ ;$$

Now provided $[T] \subseteq D$ is commutative (but *cf* [6]), we more or less repeat verbatim the introductory (1.2)-(1.11) with D in place of $\mathbb{K}$. For the analogue of (1.3) we need to augment the commutivity of (5.2):

5.3 $$[T] \cup [S] \ \textit{commutative} \Longrightarrow (ST)^\sim = T^\sim S^\sim \ .$$

If the coefficient ring D is an algebra, and also carries an adjugate mapping $d \mapsto d^\sim$, then there will be two kinds of adjugate on $E = D^{2\times 2}$:

5.4 $$T \mapsto \mathrm{adj}_D(T) \in E \ ; \ T \mapsto \det_D(T)^\sim \mathrm{adj}_D(T) \in E \ ,$$

with induced determinants

5.5 $$T \mapsto \det_D(T) \in D \ ; \ T \mapsto |\det_D(T)| \in \mathbb{K} \ .$$

When the ring D is not commutative, the invertibility condition (2.8) subdivides: with $E = D^{2\times 2}$ we have [8],[9] two implications:

5.6 $$T \in E^{-1}_{left} \Longleftrightarrow \det_D(T) \in D^{-1}_{left}$$

and

5.7 $$T \in E^{-1}_{right} \Longleftrightarrow \det_D(T) \in D^{-1}_{right} \ .$$

Definition 5.1. *If*

5.8 $$T = \begin{pmatrix} a & m \\ n & b \end{pmatrix} \in E = D^{2\times 2}$$

then (1.6)*,* (1.10)*,*

5.9 $$\mathrm{adj}_D(T) = \begin{pmatrix} b & -m \\ -n & a \end{pmatrix} \in E \ , \ \det_D(T) = ab - mn = ba - nm \in D \ ;$$

$\mathrm{adj}_{\mathbb{K}}(T) \in E$ and $\det_{\mathbb{K}}(T) \in \mathbb{K}$ are rather more complicated. The relationship between the two kinds of determinant can be [16],[17],[14] rather simple. For example if

5.10 $$D = \mathbb{K}^{2\times 2} \ ; \ E = D^{2\times 2} = \mathbb{K}^{4\times 4}$$

then

Theorem 5.2. *If $T \in E$ is given by* (5.8)*, with $D = \mathbb{K}^{2\times 2}$, then*

5.11 $$\det_{\mathbb{K}}(T) = \det_{\mathbb{K}}\det_D(T) = \det_{\mathbb{K}}(ab - mn) \in \mathbb{K}$$

and

5.12 $$\mathrm{adj}_{\mathbb{K}}(T) = \big(\mathrm{adj}_{\mathbb{K}}\det_D(T)\big)\mathrm{adj}_D(T) = \mathrm{adj}_{\mathbb{K}}(ab - mn) \begin{pmatrix} b & -m \\ -n & a \end{pmatrix} \in E \ .$$

Proof. Towards (5.12), writing, for $d \in D$, $\mathrm{adj}_{\mathbb{K}}(d) = d^{\sim}$ and $\det_{\mathbb{K}}(d) = |d|$, (1.8) and (1.3) together with the commutivity say that the right hand side of (5.12) takes the form

$$\begin{pmatrix} (ab - mn)^{\sim} & 0 \\ 0 & (ab - mn)^{\sim} \end{pmatrix} \begin{pmatrix} b & -m \\ -n & a \end{pmatrix} = \begin{pmatrix} a^{\sim}|b| - m^{\sim}bn^{\sim} & -a^{\sim}mb^{\sim} + |m|n^{\sim} \\ -b^{\sim}na^{\sim} + |n|m^{\sim} & |a|b - m^{\sim}an^{\sim} \end{pmatrix}$$

□

We remark that while (5.11) obviously follows from (5.12), the implication also runs the other way: for (5.11) says that in particular that

$$\mathrm{adj}_{\mathbb{K}}(T)T = \mathrm{adj}_{\mathbb{K}}(\det_D(T))\det_D(T) = \mathrm{adj}_{\mathbb{K}}(\det_D(T))\mathrm{adj}_D(T)T \in D \in E \ .$$

If however it should happen that $T \in E$ is "epimorphic", in the sense that there is implication

5.13 $$UT = O \Longrightarrow U = O \ ,$$

this gives back (5.12).

If we now extend the whole argument from D and E to polynomials $D[z]$ and $E[z]$, and then [KSW] replace $T \in E$ by $T - zI \in E[z]$, then we get the epimorphism (5.13).

Theorem 5.2 extends [17] from $D = \mathbb{K}^{2\times 2}$ and $E = D^{2\times 2}$ to $D = \mathbb{K}^{\ell\times\ell}$ and $E = D^{k\times k}$:

Theorem 5.3. *Suppose $a \in D$ and $b \in D^{k\times k}$, where D is a linear algebra with identity, then if $T \in E$ is given by the analogue of* (5.8) *then*

5.14 $$\begin{pmatrix} 1 & 0 \\ -n & \triangle_a \end{pmatrix} T = \begin{pmatrix} a & m \\ 0 & \triangle_a b - nm \end{pmatrix} ,$$

and hence inductively (5.11).

Proof. Here $\triangle_a \in B$ is the "diagonal" element aI_B generated by the scalar $a \in D$. If $a \in D$ is not a zero-divisor, so that also

$$\begin{pmatrix} 1 & 0 \\ 0 & \triangle_a \end{pmatrix} \in D \times B$$

is not a zero-divisor, then the same will apply to the relevant "determinants". Indeed if $T \in E = D^{k+1\times k+1}$ is given by (5.8), with $a \in D$ and $b \in D^{k\times k}$ then, inductively, we suppose that Theorem 5.3 holds with $b' \in D^{k\times k}$ in place of T: now [KSW] gives (5.14). It follows

$$\det_D \begin{pmatrix} 1 & a \\ -m & \triangle_a \end{pmatrix} \det_D(T) = \det_D \begin{pmatrix} a & m \\ 0 & \triangle_a b - nm \end{pmatrix} .$$

That is to say

$$a^k \det_D(T) = a \ \det_D(\triangle_a b - nm) ,$$

and now applying the inductive hypothesis to $b' = \triangle_a b - nm$ gives

$$|a|^k |\det_D(T)| = |a| \ |\triangle_a b - nm| .$$

Provided $a \in D$ is not a zero divisor, this gives Theorem 5.3 for T. Once again we replace D and E by $D[z]$ and $E[z]$ and replace T by $T - zI$ $\square$

For example Theorem 5.2 applies to the 4×4 version of (3.12): with $k = 3$ in (4.7)

5.15 $$T = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \omega \\ \omega & 0 \end{pmatrix} = \varpi_3 \Longrightarrow \det_D(T) = -\omega^2 = -1 \in D \ , \ \mathrm{adj}_D(T) = -T \in E \ .$$

For another example

5.16 $$T = d \in D \Longrightarrow \det_D(T) = d \ ; \ \mathrm{adj}_D(T) = 1 \ ,$$

for which Theorem 5.2 is trivial.

6. Quaternions

Complex numbers are traditionally introduced "rigourously" to real analysts as the cartesian product $\mathbb{R}^2$ with a fancy multiplication on board. We are here able to see them as a subalgebra of the 2×2 real matrices $\mathbb{R}^{2\times 2}$:

6.1 $$\mathbb{C} = \mathbb{R} + \mathbb{R}\iota \subseteq \mathbb{R}^{2\times 2} ,$$

where

6.2 $$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ , \ \iota = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ ,$$

giving, for arbitrary $(a, b) \in \mathbb{R}^2$,

6.3 $$a + b\iota = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \ .$$

Theorem 3.3 tells us that ι, the mysterious "square root of minus one", is not the product of two real triangles. In the notation of (5.2), the "complex conjugate" of $a + b\iota$ is given by the adjugate matrix:

6.4 $$(a + b\iota)^{\sim} = a - b\iota = (a + b\iota)^* \ .$$

Of course

6.5 $$\iota^2 = -1 \ .$$

Complex number multiplication is commutative, and we have the vital observation that

6.6 $$0 \neq a + b\iota \Longrightarrow \exists (a + b\iota)^{-1} \ :$$

every non-zero complex number is invertible. Also - here our notation begins to crack - the determinant is the *square* of the modulus:

6.7 $$\det(a + b\iota) = a^2 + b^2 \ .$$

It is also possible to represent Hamilton's *quaternions* $\mathbb{H}$ as 2×2 complex matrices: with

6.8 $$i = \begin{pmatrix} \iota & 0 \\ 0 & -\iota \end{pmatrix} \ , \ j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ , \ k = \begin{pmatrix} 0 & \iota \\ \iota & 0 \end{pmatrix}$$

we find

6.9 $$i^2 = j^2 = k^2 = ijk = -1 \ .$$

The general quaternion is the 2×2 complex matrix, with of course commuting entries,

6.10 $$T = t + xi + yj + zk = \begin{pmatrix} t + x\iota & y + z\iota \\ -y + z\iota & t - x\iota \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \in \mathbb{C}^{2 \times 2} \cong \mathbb{R}^{4 \times 4} \ .$$

If they were good for nothing else in mathematics, quaternions would earn their place for making sense of three dimensional "vector products" and "scalar triple products". Writing out the adjugate and the determinant we can see that a non zero quaternion again has a two-sided inverse quaternion. The quaternion (6.10) of course has both a "real" and a "complex" determinant:

6.11 $$\det_{\mathbb{C}}(T) = (t + x\iota)(t - x\iota) - (y + z\iota)(-y + z\iota) = |\alpha|^2 + |\beta|^2 \ ;$$

the real determinant is also, by (5.11), the real determinant of the complex determinant. Since the complex determinant is itself "real", and indeed positive, the real and the complex determinants coincide. The "complex" adjugate is given by

6.12 $$\mathrm{adj}_{\mathbb{C}}(T) = t - (xi + yj + zk) = \begin{pmatrix} \overline{\alpha} & -\beta \\ \overline{\beta} & \alpha \end{pmatrix} \ .$$

For the "real" adjugate we can go back to (5.12):

6.13 $$\mathrm{adj}_{\mathbb{R}}(T) = \big(\mathrm{adj}_{\mathbb{R}}\mathrm{det}_{\mathbb{C}}(T)\big)\mathrm{adj}_{\mathbb{C}}(T) = (t^2 + x^2 + y^2 + z^2)(t - (xi + yj + zk)) \ .$$

Roger Penrose observes [19] that, in contrast to complex numbers, the quaternion conjugate is in a sense "holomorphic", which somehow puts paid to "quaternion analysis":

6.14 $$2T^* + T + iTi + jTj + kTk = 0 \ .$$

We can also represent in this way more general "quadratic surds" [20]. If $\mathbb{Q} \subseteq \mathbf{F} \subseteq \mathbb{R}$ is a subfield of the reals, with

6.15 $$0 < r \in \mathbb{R} \setminus \{t^2 : t \in \mathbf{F}\}$$

then

6.16 $$c + d\sqrt{r} \in \mathbf{F} + \mathbf{F}\sqrt{r} \longleftrightarrow \begin{pmatrix} c + da & dm \\ dn & c - da \end{pmatrix} \in \mathbf{F}^{2\times 2}$$

with

6.17 $$mn = r - a^2 \ ;$$

the general solution in $\mathbf{F}^{2\times 2}$ of the equation $T^2 = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}$ is given by

6.18 $$T = \begin{pmatrix} a & m \\ n & -a \end{pmatrix} \ ,$$

subject to (6.17).

7. Bilateral Hankel operators

There is a similar pattern lurking in the theory of Hankel and Toeplitz operators. Toeplitz operators on the unit circle $\mathbb{S} \subseteq \mathbb{C}$ can be seen as truncations of *Laurent operators*, which collectively form the commutant of the bilateral forward shift:

7.1 $$D = \{L_\varphi : \varphi \in L_\infty(\mathbb{S})\} = \{T \in B(L_2(\mathbb{S})) : TU = UT\} \ .$$

where

7.2 $$U = L_z : f \mapsto z \cdot f : L_2(\mathbb{S}) \to L_2(\mathbb{S})$$

is the bilateral forward shift, and $z \in L_\infty(\mathbb{S})$ the fundamental complex co-ordinate. The spectral theory of a Laurent operator L_φ is identical to that, the *essential range*, of its "symbol" φ:

7.3 $$\sigma(L_\varphi) = \sigma(\varphi) = \varphi_{ess}(\mathbb{S}) \ .$$

The *bilateral Hankel operators*, those which intertwine the forward and backward shifts,

7.4 $$TU = U^*T \ .$$

do not constitute a subalgebra of $B(L_2(\mathbb{S}))$, and are best understood [21],[18],[15] by adding them to Laurent operators:

7.5 $$E = \{L_\varphi + L_\psi S : \{\varphi, \psi\} \subseteq L_\infty(\mathbb{S})\} \ .$$

Here the bilateral Hankel operator S is defined by setting

7.6 $$f \in L_2(\mathbb{S}) \ , \ \lambda \in \mathbb{S} \Longrightarrow (Sf)(\lambda) = f(-\lambda) \ .$$

The spectral theory of bilateral Hankel operators is derived from the correspondence

7.7 $$A + BS \longleftrightarrow \begin{pmatrix} A & B \\ SAS & SBS \end{pmatrix} \ ,$$

which gives rise to a "determinant"

7.8 $$\det_D(A + BS) = ASBS - BSAS \in D \ ,$$

and an "adjugate"

7.9 $$\mathrm{adj}_D(A + BS) = SBS - AS \in E \ .$$

In turn Laurent and bilateral Hankel operators can be represented as 2×2 matrices of Toeplitz and Hankel operators: represented, via Fourier series, as operators on $\ell_2 \equiv \ell_2(\mathbb{N})$, the *Toeplitz* and *Hankel* operators intertwine the forward and backward unilateral shifts

7.10 $$u : (x_1, x_2 \ldots) \mapsto (0, x_1, x_2, \ldots) \ , \ v : (x_1, x_2, \ldots) \mapsto (x_2, x_3, \ldots) \ ;$$

formally $a \in B(\ell_2)$ is Toeplitz provided

7.11 $$vau = a \ ,$$

and is Hankel provided

7.12 $$va = au \ .$$

For example

7.13 $$U = \begin{pmatrix} u & 1 - uv \\ 0 & v \end{pmatrix} \ ; \ V = \begin{pmatrix} v & 0 \\ 1 - uv & u \end{pmatrix} \ .$$

Now ([15] Theorem 2.5)

7.14 $$\begin{pmatrix} a & m \\ n & b \end{pmatrix} \text{ Laurent} \Longrightarrow a, b \text{ Toeplitz and } m, n \text{ Hankel} \ ,$$

and

7.15 $$\begin{pmatrix} a & m \\ n & b \end{pmatrix} \text{ bilateral Hankel} \Longrightarrow a, b \text{ Hankel and } m, n \text{ Toeplitz} \ .$$

Thus Laurent and bilateral Hankel operators can also be thought of as 4×4 matrices of Hankel and Toeplitz operators; of course here neither the entries, nor the 2×2 submatrices, mutually commute. The spectral theory of Toeplitz and Hankel operators is much more complicated than that of bilateral Hankel operators.

8. Tensor products

If cofactors $a \mapsto a^\sim$ and $b \mapsto b^\sim$ on linear algebras A and B satisfy the conditions (1.2)-(1.4) then so does the mapping

8.1 $$a \otimes b \mapsto a^\sim \otimes b^\sim \ : A \otimes B \to A \otimes B$$

on the *tensor product* [9] $A \otimes B$, with associated determinant $|a|\ |b|$. If in particular $A = \mathbb{K}^{k\times k}$ then

8.2 $$B \otimes A \cong A \otimes B \cong B^{k\times k}\ .$$

If in particular $k = 2$ and the entries mutually commute, we are back in the situation of Definition 5.1, with $D = B$; we shall then write

8.3 $$T = \begin{pmatrix} a & m \\ n & b \end{pmatrix} \Longrightarrow T \otimes S = \begin{pmatrix} aS & mS \\ nS & bS \end{pmatrix}\ .$$

Generally if $a \in A = \mathbb{K}^{k\times k}$ and $b \in B$ then

Theorem 8.1. *If $a \in A = \mathbb{K}^{k\times k}$ and $b \in B$ then*

8.4 $$\det_B(a \otimes b) = |a|b^k \equiv \det_{\mathbb{K}}(a)b^k \in B$$

and

8.5 $$\mathrm{adj}_B(a \otimes b) = a^\sim \otimes b^{k-1} \equiv \mathrm{adj}_{\mathbb{K}}(a) \otimes b^{k-1} \in A \otimes B\ .$$

Proof.

8.6 $$a \otimes b = (a \otimes 1_B)(1_A \otimes b) \in A \otimes B \cong B^{k\times k}$$

with

8.7 $$\det_B(a \otimes 1_B) = |a|1_B \in B$$

and

8.8 $$\det_B(1_A \otimes b) = b^k \in B$$

and hence

8.7 $$\det_B(a \otimes b) = |a|b^k \equiv \det_{\mathbb{K}}(a)b^k \in B\ .$$

It follows

8.8 $$\mathrm{adj}_B(a \otimes b) = a^\sim \otimes b^{k-1} \equiv \mathrm{adj}_{\mathbb{K}}(a) \otimes b^{k-1} \in A \otimes B\ .$$

□

If we now specialize to also $B = \mathbb{K}^{\ell\times\ell}$, then

Theorem 8.2. *If $a \in A = \mathbb{K}^{k\times k}$ and $b \in B = \mathbb{K}^{\ell\times\ell}$ then*

8.9 $$(a \otimes b)^\sim = (a \otimes 1_B)^\sim(1_A \otimes b)^\sim = |a|^{\ell-1}|b|^{k-1}(a^\sim \otimes b^\sim) \in A \otimes B\ .$$

Proof. This is Theorem 3.1 together with Theorem 5.3 □

Alternatively prove it for $a\otimes 1_B$, using Theorem 3.1 and induction on ℓ; similarly for $1_A \otimes b$ with induction on k. For example "scalar multiplication"

8.10 $$T \mapsto \alpha T = \alpha \otimes T$$

is a tensor product, if we interpret $\alpha \in \mathbb{K} = \mathbb{K}^{1\times 1}$ as a 1×1 matrix.

For another example the matrix $T = \varpi_3$ of (5.15) is the tensor product

8.11 $$\begin{pmatrix} 0 & \omega \\ \omega & 0 \end{pmatrix} = \omega \otimes \omega\ .$$

9. Spatial determinants

For a more "spatial" version of the the determinant, and the adjugate, of a square matrix, we recall the *Koszul complex* [10],[12] of a pair of commuting linear operators, or more general algebra elements: If $t = (t_1, t_2) \in A^2$ then this is given [10],[12] by the sequence of matrices $(0, T^\sim, T, 0)$, where, as in (4.8),

9.1 $$T = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} \ , \ T^\sim = \begin{pmatrix} t_2 & -t_1 \end{pmatrix} \ .$$

As we recalled in the Introduction, in the traditional recipe [1] the adjugate of a square matrix is constructed by carrying out two separate operations, which indeed commute with one another. Firstly each entry in the matrix is replaced by the determinant of the matrix which is revealed when the row and the column intersecting at that point are deleted; then secondly each entry is moved across the diagonal and either has its sign changed or not. This determinant is of course what is traditionally referred to as the "cofactor" of the particular entry; our matrix-valued cofactor is therefore an *abus de language*. The second of these two operations by itself gives the adjugate of a 2×2 matrix, but can also [10] be carried out on an arbitrary rectangular matrix, where it satisfies the condition (1.7) and (1.8), and indeed also (1.3). This what we have just done in (9.1), but earlier in (4.8) to the row and the column of (4.7).
If now $p = (p_1, p_2)$ is a pair of two-variable polynomials without constant term then the Koszul complex $(0, S^\sim, S, 0)$ of $s = p(t)$ is related to the Koszul complex of t by factorizations

9.2 $$S = UT \ ; \ S^\sim U = |U|T \ ; \ S^\sim = T^\sim U^\sim \ ; \ U^\sim S = T|U| \ ,$$

where

9.3 $$U = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \in A^{2\times 2} \ .$$

If it should happen that the Koszul complex of s is *exact*, in the sense that there are matrices R and $R^\sim$ for which

9.4 $$R^\sim S = 1 = S^\sim R \ ,$$

and also

9.5 $$RS^\sim + SR^\sim = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ ,$$

then we maintain that also the Koszul complex of t is exact. Indeed certainly

9.6 $$(R^\sim U)T = 1 = T^\sim (U^\sim R) \ ,$$

and then also

9.7 $$RT^\sim U^\sim + UTR^\sim = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ ;$$

but we now claim that (9.7) can be replaced by

9.8 $$U^\sim RT^\sim + TR^\sim U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ .$$

Specifically, if

9.9 $$WT = 1 = T^\sim W^\sim$$

with

$$\{t_1, t_2\} \subseteq \mathrm{comm}\{w_1, w_2\} \ ,$$

then also ([12] (10.6), (10.7))

9.10 $$TW + W^{\sim}T^{\sim} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ .$$

If define U by (9.3), with mutually commuting u_{ij}, and then vary $\{t_1, t_2\} \subseteq \mathrm{comm}\{u_{11}, u_{12}, u_{21}, u_{22}\}$, then the rest of the factorization (9.2) is given uniqely by

9.11 $$U^{\sim} = \mathrm{adj}(U) \ ; \ |U| = \det(U) \ .$$

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