Functional Analysis,
Approximation and
Computation
14 (1) (2022), 47–54

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/faac

Class of (n, m)-power-D-quasi-hyponormal operators in Hilbert space

Abdelkader Benali[a]

[a] *Laboratory (LMA), Faculty of the Exact Sciences and Computer, Mathematics Department, University of Hassiba Benbouali, Chlef Algeria. B. P. 151 Hay Essalem, Chlef 02000, Algeria*

Abstract. In this paper, we introduce a new classes of operators acting on a complex hilbert space H , denoted by $[(n, m)DQH]$, called (n, m)-power-D-quasi-hyponormal associated with a Drazin invertible operator usingits Drazin inverse. Somme properties of (n, m)-power-D-quasi-hyponormal, are investigated and somme examples.

1. INTRODUCTION

Let $\mathcal{H}$ be a complex Hilbert space. Let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators defined in $\mathcal{H}$. Let T be an operator in $\mathcal{B}(\mathcal{H})$. The operator T is called normal if it satisfies the following condition $T^*T = TT^*$, i.e.,T commutes with T^*. The class of quasinormal operators was first introduced and studied by A. brown in [5] in 1953. The operator T is quasi-normal if T commutes with T^*T, i.e. $T(T^*T) = (T^*T)T$ and it is denoted by $[QN]$. A.A.S. Jibril [7, 8], in 2008 introduced the class of n power normal operators as a generalization of normal operators. The operator T is called n power normal if T^n commutes with T^* , i.e., $T^nT^* = T^*T^n$ and is denoted by $[nN]$. In the year 2011, O.A. Mahmoud Sid Ahmed introduced n power quasi normal operators [15], as a generalization of quasi normal operators. The operator T is called n power quasi normal if T^n commutes with T^*T , i.e., $T^n(T^*T) = (T^*T)T^n$ and it is denoted by $[nQN]$.
Recently in [14], the authors introduced and studied the operator $[(n, m)DN]$ and $[(n, m)DQ]$.In this search, we introduce a new class of operators T namely (n, m)-power-D-hyponormal operator for a positive integer n, m if

$$T^{*m}T(T^D)^n \geq (T^D)^nT^{*m}T, m = n = 1, 2, ...$$

denoted by $[(n, m)DQH]$.

And we in this work, we will try to apply the same results obtained in [9] for this new classes.

Definition 1.1. *An operator $T \in \mathcal{B}(H)$ be Drazin inversible operator. We said that T is (n, m)-power-D-quasi-hyponormal operator for a positive integer n, m if*

$$T^{*m}T(T^D)^n \geq (T^D)^nT^{*m}T, m = n = 1, 2, ...$$

We denote the set of all (n, m)-Power-D-quasi-hyponormal operators by $[(n, m)QH]$

2020 *Mathematics Subject Classification*. Primary 46C05 ; Secondary 47A05.
Keywords. Hilbert space; (n, m)-power-D-quasi-hyponormal operators; D-idempotent; Power-D-quasi-hyponormal operators; $(2, 2)$-power-D-quasi-normal operators; Drazin invertible operator.
Received: 15 November 2021; Accepted: 14 December 2021
Communicated by Dragan S. Djordjević
Email address: benali4848@gmail.com (Abdelkader Benali)

Remark 1.2. *Clearly* $n = m = 1$*, then* $(1,1)$*-Power-D-quasi-hyponormal operator is precisely Power-D-quasi-hyponormal operator.*

Definition 1.3. *An operator* $T \in \mathcal{B}(\mathcal{H})^D$ *is said to be* (n,m)*-power-D-quasi-hyponormal if* $T^{*m}T(T^D)^n - (T^D)^n T^{*m}T$ *is positive i.e:* $T^{*m}T(T^D)^n - (T^D)^n T^{*m}T \geq 0$ *or equivalently*

$$\langle \left(T^{*m}T(T^D)^n - (T^D)^n T^{*m}T\right)u \mid u\rangle \geq 0 \ \text{ for all } \ u \in \mathcal{H}.$$

Example 1.4. *Let* $T = \begin{pmatrix} 3 & -2 \\ 0 & -3 \end{pmatrix}, S = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \in \mathcal{B}(\mathbb{R}^2)$*. A simple computation shows that*

$$T^D = \frac{1}{9}\begin{pmatrix} 3 & -2 \\ 0 & -3 \end{pmatrix}, S^D = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, S^* = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, T^* = \begin{pmatrix} 3 & 0 \\ -2 & -3 \end{pmatrix}.$$

Then $T \in [(2,2)DQH]$*, but* $T \notin [(3,3)DQH]$ *and* S *is* $(3,2)$*-power-D-co-quasi-hyponormal , but* $S \notin [(2,2)DQH]$

Proposition 1.5. *If* S*,* $T \in \mathcal{B}(\mathcal{H})^D$ *are unitarily equivalent and if* T *is* (n,m)*-Power-D-quasi-hyponormal operators then so is* S

Proof. Let T be an (n,m)-Power-D-quasi-hyponormal operator and S be unitary equivalent of T. Then there exists unitary operator U such that $S = UTU^*$ so $S^n = UT^nU^*$
We have

$$\begin{aligned}
(S^D)^n S^{*m} S &= U(T^D)^n U^* \,(UT^mU^*)^* \,UTU^* \\
&= U(T^D)^n U^* U T^{*m} T U^* \\
&= U(T^D)^n T^{*m} T U^* \\
&\leq UT^{*m}T(T^D)^n U^* \\
&\leq (UT^mU^*)^* \,UTU^* U(T^D)^n U^* \\
&= S^{*m}S(S^D)^n
\end{aligned}$$

Hence, $S^{*m}S(S^D)^n - (S^D)^n S^{*m} S \geq 0$ □

Proposition 1.6. *Let* $T \in \mathcal{B}(\mathcal{H})^D$ *be an* (n,m)*-Power-D-quasi-hyponormal operator. Then* T^* *is* (n,m)*-Power-D-co-quasi-hyponormal operator*

Proof. Since T is (n,m)-Power-D-quasi-hyponormal operator. We have

$$(T^D)^n T^{*m} T \leq T^{*m}T(T^D)^n \Rightarrow \left((T^D)^n T^{*m} T\right)^* \leq \left(T^{*m}T(T^D)^n\right)^* \quad \Rightarrow \quad T^*T^m(T^D)^{*n} \leq (T^D)^{*n}T^*T^m.$$

Hence, T^* is (n,m)-Power-D-co-quasi-hyponormal operator. □

Proposition 1.7. *Let* $T \in \mathcal{B}(\mathcal{H})^D$ *be an* (n,m)*-Power-D-quasi-hyponormal operator. Then* T^* *is* (n,m)*-Power-D-co-quasi-hyponormal operator*

Proof. Since T is (n,m)-Power-D-quasi-hyponormal operator. We have

$$T^{*m}T(T^D)^n \geq (T^D)^n T^{*m} T \Rightarrow \left(T^{*m}T(T^D)^n\right)^* \geq \left((T^D)^n T^{*m} T\right)^* \quad \Rightarrow \quad (T^D)^{*n}T^*T^m \geq T^*T^m(T^D)^{*n}.$$

Hence, T^* is (n,m)-Power-D-co-quasi-hyponormal operator. □

Theorem 1.8. *If* T*,* T^* *are two* (n,m)*-Power-D-quasi-hyponormal operator, then* T *is an* (n,m)*-Power-D-quasi-normal operator.*

Proposition 1.9. *If T is $(2,2)$-power-D-quasi-hyponormal operator such that $T^DT^* = -T^*T^D$and $T^DT = TT^D$. Tthen T is $(2,2)$-Power-D-quasi-normal operator.*

Proof. Since $(T^D)^2T^{*2}T = T^DT^DT^*T^*T = -T^DT^*T^DT^*T = T^DT^*T^*T^DT = -T^*T^DT^*T^DT = T^{*2}T(T^D)^2$
And
$T^{*2}T(T^D)^2 = T^*T^*TT^DT^D = -T^*T^DT^*TT^D = T^DT^*T^*TT^D = -T^DT^*T^DT^*T = (T^D)^2T^{*2}T$
So
T is $(2,2)$-Power-D-quasi-hyponormal, then

$$\begin{aligned}
(T^D)^2T^{*2}T \leq T^{*2}(T^D)^2 \quad &\Rightarrow \quad T^DT^DT^*T^*T \leq T^*T^*TT^DT^D \\
&\Rightarrow \quad -T^DT^*T^DT^*T \leq -T^*T^DT^*TT^D \\
&\Rightarrow \quad T^DT^*T^*T^DT \geq T^DT^*T^*TT^D \\
&\Rightarrow \quad -T^*T^DT^*T^DT \geq -T^DT^*T^DT^*T \\
&\Rightarrow \quad -T^*T^DT^*T^DT \geq -T^DT^*T^DT^*T \\
&\Rightarrow \quad T^{*2}T(T^D)^2 \leq (T^D)^2T^{*2}T.
\end{aligned}$$

Hence $T^{*2}T(T^D)^2 = (T^D)^2T^{*2}T$. □

Example 1.10. *Let* $T = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \in \mathcal{B}(\mathbb{C}^3)$. *A simple computation, shows that ;* $T^* = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $T^D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$.

Then power-D-quasi-hyponormal operator, but

$$T^{*2}(T^D)^2 \neq (T^D)^2T^{*2} \text{and} \;\; T^*T(T^D)^2 \neq (T^D)^2T^*T.$$

Lemma 1.11. *Let $T_k, S_k \in \mathcal{B}(\mathcal{H})^D$, $k = 1,2$ such that $T_1 \geq T_2 \geq 0$ and $S_1 \geq S_2 \geq 0$, then*

$$\left(T_1 \otimes S_1\right) \geq \left(T_2 \otimes S_2\right) \geq 0.$$

Theorem 1.12. *. Let $T, S \in \mathcal{B}(\mathcal{H})^D$, such that $(S^D)^nS^*S \geq 0$ and $(T^D)^nT^*T \geq 0$, then .*
$T \otimes S$ is $(n,1)$-Power-D-quasi-hyponormal if and only if T and S are $(n,1)$-Power-D-quasi-hyponormal operators

Proof. Assume that T, S are $(n,1)$-power-D-quasi-hyponormal operators. Then

$$\begin{aligned}
(\left(T \otimes S\right)^D)^n\left(T \otimes S\right)^*\left(T \otimes S\right) &= \left(T^D \otimes S^D\right)^n\left(T^* \otimes S^*\right)\left(T \otimes S\right) \\
&= (T^D)^nT^*T \otimes (S^D)^nS^*S \\
&\leq T^*T(T^D)^n \otimes S^*S(S^D)^n \\
&= \left(T \otimes S\right)^*\left(T \otimes S\right)(\left(T \otimes S\right)^D)^n.
\end{aligned}$$

Which implies that $T \otimes S$ is $(n,1)$-power-D-quasi-hyponormal operator.

Conversely, assume that $T \otimes S$ is $(n,1)$-power-D-quasi-hyponormal operator.We aim to show that T, S are $(n,1)$-power-D-quasi-hyponormal. Since $T \otimes S$ is a $(n,1)$-power-D-quasi-hyponormal operator, we obtain

$$\begin{aligned}
(T \otimes S) \text{ is } (n,1)\text{-power-}D\text{-quasi-hyponormal} \quad &\Longleftrightarrow \quad ((\left(T \otimes S\right)^D)^n\left(T \otimes S\right)^*\left(T \otimes S\right) \leq \left(T \otimes S\right)^* \quad . \\
\left(T \otimes S\right)(\left(T \otimes S\right)^D)^n \quad & \\
&\Longleftrightarrow \quad (T^D)^nT^*T \otimes (S^D)^nS^*S \leq T^*T(T^D)^n \otimes S^*S(S^D)^n.
\end{aligned}$$

Then, there exists $d > 0$ such that

$$\begin{cases} d\ T^*T(T^D)^n \geq (T^D)^nT^*T. \\ \text{and} \\ d^{-1}S^*S(S^D)^n \geq (S^D)^nS^*S \end{cases}$$

a simple computation shows that $d = 1$ and hence

$$(T^D)^nT^*T \leq T^*T(T^D)^n \quad \text{and} \quad (S^D)^nS^*S \leq S^*S(S^D)^n.$$

Therefore, T, S are $(n, 1)$-power-D-quasi-hyponormal.
□

Proposition 1.13. *If $T, S \in \mathcal{B}(\mathcal{H})^D$ are $(n, 1)$-power-D-quasi-hyponormal operators commuting, such that such that $0 \leq (S^D)^nS^*(T^D)^nT^*TS \leq S^*(S^D)^nT^*(T^D)^nTS$ and $(T^D)^nT^*T \geq 0$, then $TS\otimes T, TS\otimes S, ST\otimes T$ and $ST\otimes S \in \mathcal{B}(\mathcal{H}\overline{\otimes}\mathcal{H})^D$ are $(n, 1)$-power-D-quasi-hyponormal if the following assertions hold:*

(1) $S^*(T^D)^n = (T^D)^nS^*$.

(2) $T^*(S^D)^n = (S^D)^nT^*$.

(3) $TS(S^D)^n(T^D)^n = (S^D)^n(T^D)^nTS$.

Proof. Assume that the conditions (1), (2) and (3) are hold. Since T and S are $(n, 1)$-power-D-quasi-hyponormal, we have

$$\begin{aligned}
(\big(TS \otimes T\big)^D)^n\big(TS \otimes T\big)^*\big(TS \otimes T\big) &= \big((TS)^D \otimes T^D\big)^n\big((TS)^* \otimes T^*\big)\big(TS \otimes T\big) \\
&= \big(((TS)^D)^n(TS)^*(TS) \otimes (T^D)^nT^*T\big) \\
&= \big(((S^D)^n(T^D)^n)S^*T^*TS \otimes (T^D)^nT^*T\big) \\
&= \big((S^D)^nS^*(T^D)^nT^*TS \otimes (T^D)^nT^*T\big) \\
&\leq \big(S^*(S^D)^nT^*(T^D)^nTS \otimes T^*T(T^D)^n\big) \\
&= \big(S^*T^*TS(S^D)^n(T^D)^n \otimes T^*T(T^D)^n\big) \\
&= \big((TS)^*(TS)((TS)^D)^n \otimes T^*T(T^D)^n\big) \\
&= \big((TS)^* \otimes T^*\big)\big((TS) \otimes T\big)\big(((TS)^D)^n \otimes (T^D)^n\big) \\
&= \big(TS \otimes T\big)^*\big(TS \otimes T\big)(\big(TS \otimes T\big)^D)^n
\end{aligned}$$

Then $TS \otimes S$ is $(n, 1)$-power-D-quasi-hyponormal operator.
In the same way, we may deduce the $(n, 1)$-power-D-quasi-hyponormal operator of $TS \otimes S, ST \otimes T$ and $ST \otimes S$. □

Theorem 1.14. *If $T, S \in \mathcal{B}(\mathcal{H})^D$ two operators commuting. Then :*

$\big(I \otimes S\big), \big(T \otimes I\big)$ are $(n, 1)$-power-D-quasi-hyponormal then $T \boxplus S$ is $(n, 1)$-power-D-quasi-hyponormal.

Proof. Firstly, observe that if $\big(I \otimes S\big), \big(T \otimes I\big)$ are $(n, 1)$-power-D-quasi-hyponormal, then we have following inequalities

$$\big((T \otimes I\big)^D)^n\big(T \otimes I\big)^*\big(T \otimes I\big) \leq \big(T \otimes I\big)^*\big(T \otimes I\big)\big((T \otimes I\big)^D)^n$$

and

$$\Big((S\otimes I)^D\Big)^n\Big(S\otimes I\Big)^*\Big(S\otimes I\Big)\leq\Big(S\otimes I\Big)^*\Big(S\otimes I\Big)\Big((S\otimes I)^D\Big)^n.$$

Then

$$\begin{aligned}
&((T\boxplus S)^D)^n(T\boxplus S)^*(T\boxplus S)\\
&\quad= \Big((T\otimes I+I\otimes S)^D\Big)^n\Big(T\otimes I+I\otimes S\Big)^*\Big(T\otimes I+I\otimes S\Big)\\
&\quad= \Big((T\otimes I)^D+(I\otimes S)^D\Big)^n\Big((T\otimes I)^*+(I\otimes S)^*\Big)\Big((T\otimes I)+(I\otimes S)\Big)\\
&\quad= ((T\otimes I)^D)^n\Big(T\otimes I)^*\Big((T\otimes I)+((T\otimes I)^D)^n(I\otimes S)^*\Big(T\otimes I)\\
&\quad+ ((I\otimes S)^D)^n(T\otimes I)^*\Big(T\otimes I)+((I\otimes S)^D)^n(I\otimes S)^*\Big(T\otimes I)\\
&\quad+ ((T\otimes I)^D)^n\Big(T\otimes I)^*\Big((I\otimes S)+((T\otimes I)^D)^n(I\otimes S)^*\Big(I\otimes S)\\
&\quad+ ((I\otimes S)^D)^n(T\otimes I)^*\Big(I\otimes S)+((I\otimes S)^D)^n(I\otimes S)^*\Big(I\otimes S)\\
&\quad\leq \Big(T\otimes I)^*\Big(T\otimes I)((T\otimes I)^D)^n+(I\otimes S)^*(I\otimes S\Big)((T\otimes I)^D)^n\\
&\quad+ (T\otimes I)^*(T\otimes I\Big)((I\otimes S)^D)^n+(I\otimes S)^*(I\otimes S\Big)((I\otimes S)^D)^n\\
&\quad= (T\boxplus S)^*(T\boxplus S)((T\boxplus S)^D)^n.
\end{aligned}$$

Then $T\boxplus S$ is $(n,1)$-power-D-quasi-hyponormal. □

Theorem 1.15. *Let $T_1, T_2, ..., T_m$ are $(n,1)$-power-D-quasi-hyponormal operator in $\mathcal{B}(\mathcal{H})^D$, such that $(T_k^D)^nT_k^*T_k \geq 0$, $\forall k\in\{1,2...m\}$. Then $(T_1\oplus T_2\oplus....\oplus T_m)$ is $(n,1)$-power-D-quasi-hyponormal operators and $(T_1\otimes T_2\otimes....\otimes T_m)$ is $(n,1)$-power-D-quasi-hyponormal operators.*

Proof. Since

$$\begin{aligned}
\Big((T_1\oplus T_2\oplus...\oplus T_m)^D\Big)^n(T_1\oplus T_2\oplus...\oplus T_m)^*(T_1\oplus T_2\oplus...\oplus T_m) &= \Big((T_1^D)^n\oplus(T_2^D)^n\oplus...\oplus(T_m^D)^n\Big)\\
&. \ \Big(T_1^*T_1\oplus T_2^*T_2\oplus...\oplus T_m^*T_m^*\Big)\\
&= ((T_1^D)^nT_1^*T_1\oplus(T_2^D)^nT_2^*T_2\oplus...\\
&\quad \oplus(T_m^D)^nT_m^*T_m)\\
&\leq (T_1^*T_1(T_1^D)^{n}{}^*_2T_2(T_2^D)^n\oplus...\\
&\quad \oplus T_m^*T_m(T_m^D)^n)\\
&= (T_1\oplus T_2\oplus...\oplus T_m)^*\\
&. \ (T_1\oplus T_2\oplus...\oplus T_m)\\
&. \ \Big((T_1\oplus T_2\oplus...\oplus T_m)^D\Big)^n.
\end{aligned}$$

Then $(T_1\oplus T_2\oplus....\oplus T_m)$ is $(n,1)$-power-D-quasi-hyponormal operators.

Now,

$$
\begin{array}{rcl}
\left((T_1 \otimes T_2 \otimes ... \otimes T_m)^D\right)^n (T_1 \otimes T_2 \otimes ... \otimes T_m)^* (T_1 \otimes T_2 \otimes ... \otimes T_m) & = & \left((T_1^D)^n \otimes (T_2^D)^n \otimes ... \otimes (T_m^D)^n\right) \\
 & . & \left(T_1^* T_1 \otimes T_2^* T_2 \otimes ... \otimes T_m^* T_m\right) \\
 & = & ((T_1^D)^n T_1^* T_1 \otimes (T_2^D)^n T_2^* T_2 \otimes ... \\
 & . & \otimes (T_m^D)^n T_m^* T_m) \\
 & \leq & (T_1^* T_1 (T_1^D)^{n*}_{2} T_2 (T_2^D)^n \otimes ... \\
 & . & \otimes T_m^* T_m (T_m^D)^n) \\
 & = & (T_1 \otimes T_2 \otimes ... \otimes T_m)^* \\
 & . & (T_1 \otimes T_2 \otimes ... \otimes T_m) \\
 & . & \left((T_1 \otimes T_2 \otimes ... \otimes T_m)^D\right)^n .
\end{array}
$$

Then $(T_1 \otimes T_2 \otimes .... \otimes T_m)$ is $(n, 1)$-power-D-quasi-hyponormal operators. □

Proposition 1.16. *If T is $(2, 1)$-power-D-quasi-hyponormal and T is D-idempotent. Then T is power-D-quasi-hyponormal operator*

Proof. Since T is $(2, 1)$-power-D-quasi-hyponormal operator, then
$(T^D)^2 T^* T \leq T^* T (T^D)^2$
since T is D-idempotent $(T^D)^2 = T^D$, wich implies
$T^D T^* T \leq T^* T T^D$
Thus T is is power-D-quasi-hyponormal operator □

Proposition 1.17. *If T is $(3, 1)$-power-D-quasi-hyponormal and T is D-idempotent. Then T is power-D-quasi-hyponormal operator*

Proof. Since T is $(3, 1)$-power-D-quasi-hyponormal operator, then
$(T^D)^3 T^* T \leq T^* T (T^D)^3$
since T is D-idempotent $(T^D)^2 = T^D$, wich implies
$(T^D) T^* T \leq T^* T T^D$
Then T is power-D-quasi-hyponormal operator □

Proposition 1.18. *If T, S are $(2, 1)$-power-D-quasi-hyponormal operators commuting, such that $T^D S^* = S^* T^D$, $T^* S + S^* T = 0$ and $T^D S - S T^D = 0$, then $S + T$ is $(2, 1)$-power-D-quasi-hyponormal operator.*

Proof. Since $T^D S - S T^D = 0$, hence $(T^D)^2 S^2 + S^2 (T^D)^2 = 0$, so $\left(S^D + T^D\right)^2 = (S^D)^2 + (T^D)^2$.

$$
\begin{array}{rcl}
\left((T+S)^D\right)^2 (S+T)^* (S+T) & = & \left((S^D)^2 + (T^D)^2\right) (S^* + T^*) (S+T) \\
 & = & (S^D)^2 S^* S + (S^D)^2 T^* T + (T^D)^2 S^* S + (T^D)^2 T^* T \\
 & = & (S^D)^2 S^* S + T^* T (S^D)^2 + S^* S (T^D)^2 + (T^D)^2 T^* T \\
 & \leq & S^* S (S^D)^2 + T^* T (S^D)^2 + S^* S (T^D)^2 +^2 T^* T (T^D) \\
 & = & (S+T)^* (S+T) \left((T+S)^D\right)^2
\end{array}
$$

Then $S + T$ is $(2, 1)$-power-D-quasi-hyponormal operator.
□

Proposition 1.19. *If T, S are $(2, 1)$-power-D-quasi-hyponormal operators commuting, such that $T^D S^* = S^* T^D$, $T^* S + S^* T = 0$ and $T^D S - S T^D = 0$, $TS = ST = S + T$ then ST is $(2, 1)$-power-D-quasi-hyponormal operator.*

Proof. Since $T^DS - ST^D = 0$, hence $(T^D)^2S^2 + S^2(T^D)^2 = 0$, so $\left(S^D + T^D\right)^2 = (S^D)^2 + (T^D)^2$.
Since,

$$\begin{aligned}
\left((ST)^D\right)^2 (ST)^* (ST) &= \left((T+S)^D\right)^2 (S+T)^* (S+T) \\
&= \left((S^D)^2 + (T^D)^2\right)(S^* + T^*)(S+T) \\
&= (S^D)^2S^*S + (S^D)^2T^*T + (T^D)^2S^*S + (T^D)^2T^*T \\
&= (S^D)^2S^*S + T^*T(S^D)^2 + S^*S(T^D)^2 + (T^D)^2T^*T \\
&\leq S^*S(S^D)^2 + T^*T(S^D)^2 + S^*S(T^D)^2 +^2 T^*T(T^D) \\
&= (S+T)^* (S+T) \left((T+S)^D\right)^2
\end{aligned}$$

Hence
$\left((ST)^D\right)^2 (ST)^* (ST) \leq (ST)^* (ST) \left((ST)^D\right)^2$.
Then ST is $(2,1)$-power-D-quasi-hyponormal operator. □

Example 1.20. *Let* $T = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, S = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \in \mathcal{B}(\mathbb{C}^2)$. *A simple computation shows that*

$$T^* = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, S^* = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}, T^D = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, S^D = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Then T is $(2,1)$*-power-D-quasi-hyponormal operator, but*

$$\left\langle \left((T^D)^2T^*T - T^*T(T^D)^2\right)\begin{pmatrix} u \\ v \end{pmatrix} \mid \begin{pmatrix} u \\ v \end{pmatrix} \right\rangle = 0.$$

For all $(u,v) \in (\mathbb{C}^2)$
and S is $(2,1)$*-power-D-quasi-hyponormal operator, but*

$$\left\langle \left((S^D)^2S^*S - S^*S(S^D)^2\right)\begin{pmatrix} u \\ v \end{pmatrix} \mid \begin{pmatrix} u \\ v \end{pmatrix} \right\rangle = 0.$$

For all $(u,v) \in (\mathbb{C}^2)$
Such that $TS + ST = 0$, $T^*S + S^*T \neq 0$ *and* $T^DS^* \neq S^*T^D$
but $S+T$ *and* ST *are* $(2,1)$*-power-D-quasi-hyponormal operator*
the following example shows that proposition (1.7) *is not necessarily true if* $T^DS^* \neq S^*T^D$

Proposition 1.21. *Let* $T, S \in \mathcal{B}(\mathcal{H})^D$ *are commuting and are* $(n,1)$*-power-D-quasi-hyponormal operators, such that* $T^DS^*S = S^*ST^D$, $S^DT^*T = T^*TS^D$, $T^*S + S^*T = 0$ *and* $(T+S)^*(T+S)$ *is commutes with*

$$\sum_{1 \leq p \leq n-1} \binom{n}{p}\left((T^D)^p(S^D)^{n-p}\right).$$

Then $(T+S)$ *is an* $(n,1)$*-power-D-quasi-hyponormal operator.*

Proof. Since

$$\begin{aligned}
\Big((T+S)^D\Big)^n\Big(T+S\Big)^*\Big(T+S\Big) &= \left[\sum_{0\le p\le n}\binom{n}{p}\Big((T^D)^p(S^D)^{n-p}\Big)\right]\Big(T+S\Big)^*\Big(T+S\Big)\\
&= (S^D)^nS^*S+\sum_{1\le p\le n-1}\binom{n}{p}\Big((T^D)^p(S^D)^{n-p}\Big)\Big(T+S\Big)^*+(T^D)^nS^*S\\
&+ (S^D)^nT^*T+(T^D)^nT^*T\\
&= (S^D)^nS^*S+\sum_{1\le p\le n-1}\binom{n}{p}\Big((T^D)^p(S^D)^{n-p}\Big)\Big(T+S\Big)^*+S^*S(T^D)^n\\
&+ T^*T(S^D)^n+(T^D)^nT^*T\\
&\le S^*S(S^D)^n+\sum_{1\le p\le n-1}\binom{n}{p}\Big((T^D)^p(S^D)^{n-p}\Big)\Big(T+S\Big)^*+S^*S(T^D)^n\\
&+ T^*T(S^D)^n+T^*T(T^D)^n\\
&\le \Big(T+S\Big)^*\Big(T+S\Big)\left[\sum_{0\le p\le n}\binom{n}{p}\Big((T^D)^p(S^D)^{n-p}\Big)\right]\\
&= \Big(T+S\Big)^*\Big(T+S\Big)\Big((T+S)^D\Big)^n.
\end{aligned}$$

Then $(T+S)$ is an $(n,1)$-power-D-quasi-hyponormal operator. □

References

[1] A.Benali, *On the class of n-Real Power Positive Operators On A Hilbert Space* , Funct. Anal. Approx. Comput. 10 (1) (2018), 41-49 .

[2] A.Aluthge, *On p-hyponormal Operators for* $0 < p < 1$, Integral Equations Operator Th. 13 (1990), 307-315.

[3] A.Benali, *On The Class Of (A,n)-Real Power Positive Operators In Semi-Hilbertian Space*, Global J. Pure Appl. Sci. 25 (2019), 161-166 .

[4] A.Benali, Ould Ahmed Mahmoud Sid Ahmed, *(* α,β*)-A-Normal Operators In Semi-Hilbertian Spaces*, Afrika Math. 06 May 2019 .

[5] A. Brown, *On a class of operators*, Proc. Amer. Math. Soc 4 (1953), 723-728.

[6] A. A.S. Jibril, *On n-Power Normal Operators*, J. Sci. Engen. 33 (2A) (2008), 247-251.

[7] A. A. S, Jibril, *On subprojection sperators*, Internat. Math. J. 4 (3) (2003), 229–238.

[8] A. A. S. Jibril, *On subprojection sperators*, Internat. Math. J. 4 (3) (2003), 229–238.

[9] C. Cherifa, A. Benali, *Class of* (A,n)*-power-Hyponormal and Quasihyponormal operators in semi-Hilbertian space*, Funct. Anal. Approx. Comput. 11 (2) (2019), 13-21.

[10] Dr.T.Veluchamy, K.M.Manikandan, *n-power-Quasi Normal operators on the Hilbert Space*, IOSR J. Math. .e-ISSN 2278-5728, p-ISSN: 2319-765WX. Volume 12,Issue 1Ver. IV 1 (jan. Feb2016), pp 06-09 .

[11] M. H. Mortad, *On The Normality of the Sum of Two Normal Operators*, Compl. Anal. Operator Th. 6 (2012), 105-112.

[12] M.Guesba and M.Nadir, *On operators for wich* $T^2 \ge -T^{*2}$, Australian J. Math. Anal. Appl. 13 (1) Article 6, pp. 1–5, 2016.

[13] Samir Al Mohammady, Sid Ahmed Ould Beinane, Ould Ahmed Mahmoud Sid Ahmed, *On*(n,m)*-A-power-normal and* (n,m)*-A-power quasi-normal operators in semi-hilbertian space* (soumis).

[14] Ould Ahmed Mahmoud Sid Ahmed and Ould Beinane sid Ahmed, *On the class of* (n,m)*-power-D-normal and* (n,m)*-power-D-quasi-normal operators*, Operators and Matrices. 13 (3) (2019), 705-732.

[15] Ould Ahmed Mahmoud Sid Ahmed, *On the class of n-power quasi-normal operators on Hilbert spaces*, Bull. Math. Anal. Appl. 3 (2), (2011), 213–228.

[16] Ould Ahmed Mahmoud Sid Ahmed, *On Some Normality-Like Properties and Bishops Property* (β) *for a Class of Operators on Hilbert Spaces Spaces*, Internat. J. Math. Math. Sci. (2012). doi:10.1155/2012/975745.

[17] Ould Ahmed Mahmoud Sid Ahmed, A. Benali, *Hyponormal And k-Quasi-hyponormal Operators On Semi-Hilbertian Spaces*, Australian J. Math. Anal. Appl. (2016). doi:10.1155/2016/975745.

[18] Riyandh R. Al-Mosawi and Hadeel A. Hassan, *A Note on Normal and n-Normal Operators* (2003)

[19] Sidi Hamidou Jah, *Class Of* (A,n)*-Power-Quasi-hyponormal Operators In Semi-Hilbertian Spaces*, Internat. J. Pure Appl. Math. 93 (1) (2014), 61-83.

[20] S.Panayappan and N.Sivamani, *On n-binormal operators*, Gen. Math. Notes, 10 (2) June 2012, pp. 1-8.