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Generalized Bicomplex Numbers

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Abstract. In this paper we introduce and study the following generalized numbers: $BF_{n,m}$ - bicomplex Fibonacci numbers; $BL_{n,m}$ - bicomplex Lucas numbers; $BJ_{n,m}$ - bicomplex Jacobsthal numbers and $Bj_{n,m}$ - bicomplex Jacobsthal-Lucas numbers. For $m = 2$, these numbers are the well-known bicomplex Fibonacci numbers; bicomplex Lucas numbers; bicomplex Jacobsthal and bicomplex Jacobsthal-Lucas numbers, respectively. Additionally, we introduce convolutions of these numbers.

1. Introduction

Recall that (see [6]) the generalized Fibonacci numbers $F_{n,m}$, and, the generalized Lucas numbers $L_{n,m}$, $(n \geq m,\ m \geq 2)$, are given by:

$$F_{n,m} = F_{n-1,m} + F_{n-m,m},\quad n \geq m, \tag{1}$$

with: $F_{0,m} = 0,\ \ F_{n,m} = 1,\ \ n = 1, \ldots, m-1$, and,

$$L_{n,m} = L_{n-1,m} + L_{n-m},\quad n \geq m, \tag{2}$$

with: $L_{0,m} = 2,\ \ L_{n,m} = 1,\ \ n = 1, \ldots, m-1.$

In that sense, the corresponding generating functions are, respectively:

$$F_m(t) = \frac{t}{1-t-t^m} = \sum_{n=0}^{\infty} F_{n,m}\, t^n,$$
$$L_m(t) = \frac{2-t}{1-t-t^m} = \sum_{n=0}^{\infty} L_{n,m}\, t^n.$$

Remark 1. The generalized Fibonacci numbers $F_{n,m}$ are one special case of the Humbert polynomials $P_n(m, x, y, p, C)$, see [9]:

$$F_{n+1,m} = P_n(m, 1/m, -1, -1, 1),$$

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where

$$\sum_{n=0}^{\infty} P_n(m,x,y,p,C)t^n = (C - mxt + yt^m)^p.$$

Furthermore, recall that the generalized Jacobsthal numbers $J_{n,m}$ and the generalized Jacobsthal Lucas numbers $j_{n,m}$, for $m \geq 2$ and $n \geq m$, (for $x = 1$ see [1], [2] and [3]; also, for $k = 1$ see [4]), are given by the following recurrence relations, respectively:

$$J_{n,m} = J_{n-1,m} + 2J_{n-m,m}, \quad n \geq m, \tag{3}$$

with: $J_{0,m} = 0, \quad J_{n,m} = 1, \ n = 1, \ldots, m-1$; and

$$j_{n,m} = j_{n-1,m} + 2j_{n-m,m}, \quad n \geq m, \tag{4}$$

with: $j_{0,m} = 2, \quad j_{n,m} = 1, \ n = 1, \ldots, m-1$.

The generating function $J_m(t)$ for the generalized Jacobsthal numbers $J_{n,m}$ is given as

$$J_m(t) = \frac{t}{1 - t - 2t^m} = \sum_{n=0}^{\infty} J_{n,m}t^n, \tag{5}$$

and the corresponding generating function $j_m(t)$ for the generalized Jacobsthal-Lucas numbers $j_{n,m}$ is given by

$$j_m(t) = \frac{2 - t}{1 - t - 2t^m} = \sum_{n=0}^{\infty} j_{n,m}t^n. \tag{6}$$

A special case of the Jacobsthal numbers can be found in [8], while a special case of the Fibonacci numbers can be found in [5] and [7]. It is not difficult to prove the following relations:

$$L_{n,m} = F_{n,m} + 2F_{n+1-m,m}, \quad n \geq m; \tag{7}$$

$$j_{n,m} = J_{n+1,m} + 2J_{n+1-m,m}. \tag{8}$$

Table 1 below illustrates some initial members of these numbers, obtained via the recurrence relations (1)–(4):

Table 1:

n	$F_{n,m}$	$L_{n,m}$	$J_{n,m}$	$j_{n,m}$
0	0	2	0	2
1	1	1	1	1
2	1	1	1	1
....	....	....	....	....
$m-1$	1	1	1	1
m	1	3	1	5
$m+1$	2	4	3	7
$m+2$	3	5	5	9
....	....	....	....	....
$2m-1$	m	$m+2$	$2m-1$	$2m+3$
$2m$	$m+1$	$m+3$	$2m+1$	$2m+13$
$2m+1$	$m+3$	$m+7$	$2m+7$	$2m+27$
....	....	....	....	....

Specially, when $m = 3, 4, 5$, using the recurrence relations (1)–(4), we get the first 13 members of these numbers in Table 2 below:

Table 2:

n	$F_{n,3}$	$F_{n,4}$	$L_{n,3}$	$L_{n,4}$	$J_{n,3}$	$J_{n,4}$	$j_{n,3}$	$j_{n,4}$	$j_{n,5}$
0	0	0	2	2	0	0	2	2	2
1	1	1	1	1	1	1	1	1	1
2	1	1	1	1	1	1	1	1	1
3	1	1	3	1	1	1	5	1	1
4	2	1	4	3	3	1	7	5	1
5	3	2	5	4	5	3	9	7	5
6	4	3	8	5	7	5	19	9	7
7	6	4	12	6	13	7	33	11	9
8	9	5	17	9	23	9	51	21	11
9	13	7	25	13	37	15	89	35	13
10	19	10	37	18	63	25	155	53	23
11	28	14	54	24	109	39	257	75	37
12	41	19	79	33	183	57	435	117	55

At this point we introduce the generalized bicomplex Fibonacci numbers, denoted as $BF_{n,m}$, and the generalized bicomplex Lucas numbers, denoted as $BL_{n,m}$, which are the main subject of this paper. Recall the quaternione bicomplex group:

.	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	k	-1	$-i$
k	k	$-j$	$-i$	1

Definition 1.1. *The generalized bicomplex Fibonacci numbers $BF_{n,m}$ are given by*

$$BF_{n,m} = F_{n,m} + iF_{n+1,m} + jF_{n+2,m} + kF_{n+3,m}, \tag{9}$$

with the initial members (by table 1):

$$BF_{0,m} = i + j + k, \;\; BF_{1,m} = \cdots = BF_{m-3,m} = 1 + i + j + k \tag{10}$$

$$BF_{m-2,m} = 1 + i + j + 2k, \;\; BF_{m-1,m} = 1 + i + 2j + 3k. \tag{11}$$

Definition 1.2. *The generalized bicomplex Lucas numbers $BL_{n,m}$ are given by*

$$BL_{n,m} = L_{n,m} + iL_{n+1,m} + jL_{n+2,m} + kL_{n+3,m}, \tag{12}$$

with:

$$BL_{0,m} = 2 + i + j + k, \;\; BL_{1,m} = \cdots = BL_{m-4,m} = 1 + i + j + k, \tag{13}$$

$$BL_{m-3,m} = 1 + i + j + 3k, \;\; BL_{m-2,m} = 1 + i + 3j + 4k, \tag{14}$$

$$BL_{m-1,m} = 1 + 3i + 4j + 5k. \tag{15}$$

By using definitions (1.1) and (1.2) and by using the well-known recurrence relations (1) and (2), we can prove that the generalized bicomplex numbers $BF_{n,m}$ and $BL_{n,m}$ satisfy the following relations:

$$BF_{n,m} = BF_{n-1,m} + BF_{n-m,m}, \;\; n \geq m, \tag{16}$$

$$BL_{n,m} = BL_{n-1,m} + BL_{n-m,m}, \;\; n \geq m, \tag{17}$$

$$BL_{n,m} = BF_{n,m} + 2BF_{n+1-m,m}, \;\; n \geq m. \tag{18}$$

We prove the following theorem.

Theorem 1.3. *The generalized bicomplex numbers $BF_{n,m}$ and $BL_{n,m}$, respectively, satisfy the following relations:*

$$\sum_{i=1}^{n} BF_{i,m} = BF_{n+m,m} - BF_{m,m}, \tag{19}$$

$$\sum_{i=1}^{n} BL_{i,m} = BL_{n+m,m} - BL_{m,m}. \tag{20}$$

Proof. We are going to prove the relation (19). Using (16), we get:

$$\begin{aligned}
&BF_{n+m,m} - BF_{m,m}\\
&= BF_{n+m-1,m} + BF_{n,m} - BF_{m,m}\\
&= BF_{n+m-2,m} + BF_{n-1,m} + BF_{n,m} - BF_{m,m}\\
&= BF_{n+m-3,m} + BF_{n-2,m} + BF_{n-1,m} + BF_{n,m} - BF_{m,m}\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&= BF_{m,m} + BF_{1,m} + BF_{2,m} + \cdots + BF_{n-1,m} + BF_{n,m} - BF_{m,m}\\
&= \sum_{i=1}^{n} BF_{i,m}.
\end{aligned}$$

The proof of (20) is similar, and follows from (17). □

2. Generating functions for the numbers $BF_{n,m}$ and $BL_{n,m}$

In this section we obtain the generating functions for the generalized bicomplex numebrs $BF_{n,m}$ and $BL_{n,m}$. We proceed to prove the following theorem.

Theorem 2.1. *The generating function $f_m(t)$ for the generalized bicomplex numbers $BF_{n,m}$ is given by*

$$f_m(t) = \frac{i + j + k + t + kt^{m-2} + (j + k)t^{m-1}}{1 - t - t^m}. \tag{21}$$

Proof. Let $f_m(t) = \sum_{n=0}^{\infty} BF_{n,m}\, t^n$ be generating function for the numbers $BF_{n,m}$. Using Definition (1.1) we get:

$$\begin{aligned}
f_m(t) &= \sum_{n=0}^{\infty} BF_{n,m}t^n\\
&= \sum_{n=0}^{\infty} F_{n,m}t^n + i\sum_{n=0}^{\infty} F_{n+1,m}t^n + j\sum_{n=0}^{\infty} F_{n+2,m}t^n + k\sum_{n=0}^{\infty} F_{n+3,m}t^n\\
&= F_m(t) + i\Big(F_{1,m} + F_{2,m}t + F_{3,m}t^2 + \dots\Big) + j\Big(F_{2,m} + F_{3,m}t + F_{4,m}t^2 + \dots\Big)\\
&\quad + k\Big(F_{3,m} + F_{4,m}t + F_{5,m}t^2 + \dots\Big)\\
&= F_m(t) + \frac{i}{t}\,(F_m(t)) + \frac{j}{t^2}\,(F_m(t) - t) + \frac{k}{t^3}\Big(F_m(t) - t^2 - t\Big)\\
&= F_m(t)\cdot\left(1 + \frac{i}{t} + \frac{j}{t^2} + \frac{k}{t^3}\right) - \frac{jt^2 + kt^2 + kt}{t^3}\\
&= \frac{t}{1 - t - t^m}\cdot\frac{t^3 + it^2 + jt + k}{t^3} - \frac{jt^2 + kt^2 + kt}{t^3}\\
&= \frac{i + j + k + t + kt^{m-2} + (j + k)t^{m-1}}{1 - t - t^m}.
\end{aligned}$$

So, the relation (21) holds. □

Corollary 2.2. *For $m = 2$, or respectively, $m = 3$, the relation (21) becomes:*

$$f_2(t) = \frac{1 + i + j + 2k + (1 + j + k)t}{1 - t - t^2},$$
$$f_3(t) = \frac{i + j + k + (k + 1)t + (j + k)t^2}{1 - t - t^3}.$$

Now, we shall prove the following theorem.

Theorem 2.3. *The generating function $l_m(t)$ for the generalized bicomplex numbers $BL_{n,m}$ is given as*

$$l_m(t) = \frac{2 + i + j + k - t + 2kt^{m-3} + (2j + k)t^{m-2} + (2i + j + k)t^{m-1}}{1 - t - t^m}. \tag{22}$$

Proof. Let $l_m(t) = \sum_{n=0}^{\infty} BL_{n,m}t^n$ be the generating function for $BL_{n,m}$. Then, using definition (1.2), we get:

$$\begin{aligned}
l_m(t) &= \sum_{n=0}^{\infty} BL_{n,m}t^n \\
&= \sum_{n=0}^{\infty} L_{n,m}t^n + i\sum_{n=0}^{\infty} L_{n+1,m}t^n + j\sum_{n=0}^{\infty} L_{n+2,m}t^n + k\sum_{n=0}^{\infty} L_{n+3,m}t^n \\
&= L_m(t) + i\left(L_{1,m} + L_{2,m}t + L_{3,m}t^2 + L_{4,m}t^3 + \dots\right) + \\
&\quad\; j\left(L_{2,m} + L_{3,m}t + L_{4,m}t^2 + L_{5,m}t^3 \dots\right) \\
&\quad\; + k\left(L_{3,m} + L_{4,m}t + L_{5,m}t^2 + L_{6,m}t^3 + \dots\right) \\
&= L_m(t) + \frac{i}{t}\left(L_m(t) - 2\right) + \frac{j}{t^2}\left(L_m(t) - t - 2\right) + \frac{k}{t^3}\left(L_m(t) - t^2 - t - 2\right) \\
&= L_m(t)\left(1 + \frac{i}{t} + \frac{j}{t^2} + \frac{k}{t^3}\right) - \frac{2it^2 + jt^2 + 2jt + kt^2 + kt + 2k}{t^3} \\
&= \frac{2 - t}{1 - t - t^m} \cdot \frac{t^3 + it^2 + jt + k}{t^3} - \frac{2it^2 + jt^2 + 2jt + kt^2 + kt + 2k}{t^3} \\
&= \frac{2 + i + j + k - t + 2kt^{m-3} + (2j + k)t^{m-2} + (2i + j + k)t^{m-1}}{1 - t - t^m},
\end{aligned}$$

thus (22) holds. □

Corollary 2.4. *For $m = 3$, in (22), we have*

$$l_3(t) = \frac{2 + i + j + 3k + (2j + k - 1)t + (2i + j + k)t^2}{1 - t - t^3}.$$

3. Generalized bicomplex Jacobsthal type numbers

In this section we introduce $BJ_{n,m}$ - the generalized bicomplex Jacobsthal numbers and $Bj_{n,m}$ - the generalized bicomplex Jacobsthal - Lucas numbers. Further, we find some interesting properties for these numbers. These numbers are generalized bicomplex numbers of the Jacobsthal type.

Definition 3.1. *The generalized bicomplex Jacobsthal numbers $BJ_{n,m}$ are given by:*

$$BJ_{n,m} = J_{n,m} + iJ_{n+1,m} + jJ_{n+2,m} + kJ_{n+3,m}, \quad n \geq m. \tag{23}$$

where:

$$\begin{cases} BJ_{0,m} = i + j + k, \\ BJ_{1,m} = \cdots = BJ_{m-3,m} = 1 + i + j + k, \\ BJ_{m-2,m} = 1 + i + j + 3k, \\ BJ_{m-1,m} = 1 + i + 3j + 5k. \end{cases} \tag{24}$$

Definition 3.2. *The generalized bicomplex Jacobsthal-Lucas numbers $Bj_{n,m}$ are given by:*

$$Bj_{n,m} = j_{n,m} + ij_{n+1,m} + jj_{n+2,m} + kj_{n+3,m}, \quad n \geq m, \tag{25}$$

where:

$$\begin{cases} Bj_{0,m} = 2 + i + j + k, \\ Bj_{1,m} = \cdots = Bj_{m-4,m} = 1 + i + j + k, \\ Bj_{m-3,m} = 1 + i + j + 5k, \\ Bj_{m-2,m} = 1 + i + 5j + 7k, \\ Bj_{m-1,m} = 1 + 5i + 7j + 9k. \end{cases} \tag{26}$$

Using the relations (23)–(26) we can prove the following equalities:

$$BJ_{n,m} = BJ_{n-1,m} + 2BJ_{n-m,m}, \quad n \geq m, \tag{27}$$
$$Bj_{n,m} = Bj_{n-1,m} + 2Bj_{n-m,m}, \quad n \geq m, \tag{28}$$
$$Bj_{n,m} = BJ_{n+1,m} + 2BJ_{n+1-m,m}, \quad n \geq m. \tag{29}$$

We prove the following theorem.

Theorem 3.3. *The generalized bicomplex numbers of the Jacobsthal type satisfy:*

$$2\sum_{i=0}^{n} BJ_{i,m} = BJ_{n+m,m} - BJ_{m-1,m}, \quad n \geq m, \quad m \geq 2; \tag{30}$$
$$2\sum_{i=0}^{n} Bj_{i,m} = Bj_{n+m,m} - Bj_{m-1,m}, \quad n \geq m, \quad m \geq 2. \tag{31}$$

Proof. We shall prove (30). Namely, by (27), we get:

$$\begin{aligned} 2BJ_{0,m} &= BJ_{m,m} - BJ_{m-1,m}, \\ 2BJ_{1,m} &= BJ_{m+1,m} - BJ_{m,m}, \\ 2BJ_{2,m} &= BJ_{m+2,m} - BJ_{m+1,m}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ 2BJ_{n-1,m} &= BJ_{m+n-1,m} - BJ_{m+n-2,m}, \\ 2BJ_{n,m} &= BJ_{m+n,m} - BJ_{m+n-1,m}. \end{aligned}$$

Taking the sum of the previous equalities gives the relation (30). Similarly, (31) follows from (28). □

4. Generating functions for the Jacobsthal type numbers

In this section we find the generating functions for the generalized bicomplex Jacobsthal numbers $BJ_{n,m}$ and the generalized bicomplex Jacobsthal - Lucas numbers $Bj_{n,m}$.

Theorem 4.1. *The generating function $\mathcal{J}_m(t)$ for the numbers $BJ_{n,m}$ is given by the following formula*

$$\mathcal{J}_m(t) = \frac{i + j + k + t + kt^{m-2} + (2j + 2k)t^{m-1}}{1 - t - 2t^m}. \tag{32}$$

Proof. To verify that $\mathcal{J}_m(t)$ is the generating function, we calculate as following

$$\begin{aligned}
\mathcal{J}_m(t) &= \sum_{n=0}^{\infty} \left(J_{n,m} + iJ_{n+1,m} + jJ_{n+2,m} + kJ_{n+3,m}\right) t^n \\
&= J_m(t) + \frac{i}{t} \cdot J_m(t) + \frac{j}{t^2} \cdot (J_m(t) - t) + \frac{k}{t^3} \cdot \left(J_m(t) - t^2 - t\right) \\
&= \frac{i + j + k + t + 2kt^{m-2} + (2j + 2k)t^{m-1}}{1 - t - 2t^m}.
\end{aligned}$$

□

Corollary 4.2. *For $m = 2$, in (32), we get $\mathcal{J}_2(t)$, i.e. the generating function for the bicomplex numbers J_n:*

$$\mathcal{J}_2(t) = \sum_{n=0}^{\infty} BJ_n\, t^n = \frac{i + j + 3k + (1 + 2j + 2k)t}{1 - t - 2t^2}.$$

We now proceed to find the generating function for the bicomplex numbers $Bj_{n,m}$.

Theorem 4.3. *The generating function $\mathfrak{j}_m(t)$ of the numbers $Bj_{n,m}$ is*

$$\mathfrak{j}_m(t) = \frac{2 + i + j + k - t + 4kt^{m-3} + (4j + 2k)t^{m-2} + (4i + 2j + 2k)t^{m-1}}{1 - t - 2t^m}. \tag{33}$$

Proof. We proceed to verify that $\mathfrak{j}_m(t)$ is the generating function:

$$\begin{aligned}
\mathfrak{j}_m(t) &= \sum_{n=0}^{\infty} Bj_{n,m} t^n \\
&= \sum_{n=0}^{\infty} \left(j_{n,m} + ij_{n+1,m} + jj_{n+2,m} + kj_{n+3,m}\right) t^n \\
&= \sum_{n=0}^{\infty} j_{n,m} t^n + i \sum_{n=0}^{\infty} j_{n+1,m} t^n + j \sum_{n=0}^{\infty} j_{n+2,m} t^n + k \sum_{n=0}^{\infty} j_{n+3,m} t^n \\
&= j_m(t) + \frac{i}{t} \left(j_m(t) - j_{0,m}\right) + \frac{j}{t^2} \left(j_m(t) - j_{1,m} t - j_{0,m}\right) \\
&\quad + \frac{k}{t^3} \left(j_m(t) - j_{2,m} t^2 - j_{1,m} t - j_{0,m}\right) \\
&= j_m(t) \left(1 + \frac{i}{t} + \frac{j}{t^2} + \frac{k}{t^3}\right) - \frac{2i}{t} - \frac{jt}{t^2} - \frac{2j}{t^2} - \frac{kt^2}{t^3} - \frac{kt}{t^3} - \frac{2k}{t^3} \\
&= \frac{2 - t}{1 - t - 2t^m} \cdot \frac{t^3 + it^2 + jt + k}{t^3} - \frac{2it^2 + jt^2 + 2jt + kt^2 + kt + 2k}{t^3} \\
&= \frac{2 + i + j + k - t + 4kt^{m-3} + (4j + 2k)t^{m-2} + (4i + 2j + 2k)t^{m-1}}{1 - t - 2t^m}.
\end{aligned}$$

Hence, the formula (33) is correct. □

Corollary 4.4. *For $m = 3$ the relation (33) becomes*

$$j_3(t) = \frac{2 + i + j + 5k + (4j + 2k - 1)t + (4i + 2j + 2k)t^2}{1 - t - 2t^3}.$$

5. Some properties of the numbers $BF_{n,m}$ and $BJ_{n,m}$

In this section we prove some recurring formulae that hold for the generalized bicomplex numbers $BF_{n,m}$ and $BJ_{n,m}$.

Theorem 5.1. *For n, m and $s \in \mathbb{N}$, the generalized bicomplex Fibonacci numbers $BF_{n,m}$ satisfy the following formula*

$$\begin{cases} BF_{n+s,m} = F_{n,m}BF_{1+s,m} + F_{n-1,m}BF_{2+s-m,m} + F_{n-2,m}BF_{3+s-m,m} + \\ + \cdots + F_{n+1-m,m}BF_{s,m}, \quad s \geq m-2. \end{cases} \tag{34}$$

Proof. Since

$$\begin{aligned}
&(1 - t - t^m)\sum_{n=1}^{\infty} BF_{n+s,m}t^n \\
&= \sum_{n=1}^{\infty} BF_{n+s,m}t^n - \sum_{n=1}^{\infty} BF_{n+s,m}t^{n+1} - \sum_{n=1}^{\infty} BF_{n+s,m}t^{n+m} \\
&= BF_{1+s,m}t + BF_{2+s,m}t^2 + \cdots + BF_{n+s,m}t^n + \ldots \\
&\ - BF_{1+s,m}t^2 - BF_{2+s,m}t^3 - \cdots - BF_{n+s-1,m}t^n - \ldots \\
&\ - BF_{1+s,m}t^{1+m} - BF_{2+s,m}t^{2+m} - BF_{3+s,m}t^{3+m} - \ldots \\
&= BF_{1+s,m}t + t^2\left(BF_{2+s,m} - BF_{1+s,m}\right) + t^3\left(BF_{3+s,m} - BF_{2+s,m}\right) \\
&\ + \cdots + t^{m-1}\left(BF_{m+s-1,m} - BF_{m+s-2,m}\right) + t^m\left(BF_{m+s,m} - BF_{m+s-1,m}\right) \\
&= BF_{1+s,m}t + BF_{2+s-m,m}t^2 + \cdots + BF_{s,m}t^m.
\end{aligned}$$

So, we get

$$\begin{aligned}
\sum_{n=1}^{\infty} BF_{n+s,m}t^n &= \frac{t}{1-t-t^m}BF_{1+s,m} + \frac{t^2}{1-t-t^m}BF_{s+2-m,m} \\
&\quad + \cdots + \frac{t^m}{1-t-t^m}BF_{s,m},
\end{aligned}$$

or

$$\begin{aligned}
BF_{n+s,m} &= F_{n,m}BF_{1+s,m} + F_{n-1,m}BF_{2+s-m,m} + F_{n-2,m}BF_{3+s-m,m} \\
&\quad + \cdots + F_{n+1-m,m}BF_{s,m}, \quad s \geq m-2, \ m \geq 2,
\end{aligned}$$

which proves formula (34), where

$$\sum_{n=0}^{\infty} F_{n,m}t^n = \frac{t}{1-t-t^m}.$$

□

Example 5.2. *For $n = 5$, $s = 3$ and $m = 4$, we get*

$$\begin{aligned}
&BF_{5+3,4} = F_{5,4}BF_{1+3,4} + F_{4,4}BF_{1,4} + F_{3,4}BF_{2,4} + F_{2,4}BF_{3,4},\\
&5 + 7i + 10j + 14k = 2(1 + 2i + 3j + 4k) + 1(1 + i + j + k)\\
&\qquad + 1(1 + i + j + 2k) + 1(1 + i + 2j + 3k)\\
&5 + 7i + 10j + 14k = 5 + 7i + 10j + 14k.
\end{aligned}$$

Theorem 5.3. *The generalized bicomplex Jacobsthal numbers $BJ_{n,m}$, for n, m and $s \in \mathbb{N}$, satisfy the following relation*

$$\begin{cases} BJ_{n+s,m} = J_{n,m}BJ_{1+s,m} + 2J_{n-1,m}BJ_{2+s-m,m} + 2J_{n-2,m}BJ_{3+s-m,m} + \\ + \cdots + 2J_{n+2-m,m}BJ_{s-1,m} + J_{n+1-m,m}BJ_{s,m}, \ s \geq m-2, \ n \geq m, \ m \geq 2. \end{cases} \tag{35}$$

Proof. Starting from

$$\begin{aligned}
&(1 - t - 2t^m)\sum_{n=1}^{\infty} BJ_{n+s,m}t^n\\
&= BJ_{1+s,m}t + BJ_{2+s,m}t^2 + BJ_{3+s,m}t^3 + \cdots + BJ_{m+s,m}t^m + \dots\\
&- BJ_{1+s,m}t^2 - BJ_{2+s,m}t^3 - \cdots - BJ_{m-1+s,m}t^m - \dots\\
&- 2BJ_{1+s,m}t^{1+m} - 2BJ_{2+s,m}t^{2+m} - 2BJ_{3+s,m}t^{3+m} - \dots\\
&= BJ_{1+s,m}t + t^2(BJ_{2+s,m} - BJ_{1+s,m}) + t^3(BJ_{3+s,m} - BJ_{2+s,m}) + \dots\\
&+ t^{m-1}(BJ_{m-1+s,m} - BJ_{m-2+s,m}) + t^m(BJ_{m+s,m} - BJ_{m+s-1,m})\\
&= BJ_{1+s,m}t + 2BJ_{2+s-m,m}t^2 + 2BJ_{3+s-m,m}t^3 + \cdots + 2BJ_{s,m}t^m, \ \text{(by (27))},
\end{aligned}$$

we get

$$\begin{aligned}
\sum_{n=1}^{\infty} BJ_{n+s,m}t^n &= \frac{t}{1-t-2t^m}BJ_{1+s,m} + \frac{2t^2}{1-t-2t^m}BJ_{2+s-m,m}\\
&+ \frac{2t^3}{1-t-2t^m}BJ_{3+s-m,m} + \cdots + \frac{2t^{m-1}}{1-t-2t^m}BJ_{s-1,m} + \frac{2t^m}{1-t-2t^m}BJ_{s,m},
\end{aligned}$$

hence

$$\begin{aligned}
BJ_{n+s,m} &= J_{n,m}BJ_{1+s,m} + 2J_{n-1,m}BJ_{2+s-m,m} + 2J_{n-2,m}BJ_{3+s-m,m}\\
&+ \cdots + 2J_{n+2-m,m}BJ_{s-1,m} + 2J_{n+1-m,m}BJ_{s,m}.
\end{aligned}$$

where

$$\sum_{n=0}^{\infty} J_{n,m}t^n = \frac{t}{1-t-2t^m}.$$

□

Example 5.4. *If $n = 4$, $s = 3$, $m = 4$, then*

$$\begin{aligned}
&BJ_{4+3,4} = J_{4,4}BJ_{4,4} + 2J_{3,4}BJ_{1,4} + 2J_{2,4}BJ_{2,4} + 2J_{1,4}BJ_{3,4},\\
&7 + 9i + 15j + 25k = 1(1 + 3i + 5j + 7k) + 2(1 + i + j + 3k) + 2(1 + i + j + k)\\
&\qquad + 2(1 + i + 3j + 5k).
\end{aligned}$$

6. Convolutions of the generalized bicomplex numbers

In this section we introduce some new generalized numbers, i.e. some convolutions of the generalized bicomplex numbers: $\mathbb{F}_{n,m}$ - convolutions of the generalized bicomplex Fibonacci numbers; $\mathbb{L}_{n,m}$ - convolutions of the generalized bicomplex Lucas numbers; $\mathbb{J}_{n,m}$ - convolutions of the generalized bicomplex Jacobsthal numbers; $\beth_{n,m}$ - convolutions of the generalized bicomplex Jacobsthal - Lucas numbers.

Definition 6.1. *With respect to the previous notation, the numbers $\mathbb{F}_{n,m}$, $\mathbb{L}_{n,m}$, $\mathbb{J}_{n,m}$ and $\beth_{n,m}$ are defined in the following manner:*

$$\mathbb{F}_{n,m} = \sum_{i=1}^{[(n+m-1)/m]} BF_{i,m}BF_{n-m(i-1),m}; \tag{36}$$

$$\mathbb{L}_{n,m} = \sum_{i=1}^{[(n+m-1)/m]} BL_{i,m}BL_{n-m(i-1),m}; \tag{37}$$

$$\mathbb{J}_{n,m} = \sum_{i=1}^{[(n+m-1)/m]} BJ_{i,m}BJ_{n-m(i-1),m}; \tag{38}$$

$$\beth_{n,m} = \sum_{i=1}^{[(n+m-1)/m]} Bj_{i,m}Bj_{n-m(i-1),m}. \tag{39}$$

Theorem 6.2. *The generating function $\mathbb{F}^m(t)$ for the numbers $\mathbb{F}_{n,m}$ is given by the following formula:*

$$\mathbb{F}^m(t) = \frac{1}{t^m}\left(f_m(t) - BF_{0,m}\right)\cdot\left(f_m(t^m) - BF_{0,m}\right), \tag{40}$$

where

$$\begin{aligned} f_m(t) &= \frac{i+j+k+t+kt^{m-2}+(j+k)t^{m-1}}{1-t-t^m},\\ f_m(t^m) &= \frac{i+j+k+t^m+kt^{m(m-2)}+(j+k)t^{m(m-1)}}{1-t^m-t^{m^2}},\\ BF_{0,m} &= i+j+k. \end{aligned}$$

Proof. We prove that $\mathbb{F}^m(t) = \sum\limits_{n=1}^{\infty}\mathbb{F}_{n,m}t^n$ satisfies (40):

$$\begin{aligned} \mathbb{F}^m(t) &= \mathbb{F}_{1,m}t + \mathbb{F}_{2,m}t^2 + \mathbb{F}_{3,m}t^3 + \cdots + \mathbb{F}_{m,m}t^m + \mathbb{F}_{m+1,m}t^{m+1} + \ldots\\ &= (BF_{1,m}BF_{1,m})t + (BF_{1,m}BF_{2,m})t^2 + (BF_{1,m}BF_{3,m})t^3 + \ldots\\ &\quad + (BF_{1,m}BF_{m,m})t^m + (BF_{1,m}BF_{m+1,m} + BF_{2,m}BF_{1,m})t^{m+1} + \ldots\\ &= \Big(BF_{1,m} + BF_{2,m}t + BF_{3,m}t^2 + BF_{4,m}t^3 + \cdots + BF_{m,m}t^{m-1} + \ldots\Big)\times\\ &\quad \Big(BF_{1,m}t + BF_{2,m}t^{m+1} + BF_{3,m}t^{2m+1} + BF_{4,m}t^{3m+1} + \ldots\Big)\\ &= \frac{1}{t^m}\left(f_m(t) - BF_{0,m}\right)\cdot\left(f_m(t^m) - BF_{0,m}\right). \end{aligned}$$

In the same manner we can obtain the generating functions for the numbers (37), (38) and (39). □

Next, we are going to prove the following theorem, where $n = mp + l$, $m \geq 2$, $l = 0, 1, \ldots, m-1$.

Theorem 6.3. *The following relations are correct:*

$$\mathbb{F}_{n,m} = \mathbb{F}_{n-1,m} + \mathbb{F}_{n-m,m} + A, \;\; n \geq m, \tag{41}$$

where

$$A = \begin{cases} BF_{p,m} \cdot BF_{0,m}, & for\ l = 0, \\ BF_{p+1,m} \cdot BF_{1,m}, & for\ l = 1, \\ 0, & for\ l = 2, \dots, m-3, \\ k \cdot BF_{p+1,m}, & for\ l = m-2, \\ (j+k) \cdot BF_{p+1,m}, & for\ l = m-1; \end{cases} \tag{42}$$

$$\mathbb{L}_{n,m} = \mathbb{L}_{n-1,m} + \mathbb{L}_{n-m,m} + B, \;\; n \geq m, \tag{43}$$

for

$$B = \begin{cases} BL_{p,m} \cdot BL_{0,m}, & for\ l = 0, \\ BL_{p+1,m} \cdot BL_{1,m}, & for\ l = 1, \\ 0, & for\ l = 2, \dots, m-4, \\ 2k \cdot BL_{p+1,m}, & for\ l = m-3, \\ (2j+k) \cdot BL_{p+1,m}, & for\ l = m-2, \\ (2i+j+k) \cdot BL_{p+1,m}, & for\ l = m-1; \end{cases} \tag{44}$$

$$\mathbb{J}_{n,m} = \mathbb{J}_{n-1,m} + 2\mathbb{J}_{n-m,m} + C, \;\; n \geq m, \tag{45}$$

where

$$C = \begin{cases} 2BJ_{p,m} \cdot BJ_{0,m}, & for\ l = 0, \\ BJ_{p+1,m} \cdot BJ_{1,m}, & for\ l = 1, \\ 0, & for\ l = 2, \dots, m-3, \\ 2k \cdot BJ_{p+1}, & for\ l = m-2, \\ (2j+2k) \cdot BJ_{p+1}, & for\ l = m-1; \end{cases} \tag{46}$$

$$\mathbb{j}_{n,m} = \mathbb{j}_{n-1,m} + 2\mathbb{j}_{n-m,m} + D, \;\; n \geq m, \tag{47}$$

where

$$D = \begin{cases} 2Bj_{p,m} \cdot Bj_{0,m}, & for\ l = 0, \\ Bj_{p+1,m} \cdot Bj_{1,m}, & for\ l = 1, \\ 0, & for\ l = 2, \dots, m-4, \\ 4k \cdot Bj_{p+1,m}, & for\ l = m-3, \\ (4j+2k) \cdot Bj_{p+1,m}, & for\ l = m-2, \\ (4i+2j+2k) \cdot Bj_{p+1,m}, & for\ l = m-1. \end{cases} \tag{48}$$

Proof. We shall prove the relations (41) and (42) using Definition (36) and the recurrence relation (16). We consider the following three cases.

1° If $n = mp$, then $[(mp+m-1)/m] = p$, $[(mp-1+m-1)/m] = p$,

$[(mp - m + m - 1)/m] = p - 1$, and:

$$\begin{aligned}\mathbb{F}_{n,m} &= BF_{1,m}BF_{n,m} + BF_{2,m}BF_{n-m,m} + \cdots + BF_{p-1,m}BF_{n-m(p-2),m}\\ &\quad + BF_{p,m}BF_{n-m(p-1),m},\\ \mathbb{F}_{n-1,m} &= BF_{1,m}BF_{n-1,m} + BF_{2,m}BF_{n-1-m,m} + \cdots + BF_{p-1,m}BF_{n-1-m(p-2),m}\\ &\quad + BF_{p,m}BF_{n-1-m(p-1),m},\\ \mathbb{F}_{n-m,m} &= BF_{1,m}BF_{n-m,m} + BF_{2,m}BF_{n-2m,m} + \cdots + BF_{p-1,m}BF_{n-m-m(p-2),m},\end{aligned}$$

hence, we get

$$\begin{aligned}\mathbb{F}_{n-1,m} + \mathbb{F}_{n-m,m} &= BF_{1,m}BF_{n,m} + BF_{2,m}BF_{n-m,m} + \cdots + BF_{p-1,m}BF_{n-m(p-2),m}\\ &\quad + BF_{p,m}BF_{n-1-m(p-1),m}\\ &= \mathbb{F}_{n,m} - BF_{p,m}BF_{n-m(p-1),m} + BF_{p,m}BF_{n-1-m(p-1),m}\\ &= \mathbb{F}_{n,m} - BF_{p,m}\left(BF_{n-m(p-1),m} - BF_{n-1-m(p-1),m}\right)\\ &= \mathbb{F}_{n,m} - BF_{p,m}\left(BF_{m,m} - BF_{m-1,m}\right) \quad \text{(by (16))}\\ &= \mathbb{F}_{n,m} - BF_{p,m}BF_{0,m}.\end{aligned}$$

So, we get the following relation

$$\mathbb{F}_{n,m} = \mathbb{F}_{n-1,m} + \mathbb{F}_{n-m,m} + BF_{p,m} \cdot BF_{0,m}.$$

2° If $n = mp + 1$, then $[(mp + 1 + m - 1)/m] = p + 1$, $[(mp + m - 1)/m] = p$, $[(mp + 1 - m + m - 1)/m] = p$, and:

$$\begin{aligned}\mathbb{F}_{n,m} &= BF_{1,m}BF_{n,m} + BF_{2,m}BF_{n-m,m} + \cdots + BF_{p,m}BF_{n-m(p-1),m}\\ &\quad + BF_{p+1,m}BF_{n-mp,m},\\ \mathbb{F}_{n-1,m} &= BF_{1,m}BF_{n-1,m} + BF_{2,m}BF_{n-1-m,m} + \cdots + BF_{p,m}BF_{n-1-m(p-1),m},\\ \mathbb{F}_{n-m,m} &= BF_{1,m}BF_{n-m,m} + BF_{2,m}BF_{n-2m,m} + \cdots + BF_{p,m}BF_{n-m-m(p-1),m},\end{aligned}$$

hence, we get

$$\mathbb{F}_{n,m} = \mathbb{F}_{n-1,m} + \mathbb{F}_{n-m,m} + BF_{p+1,m} \cdot BF_{1,m}.$$

3° Finally, for $n = mp + l$, where $l = 2, \ldots, m - 1$, then:
$[(mp + l + m - 1)/m] = p + 1$, $[(mp + l - 1 + m - 1)/m] = p + 1$, $[(mp + l - m + m - 1)/m] = p$, and

$$\mathbb{F}_{n,m} = \mathbb{F}_{n-1,m} + \mathbb{F}_{n-m,m} + BF_{p+1,m} \cdot \mathfrak{F},$$

where

$$\mathfrak{F} = BF_{l,m} - BF_{l-1,m},$$

i.e.

$$\mathfrak{F} = \begin{cases} 0, & for\ l = 2, \ldots, m - 3,\\ k, & for\ l = m - 2,\\ j + k, & for\ l = m - 1.\end{cases}$$

The relations (43)–(48) can be proved in a similar way. □

Example 6.4. *For* $n = 4 \cdot 2 + 1$, *i.e.* $m = 4$, $l = 1$ *and* $p = 2$, *we get*

$$\begin{aligned} \mathbb{F}_{9,4} &= BF_{1,4}BF_{9,4} + BF_{2,4}BF_{5,4} + BF_{3,4}BF_{1,4}, \\ \mathbb{F}_{8,4} &= BF_{1,4}BF_{8,4} + BF_{2,4}BF_{4,4}, \\ \mathbb{F}_{5,4} &= BF_{1,4}BF_{5,4} + BF_{2,4}BF_{1,4}, \end{aligned}$$

so, we get

$$\begin{aligned} \mathbb{F}_{8,4} + \mathbb{F}_{5,4} &= BF_{1,4}BF_{9,4} + BF_{2,4}BF_{5,4} \\ &= \mathbb{F}_{9,4} - BF_{3,4}BF_{1,4}, \quad ((42)\ \textit{for}\ \ l = 1). \end{aligned}$$

Example 6.5. *If* $n = 4 \cdot 3 + 2$, *i.e.* $m = 4$, $p = 3$ *and* $l = 2$, *we find*

$$\begin{aligned} \mathbb{L}_{14,4} &= BL_{1,4}BL_{14,4} + BL_{2,4}BL_{10,4} + BL_{3,4}BL_{6,4} + BL_{4,4}BL_{2,4}, \\ \mathbb{L}_{13,4} &= BL_{1,4}BL_{13,4} + BL_{2,4}BL_{9,4} + BL_{3,4}BL_{5,4} + BL_{4,4}BL_{1,4}, \\ \mathbb{L}_{10,4} &= BL_{1,4}BL_{10,4} + BL_{2,4}BL_{6,4} + BL_{3,4}BL_{2,4}, \end{aligned}$$

and

$$\begin{aligned} \mathbb{L}_{13,4} + \mathbb{L}_{10,4} &= \mathbb{L}_{14,4} + BL_{4,4}BL_{2,4} - B_{4,4}BL_{2,4} \\ &= \mathbb{L}_{14,4} - BL_{4,4} \cdot (BL_{2,4} - BL_{1,4}) \\ &= \mathbb{L}_{14,4} - (2j + k) \cdot BL_{4,4}. \end{aligned}$$

Example 6.6. *If* $n = 4 \cdot 3 + 1$, *where* $m = 4$, $p = 3$ *and* $l = 1$, *we get*

$$\begin{aligned} \mathbb{J}_{13,4} &= BJ_{1,4}BJ_{13,4} + BJ_{2,4}BJ_{9,4} + BJ_{3,4}BJ_{5,4} + BJ_{4,4}BJ_{1,4}, \\ \mathbb{J}_{12,4} &= BJ_{1,4}BJ_{12,4} + BJ_{2,4}BJ_{8,4} + BJ_{3,4}NJ_{4,4}, \\ \mathbb{J}_{9,4} &= BJ_{1,4}BJ_{9,4} + BJ_{2,4}BJ_{5,4} + BJ_{3,4}BJ_{1,4}, \end{aligned}$$

so

$$\begin{aligned} \mathbb{J}_{12,4} + 2\mathbb{J}_{9,4} &= BJ_{1,4}BJ_{13,4} + BJ_{2,4}BJ_{9,4} + BJ_{3,4}BJ_{5,4} \\ &= \mathbb{J}_{13,4} - BJ_{4,4}BJ_{1,4}, \end{aligned}$$

i.e.

$$\mathbb{J}_{13,4} = \mathbb{J}_{12,4} + 2\mathbb{J}_{9,4} + BJ_{4,4}BJ_{1,4}, \ \ (\textit{by}\ (46),\ \ l = 1).$$

Example 6.7. *If* $n = 5 \cdot 3 + 4$, *where* $m = 5$, $p = 3$ *and* $l = 4$, *we have*

$$\begin{aligned} ℷ_{19,5} &= Bj_{1,5}Bj_{19,5} + Bj_{2,5}Bj_{14,5} + Bj_{3,5}Bj_{9,5} + Bj_{4,5}Bj_{4,5}, \\ ℷ_{18,5} &= Bj_{1,5}Bj_{18,5} + Bj_{2,5}Bj_{13,5} + Bj_{3,5}Bj_{8,5} + Bj_{4,5}Bj_{3,5}, \\ ℷ_{14,5} &= Bj_{1,5}Bj_{14,5} + Bj_{2,5}Bj_{9,5} + Bj_{3,5}Bj_{4,5}, \end{aligned}$$

hence

$$\begin{aligned}\daleth_{18,5} + 2\daleth_{14,5} &= Bj_{1,5}Bj_{19,5} + Bj_{2,5}Bj_{14,5} + Bj_{3,5}Bj_{9,5} + Bj_{4,5}Bj_{3,5}\\ &= \daleth_{19,5} - Bj_{4,5}Bj_{4,5} + Bj_{4,5}Bj_{3,5}\\ &= \daleth_{19,5} - Bj_{4,5}(Bj_{4,5} - Bj_{3,5})\\ &= \daleth_{19,5} - Bj_{4,5}\cdot(4i + 2j + 2k) \quad (\textit{from table } 2).\end{aligned}$$

7. Mixed convolutions

In this section we introduce the following numbers.

Definition 7.1. *Mixed convolutions of the numbers $BF_{n,m}$, $BL_{n,m}$, $BJ_{n,m}$ and $Bj_{n,m}$, are given by:*

$$\mathfrak{f}_{n,m} = \sum_{i=1}^{[(n+m-1)/m]} BF_{i,m}BL_{n-m(i-1),m}; \tag{49}$$

$$\mathfrak{l}_{n,m} = \sum_{i=1}^{[(n+m-1)/m]} BL_{i,m}BF_{n-m(i-1),m}; \tag{50}$$

$$\mathfrak{a}_{n,m} = \sum_{i=1}^{[(n+m-1)/m]} BF_{i,m}BJ_{n-m(i-1),m}; \tag{51}$$

$$\mathfrak{b}_{n,m} = \sum_{i=1}^{[(n+m-1)/m]} BJ_{i,m}BF_{n-m(i-1),m}; \tag{52}$$

$$\mathfrak{c}_{n,m} = \sum_{i=1}^{[(n+m-1)/m]} BL_{i,m}BJ_{n-m(i-1),m}; \tag{53}$$

$$\mathfrak{d}_{n,m} = \sum_{i=1}^{[(n+m-1)/m]} BJ_{i,m}BL_{n-m(i-1),m}; \tag{54}$$

$$\mathfrak{e}_{n,m} = \sum_{i=1}^{[(n+m-1)/m]} BJ_{i,m}Bj_{n-m(i-1),m}; \tag{55}$$

$$\mathfrak{g}_{n,m} = \sum_{i=1}^{[(n+m-1)/m]} Bj_{i,m}BJ_{n-m(i-1),m}. \tag{56}$$

These numbers, (49)–(56), are mixed convolutions of the corresponding numbers.

Now, we shall prove the following theorem.

Theorem 7.2. *The numbers (49)–(56), for $n = m\cdot p + l$, where $m \geq 2$, $l = 0, 1, \ldots, m-1$, satisfy the following relations, respectively:*

$$\mathfrak{f}_{n,m} = \mathfrak{f}_{n-1,m} + \mathfrak{f}_{n-m,m} + A_1, \tag{57}$$

where

$$A_1 = \begin{cases} BF_{p,m} \cdot BL_{0,m}, & for \ \ l = 0, \\ BF_{p+1,m} \cdot BL_{1,m}, & for \ \ l = 1, \\ 0, & for \ l = 2, \ldots, m-4, \\ 2k \cdot BF_{p+1,m}, & for \ \ l = m-3, \\ (2j+k) \cdot BF_{p+1,m}, & for \ \ l = m-2, \\ (2i+j+k) \cdot BF_{p+1,m}, & for \ \ l = m-1. \end{cases} \tag{58}$$

$$\mathfrak{l}_{n,m} = \mathfrak{l}_{n-1,m} + \mathfrak{l}_{n-m,m} + B_1, \tag{59}$$

where

$$B_1 = \begin{cases} BL_{p,m} \cdot BF_{0,m}, & for \ \ l = 0, \\ BL_{p+1,m} \cdot BF_{1,m}, & for \ \ l = 1, \\ 0, & for \ \ l = 2, \ldots, m-3, \\ k \cdot BL_{p+1,m}, & for \ \ l = m-2, \\ (j+k) \cdot BL_{p+1,m}, & for \ \ l = m-1. \end{cases} \tag{60}$$

$$\mathfrak{a}_{n,m} = \mathfrak{a}_{n-1,m} + 2\mathfrak{a}_{n-m,m} + C_1, \tag{61}$$

where

$$C_1 = \begin{cases} 2BF_{p,m} \cdot BJ_{0,m}, & for \ \ l = 0, \\ BF_{p+1,m} \cdot BJ_{1,m}, & for \ \ l = 1, \\ 0, & for \ \ l = 2, \ldots, m-3, \\ 2k \cdot BF_{p+1,m}, & for \ \ l = m-2, \\ (2j+2k) \cdot BF_{p+1,m}, & for \ \ l = m-1. \end{cases} \tag{62}$$

$$\mathfrak{b}_{n,m} = \mathfrak{b}_{n-1,m} + \mathfrak{b}_{n-m,m} + D_1, \tag{63}$$

where

$$D_1 = \begin{cases} BJ_{p,m} \cdot BF_{0,m}, & for \ \ l = 0, \\ BJ_{p+1,m} \cdot BF_{1,m}, & for \ \ l = 1, \\ 0, & for \ \ l = 2, \ldots, m-3, \\ k \cdot BJ_{p+1,m}, & for \ \ l = m-2, \\ (j+k) \cdot BJ_{p+1,m}, & for \ \ l = m-1. \end{cases} \tag{64}$$

$$\mathfrak{c}_{n,m} = \mathfrak{c}_{n-1,m} + 2\mathfrak{c}_{n-m,m} + E_1, \tag{65}$$

where

$$E_1 = \begin{cases} 2BL_{p,m} \cdot BJ_{0,m}, & for \;\; l = 0, \\ BL_{p+1,m} \cdot BJ_{1,m}, & for \;\; l = 1, \\ 0, & for \;\; l = 2, \ldots, m-3, \\ 2k \cdot BL_{p+1,m}, & for \;\; l = m-2, \\ (2j + 2k) \cdot BL_{p+1,m}, & for \;\; l = m-1. \end{cases} \tag{66}$$

$$\mathfrak{d}_{n,m} = \mathfrak{d}_{n-1,m} + \mathfrak{d}_{n-m,m} + F_1, \tag{67}$$

where

$$F_1 = \begin{cases} BJ_{p,m} \cdot BL_{0,m}, & for \;\; l = 0, \\ BJ_{p+1,m} \cdot BL_{1,m}, & for \;\; l = 1, \\ 0, & for \;\; l = 2, \ldots, m-3, \\ 2k \cdot BJ_{p+1,m}, & for \;\; l = m-2, \\ (2i + j + k) \cdot BJ_{p+1,m}, & for \;\; l = m-1. \end{cases} \tag{68}$$

$$\mathfrak{e}_{n,m} = \mathfrak{e}_{n-1,m} + 2\mathfrak{e}_{n-m,m} + G_1, \tag{69}$$

where

$$G_1 = \begin{cases} 2BJ_{p,m} \cdot Bj_{0,m}, & for \;\; l = 0, \\ BJ_{p+1,m} \cdot Bj_{1,m}, & for \;\; l = 1, \\ 0, & for \;\; l = 2, \ldots, m-3, \\ 2k \cdot BJ_{p+1,m}, & for \;\; l = m-2, \\ (2j + 2k) \cdot BJ_{p+1,m}, & for \;\; l = m-1. \end{cases} \tag{70}$$

$$\mathfrak{g}_{n,m} = \mathfrak{g}_{n-1,m} + 2\mathfrak{g}_{n-m,m} + H_1, \tag{71}$$

where

$$H_1 = \begin{cases} 2Bj_{p,m} \cdot BJ_{0,m}, & for \;\; l = 0, \\ Bj_{p+1,m} \cdot BJ_{1,m}, & for \;\; l = 1, \\ 0, & for \;\; l = 2, \ldots, m-4, \\ 4k \cdot Bj_{p+1,m}, & for \;\; l = m-3, \\ (4j + 2k) \cdot Bj_{p+1,m}, & for \;\; l = m-2, \\ (4i + 2j + 2k) \cdot Bj_{p+1,m}, & for \;\; l = m-1. \end{cases} \tag{72}$$

Proof. By Definition (7.1), using methods similar to those in Theorem (6.3), we can prove the relations (57)–(72). □

Example 7.3. *If $n = 3 \cdot 3$, $m = 3$ and $l = 0$, then*

$$\begin{aligned}
\mathfrak{c}_{9,3} &= BL_{1,3}BJ_{9,3} + BL_{2,3}BJ_{6,3} + BL_{3,3}BJ_{3,3},\\
\mathfrak{c}_{8,3} &= BL_{1,3}BJ_{8,3} + BL_{2,3}BJ_{5,3} + BL_{3,3}BJ_{2,3},\\
\mathfrak{c}_{6,3} &= BL_{1,3}BJ_{6,3} + BL_{2,3}BJ_{3,3},
\end{aligned}$$

and

$$\begin{aligned}
\mathfrak{c}_{8,3} + 2\mathfrak{c}_{6,3} &= BL_{1,3}BJ_{9,3} + BL_{2,3}BJ_{6,3} + BL_{3,3}BJ_{2,3}\\
&= \mathfrak{c}_{9,3} - BL_{3,3}(BJ_{3,3} - BJ_{2,3})\\
&= \mathfrak{c}_{9,3} - 2BL_{3,3}BJ_{0,3}.
\end{aligned}$$

Example 7.4. *Let $n = 4 \cdot 3 + 3$, where $m = 4$, $p = 3$ and $l = 3$, then we get:*

$$\begin{aligned}
\mathfrak{d}_{15,4} &= BJ_{1,4}BL_{15,4} + BJ_{2,4}BL_{11,4} + BJ_{3,4}BL_{7,4} + BJ_{4,4}BL_{3,4},\\
\mathfrak{d}_{14,4} &= BJ_{1,4}BL_{14,4} + BJ_{2,4}BL_{10,4} + BJ_{3,4}BL_{6,4} + BJ_{4,4}BL_{2,4},\\
\mathfrak{d}_{11,4} &= BJ_{1,4}BL_{11,4} + BJ_{2,4}BL_{7,4} + BJ_{3,4}BL_{3,4},
\end{aligned}$$

since

$$\begin{aligned}
\mathfrak{d}_{14,4}+\mathfrak{d}_{11,4} &= BJ_{1,4}BL_{15,4} + BJ_{2,4}BL_{11,4} + BJ_{3,4}BL_{7,4} + BJ_{4,4}BL_{2,4}\\
&= \mathfrak{d}_{15,4} - BJ_{4,4}BL_{3,4} + BJ_{4,4}BL_{2,4}\\
&= \mathfrak{d}_{15,4} - BJ_{4,4}(BJ_{3,4} - BJ_{2,4})\\
&= \mathfrak{d}_{15,4} - (2i + j + k) \cdot BJ_{4,4}.
\end{aligned}$$

Example 7.5. *For $n = 4 \cdot 3 + 1$, i.e. $m = 4$, $p = 3$ and $l = 1$, we find:*

$$\begin{aligned}
\mathfrak{g}_{13,4} &= Bj_{1,4}BJ_{13,4} + Bj_{2,4}BJ_{9,4} + Bj_{3,4}BJ_{5,4} + Bj_{4,4}BJ_{1,4},\\
\mathfrak{g}_{12,4} &= Bj_{1,4}BJ_{12,4} + Bj_{2,4}BJ_{8,4} + Bj_{3,4}BJ_{4,4},\\
\mathfrak{g}_{9,4} &= Bj_{1,4}BJ_{9,4} + Bj_{2,4}BJ_{5,4} + Bj_{3,4}BJ_{1,4},
\end{aligned}$$

so

$$\begin{aligned}
\mathfrak{g}_{12,4} + 2\mathfrak{g}_{9,4} &= Bj_{1,4}BJ_{13,4} + Bj_{2,4}BJ_{9,4} + Bj_{3,4}BJ_{5,4}\\
&= \mathfrak{g}_{13,4} - Bj_{4,4}BJ_{1,4}.
\end{aligned}$$

8. Generating functions

In this section we are going to find some generating functions for the numbers (49)–(56). We start with the following theorem.

Theorem 8.1. *The generating function $f^m(t)$ for the numbers $\mathfrak{f}_{n,m}$ in (49) is given by*

$$f^m(t) = \frac{1}{t^m}\left(l_m(t) - BL_{0,m}\right) \cdot \left(f_m(t^m) - BF_{0,m}\right), \tag{73}$$

where

$$
\begin{aligned}
l_m(t) &= \frac{2+i+j+k-t+2kt^{m-3}+(2j+k)t^{m-2}+(2i+j+k)t^{m-1}}{1-t-t^m},\\
f_m(t^m) &= \frac{i+j+k+t^m+kt^{m(m-2)}+(j+k)t^{m(m-1)}}{1-t^m-t^{m^2}},
\end{aligned}
$$

$$BF_{0,m}=i+j+k,\quad BL_{0,m}=2+i+j+k.$$

Proof. If $f^m(t)$ is indeed the generating function, i.e.

$$f^m(t)=\sum_{n=1}^{\infty}\mathfrak{f}_{n,m}t^n,$$

then we get:

$$
\begin{aligned}
f^m(t) &= \sum_{n=1}^{\infty}\mathfrak{f}_{n,m}t^n = \mathfrak{f}_{1,m}t+\mathfrak{f}_{2,m}t^2+\mathfrak{f}_{3,m}t^3+\cdots+\mathfrak{f}_{m,m}t^m+\mathfrak{f}_{m+1,m}t^{m+1}+\ldots\\
&= (BF_{1,m}BL_{1,m})t+(BF_{1,m}BL_{2,m})t^2+(BF_{1,m}BL_{3,m})t^3\\
&\quad +\cdots+(BF_{1,m}BL_{m,n})t^m+(BF_{1,m}BL_{m+1,m}+BF_{2,m}BL_{1,m})t^{m+1}+\ldots\\
&= \Big(BL_{1,m}+BL_{2,m}t+BL_{3,m}t^2+\cdots+BL_{m,m}t^{m-1}+BL_{m+1,m}t^m+\ldots\Big)\times\\
&\quad \Big(BF_{1,m}t+BF_{2,m}t^{m+1}+BF_{3,m}t^{2m+1}+BF_{4,m}t^{3m+1}+\ldots\Big)\\
&= \frac{1}{t}\Big(BL_{1,m}t+BL_{2,m}t^2+BL_{3,m}t^3+\cdots+BL_{m,m}t^m+\ldots\Big)\times\\
&\quad \frac{1}{t^{m-1}}\Big(BF_{1,m}t^m+BF_{2,m}t^{2m}+BF_{3,m}t^{3m}+\ldots\Big)\\
&= \frac{1}{t^m}\left(l_m(t)-BL_{0,m}\right)\cdot\left(f_m(t^m)-BF_{0,m}\right).
\end{aligned}
$$

Hence, we get the wanted relation (73). □

Next, we are going to find the generating function for the numbers (54).

Theorem 8.2. *The generating function $d^m(t)$ for the numbers $\mathfrak{d}_{n,m}$, is given as*

$$d^m(t)=\sum_{n=1}^{\infty}\mathfrak{d}_{n,m}t^n=\frac{1}{t^m}\left(l_m(t)-BL_{0,m}\right)\cdot\left(\mathcal{J}_m(t^m)-BJ_{0,m}\right), \tag{74}$$

where

$$
\begin{aligned}
l_m(t) &= \frac{2+i+j+k-t+2kt^{m-3}+(2j+k)t^{m-2}+(2i+j+k)t^{m-1}}{1-t-t^m},\\
\mathcal{J}_m(t^m) &= \frac{i+j+k+t^m+kt^{m(m-2)}+(2j+2k)t^{m(m-1)}}{1-t^m-2t^{m^2}},\\
&\quad BL_{0,m}=2+i+j+k,\quad BJ_{0,m}=i+j+k.
\end{aligned}
$$

Proof. If $d^m(t)$ is the generating function for the numbers $\mathfrak{d}_{n,m}$, then:

$$\begin{aligned}
d^m(t) &= \sum_{n=1}^{\infty} \mathfrak{d}_{n,m} t^n = \mathfrak{d}_{1,m} t + \mathfrak{d}_{2,m} t^2 + \mathfrak{d}_{3,m} t^3 + \cdots + \mathfrak{d}_{m,m} t^m + \mathfrak{d}_{m+1,m} t^{m+1} + \ldots \\
&= \left(BJ_{1,m} BL_{1,m}\right) t + \left(BJ_{1,m} BL_{2,m}\right) t^2 + \left(BJ_{1,m} BL_{3,m}\right) t^3 + \ldots \\
&\quad + \left(BJ_{1,m} BL_{m,m}\right) t^m + \left(BJ_{1,m} BL_{m+1,m} + BJ_{2,m} BL_{1,m}\right) t^{m+1} + \ldots \\
&= \Big(BL_{1,m} + BL_{2,m} t + BL_{3,m} t^2 + \cdots + BL_{m,m} t^{m-1} + BL_{m+1,m} t^m + \ldots\Big) \times \\
&\quad \Big(BJ_{1,m} t + BJ_{2,m} t^{m+1} + BJ_{3,m} t^{2m+1} + BJ_{4,m} t^{3m+1} + \ldots\Big) \\
&= \frac{1}{t^m} \left(l_m(t) - BL_{0,m}\right) \cdot \left(\mathcal{J}_m(t^m) - BJ_{0,m}\right).
\end{aligned}$$

Remark. The generating functions for the numbers (50)–(53), (55) and (56) can be found in a similar way. □

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