Functional Analysis,
Approximation and
Computation
14 (1) (2022), 55–65

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/faac

Generation of α-dense curves in infinite dimensional Banach spaces

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Abstract. The so called α-dense curves have an important role in the approximation of the solutions of certain problems involving functions of several variables, such as global optimization, numerical integration and many others. Roughly speaking, the α-dense curves allow us, in certain sense, transform the given problem into a single variable one, thus eliminating the so called *curse of dimensionality*. In the present paper we show how construct α-dense curves in infinite dimensional Banach spaces having a Schauder basis.

1. Introduction

In what follows, $(X, \|\cdot\|)$ will be a (real) Banach space and $X^\star$ its topological dual. As by the context there will be no possible confusion, we denote by $\|\cdot\|$ the norm of $X^\star$ too, i.e. $\|x^\star\| := \sup\{|x^\star(x)| : \|x\| \leq 1\}$ for all $x^\star \in X^\star$. In the next lines we recall some basic definitions related with the Schauder basis of a Banach space. For a detailed exposition of this topic, see the books [1, 4, 12]

A sequence $(\xi_n)_{n\geq 1} \subset X$ is a *Schauder basis* for X if for each $x \in$ there is a unique sequence $(a_n)_{n\geq 1} \subset \mathbb{R}$ (depending on x) such that

$$x = \sum_{n\geq 1} a_n \xi_n. \tag{1}$$

An important observation is the following: if $(\xi_n)_{n\geq 1}$ is a Schauder basis for X and $(c_n)_{n\geq 1} \subset \mathbb{R}$ is a sequence with $c_n \neq 0$ for all $n \geq 1$, then $(c_n \xi_n)_{n\geq 1}$ is also a Schauder basis for X. Therefore, we can assume $\|\xi_n\| = 1$ for all $n \geq 1$. In this case, we will say that $(\xi_n)_{n\geq 1}$ is a *normalized Schauder basis* for X

Also, if $\xi_n^\star \subset X^\star$ satisfies the equality $\xi_n^\star(\xi_m) = \delta_{nm}$ (the Kronecker symbol), then $a_n = \xi_n^\star(x)$ for all $n \geq 1$, where the a_n's are given in (1). Let us note that such $\xi_n^\star$ exists, courtesy of the Hahn-Banach Separation Theorem (see, for instance, [22, Theorem 3.5]). Moreover, we can take such sequence in a way that $\|\xi_n^\star\| = 1$ for all $n \geq 1$. The pair $(\xi_n, \xi_n^\star)_{n\geq 1}$, with $\|\xi_n\| = \|\xi_n^\star\| = 1$ for each $n \geq 1$, is usually called a *normalized biorthogonal system* for $X \times X^\star$.

2020 *Mathematics Subject Classification*. Primary: 46B25, 41A65; Secondary: 41A29, 41A30.
Keywords. Constructive approximation; α-dense curves; Schauder basis; Approximation in Banach spaces; Optimization in Banach spaces.
Received: 23 April 2022; Accepted: 20 June 2022
Communicated by Dragan S. Djordjević
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Remark 1.1. *It is important to stress that, in general, the sequence $(\xi_n^\star)_{n\geq 1} \subset X^\star$ is not a Schauder basis for $X^\star$. For instance, the Banach spaces ℓ_∞ of bounded sequences, with the usual supremum norm, or the space $L^\infty([0,1])$ of the Lebesgue measurable functions defined on $[0,1]$ endowed the essential supremum norm, are not separable and therefore can not have a Schauder basis.*

On the other hand, in Section 2 we detail the concepts of α-dense curve and densifiable set. These curves are important in some approximation problems, because of they allow us to "tranform" (in the specified sense and under suitable conditions) a multidimensional problem into a single variable one. The parameter $\alpha > 0$ plays a crucial role in the approximation error, in the sense that a smaller α correspond a smaller approximation error. In particular, for problems defined into a densifiable set, we can take α arbitrarily small, and consequently the approximation error will be arbitrarily small.

To approximate the solution of certain problems, for instance, a global optimization problem or the numerical integration of a continuous function of several variables, say f, we need to use an algorithm or procedure which evaluate, generally many times, the function f. But, the so called *curse of dimensionality*, introduced in 1961 by Bellman [3] (see also [20]), states that the number of function evaluations needed to obtain (by using an algorithm or procedure) a given level of accuracy, grows exponentially with respect to the number of variables of f, i.e. the *dimensionality* of f. Roughly speaking, the curse of dimensionality indicates us that, at least from a computational point of view, a single variable approximation problem can be solved more efficiently than the same problem for a several variables function. So, by using α-dense curves we can eliminate, or at least mitigate, the effect of the curse of dimensionality in these problems, because of an α-dense curve is a continuous function defined on the closed unit interval $[0,1]$ and values in the domain of definition of f, and therefore $f \circ g$ is a single variable (continuous) function.

There are many examples of α-dense curves (given by explicit formulas) in certain subsets of $\mathbb{R}^N$, with $N > 1$, or to be more precise in compact hyper-cubes of $\mathbb{R}^N$. This fact will be illustrated in Section 2 (see Proposition 2.4). Also, it is possible to construct α-dense curves in some infinite dimensional metric spaces (see Example 2.3). However, the construction (by explicit formulas) of α-dense curves, for $\alpha > 0$ arbitrarily small, in infinite dimensional Banach spaces is practically non-existent. Thus, our main goal in this paper is the following (see Theorem 3.2): assuming X has a normalized biorthogonal system, given $K \subset X$ arc-connected and compact and $\alpha > 0$, construct an α-dense curve in certain subset $K_\alpha \subset X$, in such way that for each $x \in K$ we can take y belonging to the image of such α-dense curve satisfying $\|x - y\| \leq \alpha$. Really, the set K_α will be defined as a subset of a finite dimensional Banach space, but under suitable conditions we can take $K_\alpha = K$.

As application of our results, and to justify the use of α-dense curves for the sake of removing the aforementioned curse of dimensionality, we show in Section 4 an example of an optimization problem posed in an infinite dimensional Banach space, and compare the number of function evaluations used by a software to solve it with and without the use of α-dense curves.

2. α-dense curves in metric spaces

Let (Y, d) be a metric space and $\mathcal{B}(Y)$ the class of bounded subsets of Y, i.e. the class of those non-empty subsets B of Y such that $\mathrm{Diam}(B) := \sup\{d(x,y) : x, y \in Y\}$, the diameter of B, is finite. Also, we denote $I := [0,1]$.

Definition 2.1. *Let $B \in \mathcal{B}(Y)$ and $\alpha \geq 0$. A continuous function $g_\alpha : I \longrightarrow (Y, d)$ is said to be an α-dense curve in B if*

(i) $g_\alpha(I) \subset B$

(ii) For each $x \in B$ there is $y \in g_\alpha(I)$ such that $d(x,y) \leq \alpha$.

If for each $\alpha > 0$ there is an α-dense curve in B. then B is said to be densifiable.

Let us note that:

(I) If $B \in \mathcal{B}(Y)$ then fixed any $x_0 \in B$, $d(x, x_0) \leq \mathrm{Diam}(B)$. So, for any $\alpha \geq \mathrm{Diam}(B)$ there is always an α-dense curve in B.

(II) By recalling that a *space-filling curve* is a continuous function from I onto I^N (for some integer $N > 1$), is clear that a function is, precisely, a 0-dense curve in I^N. So, the α-dense curves are a generalization of the space-filling curves (see also [14]).

The above concepts were introduced in 1997 by Mora and Cherruault [16], and since then other works related with this type of curves have been published, see [5, 14, 18, 18] and the below references.

Example 2.2. *Fixed an integer $N > 1$, for each $m \geq 1$ define $\alpha := \frac{\sqrt{N-1}}{m}$ and $g_\alpha : I \longrightarrow \mathbb{R}^N$ as*

$$g_\alpha(t) := \left(t, \frac{1}{2}(1 - \cos(\pi m t)), \ldots, \frac{1}{2}(1 - \cos(\pi m^{N-1} t))\right), \quad \textit{for all } t \in I.$$

Then, g_α is an α-dense curve in I^N (see [5, Proposition 9.5.4]). We show in Figure 1 the graphs of some of these curves.

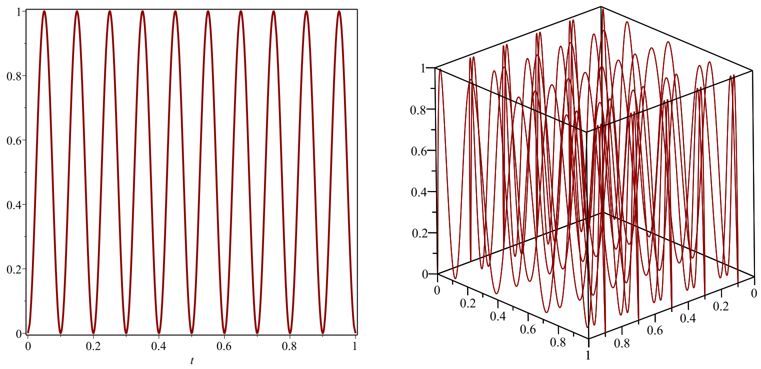


Figure 1: Graphs of the curves g_α for $N = 2, m = 20$ (left) and $N = 3, m = 10$ (right).

Example 2.3. *Let us denote by $\mathbb{R}^{\mathbb{N}}$ the product of countably many copies of $\mathbb{R}$ endowed with the metric defined by*

$$d(x, y) := \sum_{n \geq 1} 2^{-n} \frac{|x_n - y_n|}{1 + |x_n + y_n|}, \quad \textit{for all } x := (x_n)_{n\geq 1}, y := (y_n)_{n \geq 1} \in \mathbb{N}.$$

Let $\alpha > 0$ and $N \geq 1$ such that $\sum_{n \geq N+1} 2^{-n} < \frac{\alpha}{2}$. Define $g_\alpha : I \longrightarrow \mathbb{R}^{\mathbb{N}}$ by

$$g_\alpha(t) := \left(g_1(t), \ldots, g_N(t), h_{N+1}(t), h_{N+2}(t), \ldots\right) \quad \textit{for all } t \in I,$$

where $(g_1, \ldots, g_N) : I \longrightarrow I^N$ is an $\frac{\alpha}{2}$-dense curve in I^N and for each $i \geq 1$, $h_i : I \longrightarrow I$ are arbitrary continuous functions. Then, g_α is an α-dense curve in the Hilbert cube $I^{\mathbb{N}}$ (see also [17, Section 4]).

Other examples of α-dense curves can be found, for instance, in [5, 9, 21, 25].

The following result, which will be used later, can be straightforwardly proved from the involved definitions so we skip the proof.

Proposition 2.4. *Let $N > 1$, $R := \prod_{n=1}^N [a_n, b_n] \subset \mathbb{R}^N$, with $-\infty < a_n \leq b_n < \infty$, and $g_\alpha := (g_1, \ldots, g_N) : I \longrightarrow \mathbb{R}^N$ an α-dense curve in I^N, for some $\alpha > 0$. Then, the function $\tilde{g}_\alpha : I \longrightarrow \mathbb{R}^N$ defined as*

$$\tilde{g}_\alpha(t) := \left(a_1 + (b_1 - a_1) g_1(t), \ldots, a_N + (b_N - a_N) g_N(t)\right) \quad \textit{for all } t \in I,$$

is an $M\alpha$-dense curve in R, where $M := \max\{b_n - a_n : i = 1, \ldots, N\}$.

Before to continue, we recall that a set $B \subset Y$ is *precompact* (also called *totally bounded* or, if (Y, d) is complete, *relatively compact*, see, for instance, [24]) if for each $\varepsilon > 0$, B can be covered by a finite number of closed balls centered at points of B and radius ε. The following result was proved in [18], but we give here the proof to emphasize the difficulty of constructing α-dense curves, for an arbitrarily small $\alpha > 0$, in arbitrary precompact and arc-connected sets of a metric space.

Proposition 2.5. *Let $B \in \mathcal{B}(Y)$ arc-connected. Then, B is densifiable if, and only if, is precompact.*

Proof. If B is densifiable, given any $\varepsilon > 0$ there is an $\frac{\varepsilon}{2}$-dense curve in B, put $g_{\frac{\varepsilon}{2}}$. As $g_{\frac{\varepsilon}{2}}$ is compact, there is $\{y_1, \ldots, y_n\} \subset g_{\frac{\varepsilon}{2}}(I)$ such that

$$g_{\frac{\varepsilon}{2}}(I) \subset \bigcup_{i=1}^{n} \bar{B}\Big(y_i, \frac{\varepsilon}{2}\Big),$$

where $\bar{B}(z, r)$ stands for the closed ball centered at z and radius r. So, as $g_{\frac{\varepsilon}{2}}$ is an $\frac{\varepsilon}{2}$-dense curve in B, for each $x \in B$ there is $y \in g_{\frac{\varepsilon}{2}}(I)$ such that $d(x, y) \le \frac{\varepsilon}{2}$. Consequently, for such y there is $1 \le i \le n$ such that $d(x, y_i) \le d(x, y) + d(y, y_i) \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. So,

$$B \subset \bigcup_{i=1}^{n} \bar{B}(y_i, \varepsilon),$$

and therefore B is precompact.

Reciprocally, if B is precompact given $\alpha > 0$ there is $F := \{x_1, \ldots, x_m\} \subset B$ such that

$$B \subset \bigcup_{i=1}^{m} \bar{B}(x_i, \alpha).$$

Then, by defining g_α as the function joining the points of F (such definition is possible because of B is arc-connected) it is clear that g_α is an α-dense curve in B. □

We show in the next example (see also [18]) that the arc-connection property in the above result can not be removed:

Example 2.6. *Let the (compact but not arc-connected) set*

$$B := \Big\{(x, \sin(1/x)) : x \in [-1, 0) \cup (0, 1]\Big\} \cup \Big\{[0, y] : y \in [-1, 1]\Big\} \subset \mathbb{R}^2.$$

Then, given $0 < \alpha < 1$, it is immediate to check that there is not any α-dense curve in B. Consequently, B is not densifiable.

So, in view of Proposition 2.4, the class of densifiable sets in $\mathbb{R}^N$ (or, more generally, in finite dimensional Banach spaces) is large. In fact, each bounded (and consequently precompact, by the Heine-Borel Theorem, see for instance [19, Theorem 2.4.2]) and arc-connected subset of $\mathbb{R}^N$ is densifiable. However, in infinite dimensional Banach spaces the situation is very different (see [17]):

Proposition 2.7. *Let X an infinite dimensional Banach space, and U_X its closed unit ball. Then, there is not an α-dense curve in U_X for any $0 < \alpha < 1$. In others words, the minimal value of α for which we can find an α-dense curve in U_X is one.*

On the other hand, and as we have pointed out in Section 1, the main application of the α-dense curves is the approximation of the solutions of certain problems involving multi-variate functions, such as global optimization [5, 13, 15, 21, 25], numerical integration [5, 9], computation of the box-counting dimension [10], interpolation of sequences [8], approximation of the Hausdorff distance [6] and approximation of the roots of nonlinear systems [7]. Here, we focus in the application of the α-dense curves in the approximation of the solutions of (global) optimization problems. We describe briefly such method in the following lines.

Let $D \subset Y$ densifiable $J : D \longrightarrow \mathbb{R}$ continuous, such the problem

$$J_{\min} := \inf\Big\{J(x) : x \in D\Big\}, \tag{2}$$

is well defined, i.e., $J_{\min}$ is finite. For a given $\alpha > 0$ and an α-dense curve in D, put g_α, consider the following problem

$$J_{\alpha,\min} := \inf\Big\{J\Big(g_\alpha(t)\Big) : t \in I\Big\}. \tag{3}$$

By the continuity of J and the compactness of I, we infer that the above problem is well defined. Moreover, in the next result we will show that the solutions of the single variable problem (3) approximate, with arbitrarily small and controlled error, the solution of the problem (2).

By recalling that the *modulus of continuity of J of order $\varepsilon > 0$* is given by

$$\omega_J(\varepsilon) = \Big\{|J(x) - J(y) : x, y \in D, d(x, y) \le \varepsilon\Big\},$$

the following result was proved in [5, Theorem 5.4.4]:

Theorem 2.8. *With above notation,*

$$0 \le J_{\alpha,\min} - J_{\min} \le \omega_J(\alpha).$$

As D is densifiable and J is continuous, we have that $\lim_{\alpha\to 0^+} \omega_J(\alpha) = 0$. and therefore $\lim_{\alpha\to 0^+} J_{\alpha,\min} = J_{\min}$. So, and as we have pointed out above, the solution of the problem (2) can be approximated by the solution of the problem (3) with arbitrarily small and controlled error.

3. α-dense curves in Banach spaces with Schauder basis

As in Section 1, $(\xi_n, \xi_n^\star)_{n\ge 1}$ is a normalized biorthogonal system in $X \times X^\star$, where X is a (infinite dimensional if otherwise is not specified) Banach space and $X^\star$ its topological dual. The following result will be useful for our goals (see [2, Chapter III, Theorem 4.1]):

Proposition 3.1. *A closed set $K \subset X$ is compact if, and only if, given $\varepsilon > 0$ there is an integer $n_\varepsilon \ge 1$ such that*

$$\|\sum_{n\ge n_\varepsilon+1} \xi_n^\star(x)\xi_n\| \le \varepsilon, \quad \textit{for all } x \in K.$$

We know, by Proposition 2.5, that if $K \subset X$ is arc-connected and precompact, then is densifiable. But, in general, the proof of such result do not provide us an efficient method to construct (by explicit formulas) α-dense curves in K, for a prescribed $\alpha > 0$. However, our main result is the following:

Theorem 3.2. *Let $K \subset X$ arc-connected and compact and $\alpha > 0$. Then,*

(1) There is a set $K_\alpha \subset X$ and an α-dense curve g_α in K_α such that for each $x \in K$ there is $y \in g_\alpha(I)$ satisfying $\|x - y\| \le \alpha$.

(2) If for each $n \ge 1$, $\sum_{n\ge 1} \theta_n \xi_n \in K$ for each sequence of real numbers $(\theta_n)_{n\ge 1}$ obeying

$$\theta_n \in \Big[\min\{\xi_n^\star(x) : x \in K\}, \max\{\xi_n^\star(x) : x \in K\}\Big] \quad \textit{for all } t \in I,$$

then there is an α-dense curve in K.

Proof. Let $\alpha > 0$. As K is arc-connected and compact, for each $n \geq 1$, by the continuity of $\xi_n^\star$, the set $\xi_n^\star(K) \subset \mathbb{R}$ is arc-connected and compact too. So, we find that $\xi_n^\star(K) = [\underline{a}_n, \overline{a}_n]$ where

$$\underline{a}_n := \min\{\xi_n^\star(x) : x \in K\} \quad \text{and} \quad \overline{a}_n := \max\{\xi_n^\star(x) : x \in K\},$$

for each $n \geq 1$.

(1) Let $N := N(\alpha) \geq 1$ be the integer provided by Proposition 3.1 satisfying

$$\|\sum_{n\geq N+1} \xi_n^\star(x)\xi_n\| \leq \frac{\alpha}{2}, \quad \text{for all } x \in K. \tag{4}$$

For $n = 1, \dots, N$, take any $a_n \leq \underline{a}_n$ and $b_n \geq \overline{a}_n$ and define $R := \prod_{n=1}^N [a_n, b_n]$. By Proposition 2.4, we can construct an $\frac{\alpha}{2N}$-dense curve in R from an $\frac{\alpha}{2N}$-dense curve in I^N, put $g_\alpha := (g_1, \dots, g_N)$.

Now, define

$$K_\alpha := \Big\{\sum_{n=1}^N A_n\xi_n : (A_1, \dots, A_N) \in R\Big\},$$

and $\hat{g}_\alpha : I \longrightarrow X$ as

$$\hat{g}_\alpha(t) := \sum_{n=1}^N g_n(t)\xi_n, \quad \text{for all } t \in I.$$

We claim that $\hat{g}_\alpha$ is an $\frac{\alpha}{2}$-dense curve in K_α. Clearly, $\hat{g}_\alpha$ is continuous and $\hat{g}_\alpha(I) \subset K_\alpha$. Also, given $z \in K_\alpha$, put $z := \sum_{n=1}^N A_n\xi_n$ for certain $A_n \in [a_n, b_n]$, $n = 1, \dots, N$, as g_α is an $\frac{\alpha}{2N}$-dense curve in R, there is $t \in I$ such that $\|(A_1, \dots, A_N) - (g_1(t), \dots, g_N(t))\|_E \leq \frac{\alpha}{2N}$ (here, $\|\cdot\|_E$ stands for the Euclidean norm of $\mathbb{R}^N$) and consequently

$$\|z - \hat{g}_\alpha(t)\| \leq \|\sum_{n=1}^N (A_n - g_n(t))\xi_n\| \leq \sum_{n=1}^N |A_n - g_n(t)| \leq N\frac{\alpha}{2N} = \frac{\alpha}{2}, \tag{5}$$

as claimed.

On the other hand, given $x \in K$, put $x := \sum_{n\geq 1} \xi_n^\star(x)\xi_n$ with $\xi_n^\star(x) \in [\underline{a}_n, \overline{a}_n] \subseteq [a_n, b_n]$ for all $n \geq 1$, noticing (4) and (5) there exists $t \in I$ such that

$$\|x - \hat{g}_\alpha(t)\| \leq \|\sum_{n=1}^N (\xi_n^\star(x) - g_n(t))\xi_n\| + \|\sum_{n\geq N+1} \xi_n^\star(x)\xi_n\| \leq \frac{\alpha}{2} + \frac{\alpha}{2} = \alpha,$$

and this completes the proof of (1).

To prove (2), take $a_n := \underline{a}_n$, $b_n := \overline{a}_n$ and, noticing Proposition 3.1, $N := N(\alpha) \geq 1$ such that

$$\|\sum_{n\geq N+1} \xi_n^\star(x)\xi_n\| \leq \frac{\alpha}{4} \quad \text{for all } x \in K, \tag{6}$$

and let $g_\alpha := (g_1, \dots, g_N) : I \longrightarrow \mathbb{R}^N$ be an $\frac{\alpha}{2N}$-dense curve in

$$R := \prod_{n=1}^N [a_n, b_n]$$

Define

$$\hat{g}_\alpha(t) := \sum_{n=1}^N g_n(t)\xi_n + \sum_{n\geq N+1} h_n(t)\xi_n \quad \text{for all } t \in I,$$

where $h_n : I \longrightarrow [a_n, b_n]$ are arbitrary continuous functions, for each $n \geq N+1$. From our assumptions, is clear that $\hat{g}_\alpha(I) \subset K$ and also, from the continuity of g_α and the h_n's, $\hat{g}_\alpha$ is continuous. As g_α is an $\frac{\alpha}{2N}$-dense curve in R and $\xi_n^\star(x) \in [a_n, b_n]$ for all $x \in K$ and $n \geq 1$, given $x \in K$ there is $t \in I$ such that

$$\|\Big(\xi_1^\star(x), \dots, \xi_1^\star(x)\Big) - \Big(g_1(t), \dots, g_N(t)\Big)\|_E \leq \frac{\alpha}{2N}. \tag{7}$$

So, noticing (6) and (7), we have

$$\|x - \hat{g}(t)\| \leq \|\sum_{n=1}^{N} (\xi_n^\star(x) - (g_n(t))\xi_n\| + \|\sum_{n\geq N+1} (h_n(t) - \xi_n^\star(x))\xi_n\| \leq$$

$$\sum_{n=1}^{N} |\xi_n^\star(x) - g_n(t)| + \|\sum_{n\geq N+1} h_n(t)\xi_n\| + \|\sum_{n\geq N+1} \xi_n^\star(x)\xi_n\| \leq \frac{\alpha}{2} + \frac{\alpha}{2} = \alpha,$$

and therefore $\hat{g}_\alpha$ is an α-dense curve in K.

□

Let us note that the α-dense curve $\hat{g}_\alpha$ of (1) of the above result is *constructive* in the following sense:

(i) The numbers a_n and b_n in the above proof can be taken as small and as large, respectively, in such way that $a_n \leq \underline{a}_n$ and $b_n \geq \overline{a}_n$, for each $n = 1, \dots, N$. In fact, we only need to know the bounds $a \leq \inf\{\underline{a}_n : n \geq 1\}$ and $b \geq \sup\{\overline{a}_n : n \geq 1\}$ and take $a_n := a$ and $b_n := b$ for each $n \geq 1$. The curve g_α can then be constructed straightforwardly according to Proposition 2.4.

(ii) In practical cases, we do not need to know the integer $N \geq 1$ obeying (4) because of fixed $\alpha > 0$, $\frac{\alpha}{2N} \to 0$ and therefore, for arbitrarily small $\alpha > 0$, we can construct α-dense curves in K_α which allow us approximate the vectors of K in the specified sense.

Really, the set K_α given in (1) is a set of a finite dimensional linear space, to be more precise of the N-dimensional space spanned by the basis vectors $\xi_1, \dots, \xi_N$. However, under the conditions given in (2), and if we know the values of $\underline{a}_n$ and $\overline{a}_n$, for each $n \geq 1$, of the minimum and maximum, respectively, of $\xi_n^\star(K)$, we can construct an α-dense curve in K. The class of subsets K of X satisfying condition (2) in Theorem 3.2 is large, as we see in the following example.

Example 3.3. *Let $(r_n)_{n\geq 1}$ a sequence of non-negative numbers such that $\sum_{n\geq 1} r_n < \infty$. Let the set*

$$K := \Big\{x \in X : |\xi_n^\star(x)| \leq r_n, \quad \text{for all } n \geq 1\Big\}.$$

This set in non-empty. Indeed, for each $n \geq 1$ we have $r_n\xi_n \in K$. Also, it is convex and compact (the convexity is clear, and the compactness follows directly by Proposition 3.1). Now, given any sequence $(\theta_n)_{n\geq 1}$, with $\theta_n \in [-r_n, r_n]$ for each $n \geq 1$, we have

$$x := \sum_{n\geq 1} \theta_n \xi_n \in K.$$

because of $|\xi_n^\star(x)| = |\theta_n| \leq r_n$ for all $n \geq 1$. Therefore, the set K fulfills the condition (2) in Theorem 3.2.

As we have pointed out in Section 2, the α-dense curves can be used to approximate the solution of some problems involving several variable functions by the solutions of problems involving a single variable function. In the case of problems posed in infinite dimensional Banach spaces, and under conditions of Theorem 3.2, we can also use the α-dense curves described in such theorem to approximate the given problem. However, in general and for computational reasons, if we use the α-dense curves proposed in (2) of Theorem 3.2 in some problem like those described in Section 2, these curves need to be *truncated*, that is to say, we need to consider the function

$$\hat{g}_\alpha(t) := \sum_{n=1}^{M} f_n(t)\xi_n \quad \text{for all } t \in I,$$

for some integer $M \geq 1$ where $f_n(t) := g_n(t)$ for $n \leq N$ and $f_n(t) := h_n(t)$ otherwise, instead of the given one in the proof of (2) of Theorem 3.2. For instance, if we have the problem

$$J_{\min} := \min\Big\{J(x) : x \in K\Big\}, \tag{8}$$

where K satisfies the conditions of (2) of Theorem 3.2 and $J : K \longrightarrow \mathbb{R}$ is continuous, we can use the function

$$\hat{g}_\alpha(t) := \sum_{n=1}^{N} g_n(t)\xi_n \quad \text{for all } t \in I,$$

where, for a fixed $\alpha > 0$, $(g_1, \ldots, g_N)$ is an $\frac{\alpha}{2N}$-dense curve in the hyper-cube

$$R := \prod_{n=1}^{N} \Big[\min\{\xi_n^\star(x) : x \in K\}, \max\{\xi_n^\star(x) : x \in K\}\Big].$$

Then, we posed the problem

$$J_{\alpha,N,\min} := \min\Big\{J\Big(\hat{g}_\alpha(t)\Big) : t \in I\Big\}, \tag{9}$$

which is a (well defined) optimization problem in the N-dimensional Banach space spanned by $\{\xi_1, \ldots, \xi_n\}$. By Theorem 2.8, is clear that

$$0 \leq J_{\alpha,N,\min} - J_{\min} \leq \omega_J(\alpha),$$

for each $N \geq 1$. Therefore, we can approximate the solution of (8) by the solutions of (9), with arbitrarily (and controlled) small error.

Let us note that the above method is, in forms, similar to others such as the so called *Ritz method* (see, for instance, [11, Chapter 8]) which approximates the solutions of the problem (8) by the solutions of problems posed in finite dimensional sub-spaces of X.

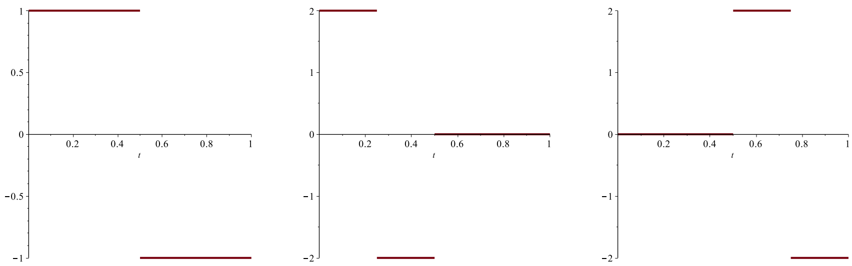
4. A numerical example

In this section, and to conclude our exposition, we will apply our results to approximate the solution of an optimization problem posed in the Banach space $L^1(I)$ of the Lebesgue integrable functions defined on I, with its usual norm $\|x\| := \int_0^1 |x(t)|dt$, for all $x \in L^1(I)$. The dual of $L^1(I)$ is $L^\infty(I)$ (see Remark 1.1). In this case, the vectors $x^\star \in L^\infty(I)$ are defined as

$$x^\star(x) := \int_0^1 x^\star(t)x(t)dt, \quad \text{for all } x \in L^1(I).$$

Define $\chi_1(t) = \chi_1^\star(t) = 1$ for all $t \in I$, and for $k \geq 0$ and $1 \leq i \leq 2^k$, $n := 2^k + i$ let

$$\chi_n(t) := \begin{cases} 2^k, & t \in \left(\frac{2i-2}{2^{k+1}}, \frac{2i-1}{2^{k+1}}\right) \\ -2^k, & t \in \left(\frac{2i-1}{2^{k+1}}, \frac{2i}{2^{k+1}}\right) \\ 0, & \text{otherwise} \end{cases}, \quad \chi_n^\star(t) := \begin{cases} 1, & t \in \left(\frac{2i-2}{2^{k+1}}, \frac{2i-1}{2^{k+1}}\right) \\ -1, & t \in \left(\frac{2i-1}{2^{k+1}}, \frac{2i}{2^{k+1}}\right) \\ 0, & \text{otherwise} \end{cases}$$



(a) Graph of $\chi_2(t)$ (b) Graph of $\chi_3(t)$ (c) Graph of $\chi_4(t)$

Figure 2: Graphs of the functions $\chi_n(t)$ for the indicated values of n

for all $t \in$. Then, $(\chi_n)_{n\geq 1}$ is called the Haar system, and is a normalized Schauder basis for $L^p(I)$ (see [1, 4]). As $\int_0^1 \chi_n(t)\chi_m^\star(t)dt = \delta_{nm}$ for each $n \neq m$, we find that $(\chi_n, \chi_n^\star)_{n\geq 1} \subset L^1(I) \times L^\infty(I)$ is a normalized biorthogonal system. We show in Figure 2 the graphs of the functions $\chi_n(t)$ for the indicated values of n.

Next, consider the set

$$K := \Big\{x \in L^1(I) : |\chi_n^\star(x)| \leq \frac{1}{n^2}, n \geq 1\Big\},$$

which is, according to Example 3.3, convex and compact. Define $J : K \longrightarrow \mathbb{R}$ as

$$J(x) := \frac{1}{1 + \|x\|} \quad \text{for all } x \in K,$$

and let the optimization problem

$$J_{\min} := \min\Big\{J(x) : x \in K\Big\}, \tag{10}$$

The above problem is well defined. In fact, is clear that $J_{\min} = 1/2$ and this value is attained at each $x \in K$ with $\|x\| = 1$. To solve the problem, and following the considerations of Section 3 we consider, for each $N > 1$ and $\alpha := 0.1$, the $\frac{\alpha}{2N}$-curve in $R := \prod_{n=1}^N [-1/n^2, 1/n^2]$ (see Proposition 2.4) given by

$$g_\alpha(\tau) := \Big(g_{\alpha,1}(\tau), \dots, g_{\alpha,N}(\tau)\Big) :=$$

$$\Big(-1 + 2\tau, \frac{-\cos(\pi m \tau)}{2^2}, \dots, \frac{-\cos(\pi m^{N-1}\tau)}{2^N}\Big),$$

for each $N > 1$ and all $t \in I$, and $m := \left\lceil \frac{4N\sqrt{N-1}}{\alpha} \right\rceil$, where $\lceil r \rceil$ stands for the smaller integer greater or equal than r. This is an $\frac{\alpha}{2N}$-curve in R (see also Example 2.2).

Then, by defining $\hat{g}_\alpha(\tau) := \sum_{n=1}^N g_{\alpha,n}(\tau)\chi_n$, to solve (10) we pose the problem

$$J_{\alpha,N,\min} := \min\Big\{J\Big(\hat{g}_\alpha(\tau)\Big) : \tau \in I\Big\}, \tag{11}$$

which (again, by the considerations of Section 3) approximates the solutions of (10). Also, to justify the use of α-dense curves in the approximation of the solution of problem (10), i.e. the use of the problems (11), we also consider the optimization problem

$$J_{N,\min} := \min\Big\{J\Big(\sum_{n=1}^N x_n\chi_n(t)\Big) : (x_1, \dots, x_N) \in R\Big\}. \tag{12}$$

N	F.E. for (12)	F.E. for (11)	Improvement Ratio
2	188	105	1.71
3	341	89	3.83
4	417	106	3.93
5	544	80	6.80

Table 1: Comparison of the obtained results for the problems (11) and (12), for the indicated values of N.

We will use the ®Maple package *DirectSearch* to solve the problems (11) and (12). For both problems, and the indicated values of N, we find in each case the *exact* solution, that is, we find $\tau_{N,\min} \in I$ and $(x_{N,\min}^{(1)}, \ldots, x_{N,\min}^{(N)}) \in R$ such that

$$J_{\alpha,N,\min} = J\Big(\hat{g}_\alpha(\tau_{N,\min})\Big) = J\Big(\sum_{n=1}^{N} x_{N,\min}^{(n)} \chi_n(t)\Big) = J_{N,\min} = \frac{1}{2} = J_{\min},$$

for the tested values of N. As we can see in Table 1 the number of function evaluations (F.E.) for the problem (11) is considerably smaller than the F.E. to solve the problem (12). This fact is explained, of course, from the *curse of dimensionality* mentioned in Section 1. The column *Improvement Ratio* indicates the quotient of the F.E. to solve (12) by the F.E. to solve (11).

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