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Topological Degree Theory for a Class of Nonlinear Degenerate Elliptic Problems in Weighted Sobolev Spaces

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Abstract. This article is devoted to study the existence of weak solutions to a Dirichlet boundary value problems for the following nonlinear degenerate elliptic problems of the type:

$$-\text{div}\Big[\mathcal{B}(x,u,\nabla u)+\mathcal{A}(x,\nabla u)\Big]=\lambda\mathcal{G}(x,u,\nabla u)+a(x)|u|^{q-2}u,$$

where $\mathcal{A}$ and $\mathcal{B}$ are Caratéodory functions that satisfy some conditions, and $\mathcal{G}(x,s,\eta)$ is a nonlinear term satisfying only the growth condition on η. our method consists in transforming this Dirichlet boundary value problem with nonlinearity into a new one governed by a Hammerstein equation. Then, we use a topological degree theory developed by Berkovits for operators of generalized monotone type.

1. Introduction and motivation

Topological degree theory has the potential to be one of the most powerful methods for solving nonlinear elliptic and parabolic equations. Leray-Schauder [21] introduced this for the first time in their study of the nonlinear equations for compact perturbations of the identity in infinite-dimensional Banach spaces. Browder [7] constructed a topological degree for operators of class (S_+) in reflexive Banach spaces with the Galerkin method, see also [33, 34]. For more information on the history of this theory the reader should consult the following works [3, 8, 11–16].

The aim of this article is to study the following Dirichlet boundary value problems associated to the nonlinear degenerate elliptic equations

$$\begin{cases} -\text{div}\Big[\mathcal{B}(x,u,\nabla u)+\mathcal{A}(x,\nabla u)\Big]=\lambda\mathcal{G}(x,u,\nabla u)+a(x)|u|^{q-2}u & \text{in} \quad \Omega,\\ u=0 & \text{on} \quad \partial\Omega. \end{cases} \tag{1}$$

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Here $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded open domain with Lipschitz boundary, λ is a real parameter and $p, q \in]2; \infty[$ such that $q < p$.

We assume that $\mathcal{B} : \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ and $\mathcal{A} : \Omega \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ are Carathéodory's functions that satisfy some conditions(that will be specified in the following pages), and the nonlinear term $\mathcal{G} : \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}$ is a Carathéodory function satisfy suitable growth assumption. Finally, we mention that the $a \in L^\infty(\Omega)$ satisfying :

$$|a(x)| \leq \gamma \sigma(x), \tag{2}$$

where γ is a positive constant and σ is a weight function in Ω which will be stated later.

The interest in studying problems like (1) relies not only on mathematical purposes but also on their significance applications in, such as, climatological model [9], image processing [23, 36], it is also applied to polymer rheology, regular variation in thermodynamics, fitting of experimental data, blood flow phenomena, aerodynamics, electro analytical chemistry, electro-dynamics of complex medium, viscoelasticity, electrical circuits, biology, control theory, capacitor theory, non-Newtonian fluids, electrorheological fluids, the flow of a fluid through a porous medium [19, 25, 30, 32]. More information on physical applications of this argument isrefered to [17, 18, 35].

We now give some background for the results in this paper. In the simplest case $\mathcal{B}(x, s, \zeta) = \zeta$, $\mathcal{A}(x, \zeta) = 0$, $q = 2$, $\lambda = 1$ and $\mathcal{G}(x, u, \nabla u) = \mu |\nabla u|^2 + f(x)$, we have the classical quasilinear elliptic problem

$$-\triangle u = a(x)u + \mu |\nabla u|^2 + f(x), \qquad u \in H_0^1(\Omega) \cap L^\infty(\Omega)$$

with a gradient dependence up to the critical growth $|\nabla u|^2$ were studied by many authors in the last forty years [5, 6, 20].

On the other hand, when $\mathcal{B}(x, s, \zeta) = |\zeta|^{p-2} \cdot \zeta$ and $\mathcal{A}(x, \zeta) = 0$, where $p \geq 2$, we obtain the p-Laplace operator, there are many classes of problems related to p-Laplace operators (see [10, 22, 24, 26–29, 31] for more details).

The rest of this article is organized as follows. in the next section some basic preliminaries and some classes of mappings of generalized (S_+) type and the recent Berkovits degree will be given. Section 3 is devoted to some necessary lemmas and basic assumptions of our problem. Finally the main results of this paper will be proven.

2. Mathematical background

This section is devoted to some essential properties of weighted Sobolev spaces, as well as some mapping classes and topological degrees needed for future developments.

2.1. The weighted Sobolev space

Suppose that Ω is a bounded open set of $\mathbb{R}^N$ $(N \geq 1)$, let p be a real number such that $1 < p < \infty$ and $w = \{w_i(x),\ 0 \leq i \leq N\}$ be a vector of weight functions, i.e., every component $w_i(x)$ is a measurable function which is positive a.e. in Ω. Further, we suppose for any $0 \leq i \leq N$ in all our considerations that

$$w_i \in L^1_{\mathrm{loc}}(\Omega) \quad \text{and} \quad w_i^{\frac{-1}{p-1}} \in L^1_{\mathrm{loc}}(\Omega). \tag{3}$$

The weighted Sobolev space, denoted by $W^{1,p}(\Omega, w)$, is defined as the space of all real-valued functions $u \in L^p(\Omega, w_0)$ such that the derivatives in the sense of distributions satisfy

$$\partial_i u \in L^p(\Omega, w_i), \quad i = 1, ..., N,$$

Note that the derivatives $\partial_i = \dfrac{\partial}{\partial x_i}$ are understood in the sense of distributions. This set of functions forms a Banach space under the norm

$$\|u\|_{1,p,w} = \Big(\int_\Omega |u(x)|^p w_0(x)\, dx + \sum_{i=1}^N \int_\Omega |\partial_i u(x)|^p\, w_i(x)\, dx \Big)^{1/p}. \tag{4}$$

The first condition in (3) implies that $C_0^\infty(\Omega)$ is a subspace of $W^{1,p}(\Omega, w)$ and consequently, we can introduce the subspace $W_0^{1,p}(\Omega, w)$ of $W^{1,p}(\Omega, w)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm (4). Furthermore, the second condition in (3) implies that $W^{1,p}(\Omega, w)$ as well as $W_0^{1,p}(\Omega, w)$ are reflexive Banach spaces.
We recall that the dual space of weighted Sobolev spaces $W_0^{1,p}(\Omega, w)$ is equivalent to $W^{-1,p'}(\Omega, w^*)$, where $w^* = \{w_i^* = w_i^{1-p'}, i = 0, \dots, N\}$ and where p' is the conjugate of p (i.e. $p' = \frac{p}{p-1}$, see [1] for more details).

Now we state the following assumption.
(A_0) The expression

$$|||u||| = \Big(\sum_{i=1}^{N}\int_\Omega |\partial_i u(x)|^p\, w_i(x)\, dx\Big)^{1/p}. \tag{5}$$

is a norm on $W_0^{1,p}(\Omega, w)$ equivalent to the norm (4).
Note that $(W_0^{1,p}(\Omega, w), |||\cdot|||)$ is a uniformly convex (and thus reflexive) Banach space.
We can find a weight function σ on Ω and a parameter q, $1 < q < \infty$, such that

$$\sigma^{1-q'} \in L^1(\Omega), \tag{6}$$

with $q' = \frac{q}{q-1}$ and such that the Hardy inequality,

$$\Big(\int_\Omega |u|^q \sigma\, dx\Big)^{1/q} \le c\Big(\sum_{i=1}^{N}\int_\Omega |\partial_i u|^p w_i\, dx\Big)^{1/p}, \tag{7}$$

holds for every $u \in W_0^{1,p}(\Omega, w)$ with a constant $c > 0$ independent of u, Moreover, the imbedding

$$W_0^{1,p}(\Omega, w) \hookrightarrow\hookrightarrow L^q(\Omega, \sigma), \tag{8}$$

determined by the inequality (7) is compact.

Remark 2.1. If we suppose that $w_0(x) \equiv 1$ and the integrability condition : There exists $\nu \in]\frac{N}{p}, +\infty[\cap]\frac{1}{p-1}, +\infty[$ such that

$$w_i^{-\nu} \in L^1(\Omega), \text{ for all } i = 1, \cdots, N \tag{9}$$

Note that the assumptions *(9)* is stronger than the second condition in *(3) then,*

$$|||u||| = \Big(\sum_{i=1}^{N}\int_\Omega |\partial_i u|^p\, w_i(x)\, dx\Big)^{1/p} \tag{10}$$

is a norm defined on $W_0^{1,p}(\Omega, w)$ and its equivalent to *(4)* and that, the imbedding

$$W_0^{1,p}(\Omega, w) \hookrightarrow L^q(\Omega), \tag{11}$$

is compact for all $1 \le q \le p_1^*$ if $p\nu < N(\nu + 1)$ and for all $q \ge 1$ if $p\nu \ge N(\nu + 1)$ where $p_1 = p\nu/\nu + 1$ *and* p_1^* *is the Sobolev conjugate of* p_1 *(see [11], pp.30-31).*

2.2. Some classes of operators and topological degree

Now, we give some results and properties from the theory of topological degree. We start by defining some classes of mappings.

In what follows, let X be a real separable reflexive Banach space and X^* be its dual space with dual pairing $\langle \cdot, \cdot \rangle$ and given a nonempty subset Ω of X.

Definition 2.2. *Let Y be another real Banach space. A operator $F : \Omega \subset X \to Y$ is said to be*

1. *bounded, if it takes any bounded set into a bounded set.*
2. *demicontinuous, if for any sequence $(u_n) \subset \Omega,\ u_n \to u$ implies $F(u_n) \rightharpoonup F(u)$.*
3. *compact, if it is continuous and the image of any bounded set is relatively compact.*

Definition 2.3. *A mapping $F : \Omega \subset X \to X^*$ is said to be*

1. *of type (S_+), if for any sequence $(u_n) \subset \Omega$ with $u_n \rightharpoonup u$ and $\limsup_{n\to\infty}\langle Fu_n, u_n - u\rangle \le 0$, we have $u_n \to u$.*
2. *quasimonotone, if for any sequence $(u_n) \subset \Omega$ with $u_n \rightharpoonup u$, we have $\limsup_{n\to\infty}\langle Fu_n,\ u_n - u\rangle \ge 0$.*

Definition 2.4. *Let $T : \Omega_1 \subset X \to X^*$ be a bounded operator such that $\Omega \subset \Omega_1$. For any operator $F : \Omega \subset X \to X$, we say that*

1. *F satisfies condition $(S_+)_T$, if for any sequence $(u_n) \subset \Omega$ with $u_n \rightharpoonup u,\ y_n := Tu_n \rightharpoonup y$ and $\limsup_{n\to\infty}\langle Fu_n,\ y_n - y\rangle \le 0$, we have $u_n \to u$.*
2. *F has the property $(QM)_T$, if for any sequence $(u_n) \subset \Omega$ with $u_n \rightharpoonup u,\ \ y_n := Tu_n \rightharpoonup y$, we have $\limsup_{n\to\infty}\langle Fu_n, y - y_n\rangle \ge 0$.*

In the sequel, let $\mathcal{O}$ be the collection of all bounded open set in X. For any $\Omega \subset X$, we consider the following classes of operators:

$$
\begin{aligned}
\mathcal{F}_1(\Omega) &:= \{F : \Omega \to X^* \backslash F \text{ is bounded, demicontinuous}\\
&\qquad\qquad\qquad\qquad \text{and satifies condition}(S_+)\},\\
\mathcal{F}_{T,B}(\Omega) &:= \{F : \Omega \to X \backslash F \text{ is bounded, demicontinuous}\\
&\qquad\qquad\qquad\qquad \text{and satifies condition}(S_+)_T\},\\
\mathcal{F}_T(\Omega) &:= \{F : \Omega \to X\backslash\ F \text{ is demicontinuous and satifies condition}(S_+)_T\},\\
\mathcal{F}_B(X) &:= \{F \in \mathcal{F}_{T,B}(\overline{E})\backslash\ E \in \mathcal{O},\ \mathrm{T} \in \mathcal{F}_1(\overline{\mathrm{E}})\}.
\end{aligned}
$$

Lemma 2.5. *$\Big($[4], Lemmas 2.2 and 2.4$\Big)$ Let $T \in \mathcal{F}_1(\overline{E})$ be continuous and $S : D_S \subset X^* \to X$ be demicontinuous such that $T(\overline{E}) \subset D_s$, where E is a bounded open set in a real reflexive Banach space X. Then the following statements are true :*

1. *If S is quasimonotone, then $I + SoT \in \mathcal{F}_T(\overline{E})$, where I denotes the identity operator.*
2. *If S is of class (S_+), then $SoT \in \mathcal{F}_T(\overline{E})$.*

Definition 2.6. *Suppose that E is bounded open subset of a real reflexive Banach space X, $T \in \mathcal{F}_1(\overline{E})$ be continuous and let $F, S \in \mathcal{F}_T(\overline{E})$. The affine homotopy $\Lambda : [0, 1] \times \overline{E} \to X$ defined by*

$$\Lambda(t, u) := (1 - t)Fu + tSu, \quad \text{for} \qquad (t, u) \in [0, 1] \times \overline{E}$$

is called an admissible affine homotopy with the common continuous essential inner map T.

Remark 2.7. *[4]* The above affine homotopy satisfies condition $(S_+)_T$.

Next, we give the Berkovits topological degree for the class $\mathcal{F}_B(X)$ for more details see [4].

Theorem 2.8. *Let*

$$M = \Big\{(F, E, h)\backslash E \in \mathcal{O},\ T \in \mathcal{F}_1(\overline{E}),\ F \in \mathcal{F}_{T,B}(\overline{E}),\ h \notin F(\partial E)\Big\}.$$

Then, there exists a unique degree function $d : M \longrightarrow \mathbb{Z}$ that satisfies the following properties:

1. (Normalization) *For any $h \in E$, we have $d(I, E, h) = 1$.*
2. (Additivity) *Let $F \in \mathcal{F}_{T,B}(\overline{E})$. If E_1 and E_2 are two disjoint open subsets of E such that $h \notin F(\overline{E} \backslash (E_1 \cup E_2))$, then we have*
$$d(F, E, h) = d(F, E_1, h) + d(F, E_2, h).$$
3. (Homotopy invariance) *If $\Lambda : [0,1] \times \overline{E} \to X$ is a bounded admissible affine homotopy with a common continuous essential inner map and h: $[0,1] \to X$ is a continuous path in X such that $h(t) \notin \Lambda(t, \partial E)$ for all $t \in [0,1]$, then the value of $d(\Lambda(t,\cdot), E, h(t))$ is constant for all $t \in [0,1]$.*
4. (Existence) *If $d(F, E, h) \neq 0$, then the equation $Fu = h$ has a solution in E.*

3. Hypotheses and technical Lemmas

In this section, we focus our attention on the basic hypotheses and the operators associated with our problem to prove the existence results.

Throughout this paper, we assume that the functions $\mathcal{A}$ and $\mathcal{B}$ satisfying the following assumptions, for a. e. in $x \in \Omega$ and all $\zeta, \zeta' \in \mathbb{R}^N$, $(\zeta \neq \zeta')$ and $s \in \mathbb{R}$.

(A_1) $\quad \mathcal{B}(x,s,\zeta).\zeta \geq \alpha_1 \sum_{i=1}^{N} w_i |\zeta_i|^p$ and $\mathcal{A}(x,\zeta).\zeta \geq \alpha_2 \sum_{i=1}^{N} w_i |\zeta_i|^p$.

(A_2) $\quad |\mathcal{A}_i(x,\zeta)| \leq \beta_2 w_i^{1/p}\Big(k_2(x) + \sum_{j=1}^{N} w_j^{1/p'} |\zeta_j|^{p-1}\Big)$ and

$$|\mathcal{B}_i(x,s,\zeta)| \leq \beta_1 w_i^{1/p}\Big(k_1(x) + \sigma^{1/p'}|s|^{p-1} + \sum_{j=1}^{N} w_j^{1/p'} |\zeta_j|^{p-1}\Big).$$

(A_3) $\quad (\mathcal{B}(x,s,\zeta) - \mathcal{B}(x,s,\zeta'))(\zeta - \zeta') > 0$ and $[\mathcal{A}(x,\zeta) - \mathcal{A}(x,\zeta')](\zeta - \zeta') > 0$.

where α_i, $\beta_i (i = 1,2)$ are some positive constants and k_1, k_2 are a positive functions in $L^{p'}(\Omega)$ (p' is the conjugate exponent of p).

Furthermore, the Carathéodory's functions $\mathcal{G}$ satisfies only the growth condition, for all $t \in \mathbb{R}^N$, $s \in \mathbb{R}$ and a.e. $x \in \Omega$

(H_1) $$\Big|\mathcal{G}(x,s,\eta)\Big| \leq \varrho\sigma^{1/q}(e(x) + \sigma^{1/q'}|s|^{q-1} + \sum_{j=1}^{N} w_j^{1/q'} |\eta_j|^{q-1}),$$

where ϱ is a positive constant, $e(x)$ is a positive function in $L^{q'}(\Omega)$.

Now, we give some properties of the related operators which will be used later.

Lemma 3.1. *[2] Let $g \in L^r(\Omega)$ and $g_n \subset L^r(\Omega)$ such that $\|g_n\|_r \leq C$, $1 < r < \infty$, If $g_n(x) \to g(x)$ a.e. in Ω then $g_n \rightharpoonup g$ weakly in $L^r(\Omega)$.*

Let us consider the following nonlinear operator $\mathcal{L}$ defined from $W_0^{1,p}(\Omega, w)$ into its dual by

$$\langle \mathcal{L}u, v\rangle = \sum_{i=1}^{N} \int_{\Omega} \Big(\mathcal{A}_i(x,\nabla u) + \mathcal{B}_i(x,u,\nabla u)\Big)\partial_i v \, dx,$$

for all $u, v \in W_0^{1,p}(\Omega, w)$.

Lemma 3.2. *Assume that the assumptions $(A_0) - (A_3)$ hold, then $\mathcal{L}$ is bounded, coercive, continuous and of type (S_+).*

Proof. Firstly, we show that the operator $\mathcal{L}$ is bounded.

Let $u, v \in W_0^{1,p}(\Omega, w)$. From the Hölder's inequality we obtain

$$
\begin{aligned}
&| < \mathcal{L}u, v > | \\
&\quad \leq \sum_{i=1}^{N} \int_{\Omega} |\mathcal{A}_i(x, \nabla u) + \mathcal{B}_i(x, u, \nabla u)| w_i^{-1/p} |\partial_i v| w_i^{1/p} dx \\
&\quad \leq \sum_{i=1}^{N} \int_{\Omega} |\mathcal{A}_i(x, \nabla u)| w_i^{-1/p} |\partial_i v| w_i^{1/p} dx \\
&\qquad + \sum_{i=1}^{N} \int_{\Omega} |\mathcal{B}_i(x, u, \nabla u)| w_i^{-1/p} |\partial_i v| w_i^{1/p} dx \\
&\quad \leq \sum_{i=1}^{N} \Big(\int_{\Omega} \Big|\mathcal{A}_i(x, \nabla u) w_i^{-1/p}\Big|^{p'} dx \Big)^{1/p'} \Big(\int_{\Omega} |\partial_i v|^p w_i dx \Big)^{1/p} \\
&\qquad + \sum_{i=1}^{N} \Big(\int_{\Omega} \Big|\mathcal{B}_i(x, u, \nabla u) w_i^{-1/p}\Big|^{p'} dx \Big)^{1/p'} \Big(\int_{\Omega} |\partial_i v|^p w_i dx \Big)^{1/p} \\
&\quad \leq \Big(\sum_{i=1}^{N} \int_{\Omega} \Big|\mathcal{A}_i(x, \nabla u) w_i^{-1/p}\Big|^{p'} dx \Big)^{1/p'} \Big(\sum_{i=1}^{N} \int_{\Omega} |\partial_i v|^p w_i dx \Big)^{1/p} \\
&\qquad + \Big(\sum_{i=1}^{N} \int_{\Omega} \Big|\mathcal{B}_i(x, u, \nabla u) w_i^{-1/p}\Big|^{p'} dx \Big)^{1/p'} \Big(\sum_{i=1}^{N} \int_{\Omega} |\partial_i v|^p w_i dx \Big)^{1/p} \\
&\quad = \Big(\sum_{i=1}^{N} \int_{\Omega} |\mathcal{A}_i(x, \nabla u)|^{p'} w_i^{1-p'} dx \Big)^{1/p'} \Big(\sum_{i=1}^{N} \int_{\Omega} |\partial_i v|^p w_i dx \Big)^{1/p} \\
&\qquad + \Big(\sum_{i=1}^{N} \int_{\Omega} |\mathcal{B}_i(x, u, \nabla u)|^{p'} w_i^{1-p'} dx \Big)^{1/p'} \Big(\sum_{i=1}^{N} \int_{\Omega} |\partial_i v|^p w_i dx \Big)^{1/p} \\
&\quad = \sum_{i=1}^{N} \Big[\|\mathcal{A}_i(x, \nabla u)\|_{L^{p'}(\Omega, w_i^{1-p'})} + \|\mathcal{B}_i(x, u, \nabla u)\|_{L^{p'}(\Omega, w_i^{1-p'})} \Big] |||v|||.
\end{aligned}
$$

On the other hand, by the growth condition (A_2) and the Hardy inequality (7), we can easily prove that $\|\mathcal{A}_i(x, \nabla u)\|_{L^{p'}(\Omega, w_i^{1-p'})}$ and $\|\mathcal{B}_i(x, u, \nabla u)\|_{L^{p'}(\Omega, w_i^{1-p'})}$ are bounded for all u in $W_0^{1,p}(\Omega, w)$. Thus $\langle \mathcal{L}u, v \rangle \leq const\, |||v|||$, which implies that $\mathcal{L}$ is bounded.

Secondly, we show that $\mathcal{L}$ is coercive. Let $v \in W_0^{1,p}(\Omega, w)$, we have from (A_1)

$$
\begin{aligned}
\frac{\langle \mathcal{L}v, v \rangle}{|||v|||} &= \frac{\displaystyle\int_{\Omega} \Big(\mathcal{A}(x, \nabla v) + \mathcal{B}(x, v, \nabla v) \Big) . \nabla v dx}{|||v|||} \\
&\geq (\alpha_1 + \alpha_2) \frac{\displaystyle\sum_{i=1}^{N} \int_{\Omega} |\nabla v|^p w_i dx}{|||v|||} \\
&\geq (\alpha_1 + \alpha_2) |||v|||^{p-1}.
\end{aligned}
$$

Hence,

$$
\frac{\langle \mathcal{L}v, v \rangle}{|||v|||} \to \infty \quad \text{as} \quad |||v||| \to \infty.
$$

Therefore $\mathcal{L}$ is coercive.

Next, we prove that $\mathcal{L}$ is continuous, let $u_n \to u$ in $W_0^{1,p}(\Omega, w)$. Thus $\nabla u_n \to \nabla u$ in $\prod_{i=1}^{N} L^p(\Omega, w_i)$. Then there exist a subsequence (u_k) of (u_n) and measurable functions R in $L^p(\Omega, w_0)$ and h in $\prod_{i=1}^{N} L^p(\Omega, w_i)$ such that

$$u_k(x) \to u(x) \quad \text{and} \quad \nabla u_k(x) \to \nabla u(x),$$

$$|u_k(x)| \leq R(x) \quad \text{and} \quad |\nabla u_k(x)| \leq h(x)$$

for $a.e.x \in \Omega$ and all $k \in \mathbb{N}$. Since $\mathcal{A}$ and $\mathcal{B}$ satisfies the Carathéodory condition, we obtain

$$\mathcal{A}(x, \nabla u_k) \to \mathcal{A}(x, \nabla u) \qquad \text{a.e. } x \in \Omega, \tag{12}$$

$$\mathcal{B}(x, u_k, \nabla u_k) \to \mathcal{B}(x, u, \nabla u) \qquad \text{a.e. } x \in \Omega. \tag{13}$$

According to (A_2), we have for all $i = 1, \cdots, N$

$$|\mathcal{A}_i(x, \nabla u_k)| \leq \beta_2\, w_i^{1/p'} \Big(k_2(x) + \sum_{j=1}^{N} w_j^{1/p'}\, |h_j|^{p-1} \Big),$$

$$|\mathcal{B}_i(x, u_k, \nabla u_k)| \leq \beta_1\, w_i^{1/p'} \Big(k_1(x) + \sigma^{1/p'}\, |R(x)|^{p/p'} + \sum_{j=1}^{N} w_j^{1/p'}\, |h_j|^{p-1} \Big),$$

for a.e. $x \in \Omega$ and for all $k \in \mathbb{N}$. As

$$\beta_2\, w_i^{1/p'} \Big(k_2(x) + \sum_{j=1}^{N} w_j^{1/p'}\, |h_j|^{p-1} \Big) \in \prod_{i=1}^{N} L^{p'}(\Omega, w_i^{1-p'}),$$

and

$$\beta_1\, w_i^{1/p'} \Big(k_1(x) + \sigma^{1/p'}\, |R(x)|^{p/p'} + \sum_{j=1}^{N} w_j^{1/p'}\, |h_j|^{p-1} \Big) \in \prod_{i=1}^{N} L^{p'}(\Omega, w_i^{1-p'}),$$

by using the dominated convergence theorem, (12) and (13), we obtain

$$\mathcal{A}(x, \nabla u_k) \to \mathcal{A}(x, \nabla u) \qquad \text{in} \quad \prod_{i=1}^{N} L^{p'}(\Omega, w_i^{1-p'}),$$

$$\mathcal{B}(x, u_k, \nabla u_k) \to \mathcal{B}(x, u, \nabla u) \qquad \text{in} \quad \prod_{i=1}^{N} L^{p'}(\Omega, w_i^{1-p'}).$$

And so on the entire sequence $\mathcal{A}(x, \nabla u_n)$ converges to $\mathcal{A}(x, \nabla u)$ in $\prod_{i=1}^{N} L^{p'}(\Omega, w_i^{1-p'})$ and $\mathcal{B}(x, u_n, \nabla u_n)$ converges to $\mathcal{B}(x, u, \nabla u)$ in $\prod_{i=1}^{N} L^{p'}(\Omega, w_i^{1-p'})$.

Therefore, for all $v \in W_0^{1,p}(\Omega, w)$ we have $\langle \mathcal{L} u_n, v \rangle \to \langle \mathcal{L} u, v \rangle$, which implies that the operator $\mathcal{L}$ is continuous.
It remains to show that the operator $\mathcal{L}$ is of type (S_+).

Let $(u_n)_n$ be a sequence in $W_0^{1,p}(\Omega, w)$ such that

$$\begin{cases} u_n \rightharpoonup u \qquad \text{in } W_0^{1,p}(\Omega, w) \\ \limsup_{n\to\infty} \langle \mathcal{L}u_n, u_n - u\rangle \leq 0. \end{cases} \tag{14}$$

We shall prove that $u_n \to u$ in $W_0^{1,p}(\Omega, w)$.

Since $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega, w)$, then $(u_n)_n$ is a bounded sequence in $W_0^{1,p}(\Omega, w)$,then there exist a subsequence still denoted by $(u_n)_n$ such that $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega, w)$,

$$u_n \to u \quad \text{in } L^p(\Omega, w_0) \quad \text{and a.e in } \Omega.$$

Under (A_3) and (14), we obtain

$$\lim_{n\to\infty} \langle \mathcal{L}u_n,\ u_n - u\rangle = \lim_{n\to\infty} \langle \mathcal{L}u_n - \mathcal{L}u,\ u_n - u\rangle = 0. \tag{15}$$

Let $D_n = \Big[\mathcal{B}(x, u_n, \nabla u_n) - \mathcal{B}(x, u_n, \nabla u)\Big].\Big(\nabla u_n - \nabla u\Big)$. Thus, from (15) $D_n \to 0$ in $L^1(\Omega)$ and for a subsequence $D_n \to 0$ a.e. in Ω.

Since $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega, w)$ and a.e in Ω, there exists a subset B of Ω, of zero measure, such that for $x \in \Omega\backslash B$, $|u(x)| < \infty$, $|\nabla u(x)| < \infty$, $k(x) < \infty$, $u_n(x) \to u(x)$, $D_n(x) \to 0$.

If we take $\zeta_n = \nabla u_n, \zeta = \nabla u$, we get

$$\begin{aligned} D_n(x) &= \Big[\mathcal{B}(x, u_n, \zeta_n) - \mathcal{B}(x, u_n, \zeta)\Big].\Big(\zeta_n - \zeta\Big) \\ &\geq \alpha_1 \sum_{i=1}^N w_i |\zeta_n^i|^p + \alpha_1 \sum_{i=1}^N w_i |\zeta^i|^p \\ &\quad - \sum_{i=1}^N \beta_1 w_i^{1/p} \Big(k_1(x) + \sigma^{1/p'} |u_n|^{p/p'} + \sum_{j=1}^N w_j^{1/p'} |\zeta_n^j|^{p-1}\Big)|\zeta^i| \\ &\quad - \sum_{i=1}^N \beta_1 w_i^{1/p} \Big(k_1(x) + \sigma^{1/p'} |u_n|^{p/p'} + \sum_{j=1}^N w_j^{1/p'} |\zeta^j|^{p-1}\Big)|\zeta_n^i| \\ &\geq \alpha_1 \sum_{i=1}^N w_i |\zeta_n^i|^p - C_x\Big[1 + \sum_{j=1}^N w_j^{1/p'} |\zeta_n^j|^{p-1} + \sum_{i=1}^N w_i^{1/p} |\zeta_n^i|\Big], \end{aligned}$$

where C_x is a constant which depends on x, but does not depend on n. Since $u_n(x) \to u(x)$ we have $|u_n(x)| \leq M_x$, where M_x is some positive constant. Then by a standard argument $|\zeta_n|$ is bounded uniformly with respect to n, we deduce that

$$D_n(x) \geq \sum_{i=1}^N |\zeta_n^i|^p \Big(\alpha_1 w_i - \frac{C_x}{N|\zeta_n^i|^p} - \frac{C_x w_i^{1/p'}}{|\zeta_n^i|} - \frac{C_x w_i^{1/p}}{|\zeta_n|^{p-1}}\Big).$$

If $|\zeta_n| \to \infty$ (for a subsequence), then $D_n(x) \to \infty$ which gives a contradiction. Let now ζ^* be a cluster point of ζ_n. We have $|\zeta^*| < \infty$ and by using the continuity of $\mathcal{L}$ we obtain

$$\Big(\mathcal{B}(x, u_n, \zeta^*) - \mathcal{B}(x, u_n, \zeta)\Big)\Big(\zeta^* - \zeta\Big) = 0.$$

Analogously, if we choose

$$\Lambda_n(x, t) = \Big(\mathcal{A}(x, \nabla u_n) - \mathcal{A}(x, \nabla u)\Big).\Big(\nabla u_n - \nabla u\Big),$$

and we take $\zeta_n = \nabla u_n$ and $\zeta = \nabla u$, then, by the same arguments used above, we obtain

$$\Big(\mathcal{A}(x,\zeta^*) - \mathcal{A}(x,\zeta)\Big)\Big(\zeta^* - \zeta\Big) = 0.$$

In the light of (A_3), we get $\zeta^* = \zeta$. The uniqueness of the cluster point implies

$$\nabla u_n(x) \longrightarrow \nabla u(x) \qquad \text{a.e. in } \Omega.$$

Since the sequence $\mathcal{A}(x,\nabla u_n)$ and $\mathcal{B}(x,u_n,\nabla u_n)$ are bounded in $\prod_{i=1}^{N} L^{p'}(\Omega, w_i^{1-p'})$, and

$$\mathcal{A}(x,\nabla u_n) \longrightarrow \mathcal{A}(x,\nabla u) \text{ a.e. in } \Omega,$$

$$\mathcal{B}(x,u_n,\nabla u_n) \longrightarrow \mathcal{B}(x,u,\nabla u) \text{ a.e. in } \Omega,$$

then, by Lemma 3.1 we have

$$\mathcal{A}(x,\nabla u_n) \rightharpoonup \mathcal{A}(x,\nabla u) \quad \text{in } \prod_{i=1}^{N} L^{p'}(\Omega, w_i^{1-p'}) \quad \text{a.e. in } \Omega,$$

$$\mathcal{B}(x,u_n,\nabla u_n) \rightharpoonup \mathcal{B}(x,u,\nabla u) \quad \text{in } \prod_{i=1}^{N} L^{p'}(\Omega, w_i^{1-p'}) \quad \text{a.e. in } \Omega.$$

If we pose

$$\overline{\rho}_n = \Big(\mathcal{A}(x,\nabla u_n) + \mathcal{B}(x,u_n,\nabla u_n)\Big)\cdot \nabla u_n,$$

$$\overline{\rho} = \Big(\mathcal{A}(x,\nabla u) + \mathcal{B}(x,u,\nabla u)\Big)\cdot \nabla u,$$

we can write

$$\overline{\rho}_n \to \overline{\rho} \quad \text{in } L^1(\Omega).$$

Thanks to (A_3), we obtain

$$\overline{\rho}_n \geq (\alpha_1+\alpha_2)\sum_{i=1}^{N} w_i|\partial_i u_n|^p \quad \text{and} \quad \overline{\rho} \geq (\alpha_1+\alpha_2)\sum_{i=1}^{N} w_i|\partial_i u|^p.$$

In view of $\tau_n = \sum_{i=1}^{N} w_i|\partial_i u_n|^p$, $\tau = \sum_{i=1}^{N} w_i|\partial_i u|^p$, $\rho_n = \dfrac{\overline{\rho}_n}{(\alpha_1+\alpha_2)}$ and $\rho = \dfrac{\overline{\rho}}{(\alpha_1+\alpha_2)}$, we have

$$\rho_n \geq \tau_n \quad \text{and} \quad \rho \geq \tau.$$

Then by Fatou's lemma, we get

$$\int_\Omega 2\,\rho\, dx \leq \liminf_{n\to\infty} \int_\Omega \rho + \rho_n - |\tau_n - \tau| dx,$$

i.e.,

$$0 \leq -\limsup_{n\to\infty} \int_\Omega |\tau_n - \tau|\, dx.$$

So

$$0 \leq \liminf_{n\to\infty} \int_\Omega |\tau_n - \tau|\, dx \leq \limsup_{n\to\infty} \int_\Omega |\tau_n - \tau|\, dx \leq 0,$$

consequently

$$\nabla u_n \longrightarrow \nabla u \quad \text{in} \quad \prod_{i=1}^{N} L^p(\Omega, w_i). \tag{16}$$

Therefore $u_n \longrightarrow u$ in $W_0^{1,p}(\Omega, w)$, which completes the proof. □

Lemma 3.3. *Under the condition* (H_1)*, the operator*

$$\mathcal{F} : W_0^{1,p}(\Omega, w) \to W^{-1,p'}(\Omega, w^{1-p'})$$

defined by

$$\langle \mathcal{F}\, u,\ v\rangle = -\int_\Omega \Big(a(x)|u|^{q-2}u + \lambda \mathcal{G}(x, u, \nabla u)\Big) v\, dx, \quad \forall u,\ v \in W_0^{1,p}(\Omega, w)$$

is compact.

Proof. We split the proof into three steps.

First step

Let us define an operator $C : W_0^{1,p}(\Omega, w) \to L^{q'}(\Omega, \sigma^*)$ by

$$C\, u(x) := -a(x)|u(x)|^{q-2}u(x) \quad \text{for} \quad u \in W_0^{1,p}(\Omega, w) \quad \text{and} \quad x \in \Omega.$$

It is obvious that C is continuous. Next we show that C is bounded.

Let $u \in W_0^{1,p}(\Omega, w)$, from the Hardy inequality and the condition (2) we obtain

$$\begin{aligned}
\|C\, u\|_{q',\sigma^*}^{q'} &= \int_\Omega | - a(x)|u|^{q-2}u|^{q'} \sigma^* dx \leq \gamma^{q'} \int_\Omega |\sigma|u|^{q-2}u|^{q'} \sigma^{1-q'} dx \\
&\leq \gamma^{q'} \int_\Omega |u|^{(q-1)q'} \sigma dx = \gamma^{q'} \int_\Omega |u|^q \sigma dx \\
&\leq C' \sum_{i=1}^{N} \int_\Omega |\partial_i u|^p\, w_i\, dx = C' |||u|||^p.
\end{aligned}$$

This implies that C is bounded on $W_0^{1,p}(\Omega, w)$.

Second step

We prove that the operator $\mathcal{T}$ defined from $W_0^{1,p}(\Omega, w)$ into $L^{q'}(\Omega, \sigma^*)$ by

$$\mathcal{T}\, u(x) := -\lambda \mathcal{G}(x, u, \nabla u) \quad \text{for} \quad u \in W_0^{1,p}(\Omega, w)) \quad \text{and} \quad x \in \Omega$$

is bounded and continuous. Let $u \in W_0^{1,p}(\Omega, w)$, from the growth condition (H_1) we get

$$\begin{aligned}
\|\mathcal{T}u\|_{q',\sigma^*}^{q'} &\leq \int_\Omega |\lambda \mathcal{G}(x,u,\nabla u)|^{q'} \sigma^* dx \\
&\leq (\varrho\lambda)^{q'} \int_\Omega \sigma^{q'/q}\Big(e(x) + |u|^q\sigma + \sum_{i=1}^N w_i|\partial_i u|^p\Big)\sigma^{1-q'} dx \\
&\leq (\varrho\lambda)^{q'} \int_\Omega e(x)^{q'} dx + (\varrho\lambda)^{q'}\Big(\int_\Omega |u|^q \sigma dx + \sum_{i=1}^N \int_\Omega |\partial_i u|^p w_i\, dx\Big) \\
&\leq (\varrho\lambda)^{q'} \int_\Omega e(x)^{q'} dx + C_H \sum_{i=1}^N \int_\Omega |\partial_i u|^p w_i dx + (\varrho\lambda)^{q'} \sum_{i=1}^N \int_\Omega |\partial_i u|^p w_i dx \\
&\leq (\varrho\lambda)^{q'} \|e\|_{q'}^{q'} + C_H |||u|||^p + (\varrho\lambda)^{q'} |||u|||^p \\
&\leq C_m(1 + |||u|||^p),
\end{aligned}$$

where $C_m = \max\Big((\varrho\lambda)^{q'}\|e\|_{q'}^{q'}, (\varrho\lambda)^{q'} + C_H\Big)$. Thus $\mathcal{T}$ is bounded on $W_0^{1,p}(\Omega, w)$.

It remains to show that $\mathcal{T}$ is continuous, let $u_n \to u$ in $W_0^{1,p}(\Omega, w)$, then $u_n \to u$ in $L^p(\Omega, w_0)$ and $\nabla u_n \to \nabla u$ in $\prod_{i=1}^N L^p(\Omega, w_i)$. Therefore there exist a subsequence denoted again by (u_n) and measurable functions l in $L^p(\Omega, w_0)$ and m in $\prod_{i=1}^N L^p(\Omega, w_i)$ such that

$$\begin{aligned}
u_n(x) \to u(x) \quad &\text{and} \quad \nabla u_n(x) \to \nabla u(x), \\
|u_n(x)| \leq l(x) \quad &\text{and} \quad |\nabla u_n(x)| \leq |m(x)|
\end{aligned}$$

for a.e. $x \in \Omega$ and all $n \in \mathbb{N}$. As g satisfies the Carathéodory condition, we get

$$\mathcal{G}(x, u_n(x), \nabla u_n(x)) \to \mathcal{G}(x, u(x), \nabla u(x)) \quad \text{a.e. } x \in \Omega. \tag{17}$$

In view of (H_1) we have

$$|\mathcal{G}(x, u_n(x), \nabla u_n(x))| \leq \varrho\sigma^{1/q}(e(x) + \sigma^{1/q'}|l(x)|^{q/q'} + \sum_{i=1}^N w_i^{1/q'}|m(x)|^{p/q'})$$

for a.e. $x \in \Omega$ and for all $k \in \mathbb{N}$.

Since

$$\varrho\sigma^{1/q}(e(x) + \sigma^{1/q'}|l(x)|^{q/q'} + \sum_{i=1}^N w_i^{1/q'}|m(x)|^{p/q'}) \in L^{q'}(\Omega, \sigma^*),$$

thanks to (17) we obtain

$$\int_\Omega |\mathcal{G}(x, u_k(x), \nabla u_k(x)) - \mathcal{G}(x, u(x), \nabla u(x))|^{p'} \sigma^* dx \longrightarrow 0.$$

The dominated convergence theorem implies that

$$\mathcal{T}u_k \to \mathcal{T}u \quad \text{in} \quad L^{q'}(\Omega, \sigma^*).$$

from where the entire sequence $(\mathcal{T}u_n)$ converges to $\mathcal{T}u$ in $L^{q'}(\Omega, \sigma^*)$ and then $\mathcal{T}$ is continuous.

Third step

Due to the compact embedding $I : W_0^{1,p}(\Omega, w) \hookrightarrow L^{q'}(\Omega, \sigma^*)$, it is known that the adjoint operator $I^* : L^{q'}(\Omega, \sigma^*) \hookrightarrow W^{-1,p'}(\Omega, w^{1-p'})$ is also compact. Hence, the compositions $I^* o C$ and $I^* o \mathcal{T}$ from $W_0^{1,p}(\Omega, w)$ into $W^{-1,p'}(\Omega, w^{1-p'})$ are obviously compact. We conclude that $S = I^* o C + I^* o \mathcal{T}$ is compact. Which completes the present proof. □

4. Main results

This section is devoted to transform this strongly nonlinear Dirichlet boundary value problem (1) into a new one governed by a Hammerstein equation. Then, we establish the existence of weak solutions for our problem by using the Berkovits topological degree theory introduced in section 2.

Let us recall that a weak solution of problem (1) is any $u \in W_0^{1,p}(\Omega, w)$ such that

$$\int_\Omega \Big(\mathcal{B}(x,u,\nabla u) + \mathcal{A}(x,\nabla u)\Big).\nabla v dx - \lambda \int_\Omega \mathcal{G}(x,u,\nabla u) v dx = \int_\Omega a(x)|u|^{q-2} u v dx,$$

for all $v \in W_0^{1,p}(\Omega, w)$.

Our main result is the following.

Theorem 4.1. *Assume that* $(A_0) - (A_3)$ *and* (H_1) *are satisfied. Then, the problem* (1) *admits at least one weak solution* u *in* $W_0^{1,p}(\Omega, w)$.

Proof. Let $\mathcal{L}$, $\mathcal{F}$ be two operators from $W_0^{1,p}(\Omega, w)$ into its dual $W^{-1,p'}(\Omega, w^{1-p'})$ as defined in Lemma 3.2 and Lemma 3.3 respectively. Then $u \in W_0^{1,p}(\Omega, w)$ is a weak solutions of the problem (1) if and only if

$$\mathcal{L}u = -\mathcal{F}u. \tag{18}$$

Note that, by Lemma 3.3 the operator $\mathcal{F}$ is bounded, quasimonotone and continuous. In view of the properties of the operator $\mathcal{L}$ given in Lemma 3.2 and as the operator $\mathcal{L}$ is strictly monotone. By using the Minty-Browder Theorem (see [35], Theorem 26 A), the inverse operator $G := \mathcal{L}^{-1} : W^{-1,p'}(\Omega, w^{1-p'}) \to W_0^{1,p}(\Omega, w)$ is bounded, continuous and satisfies condition (S_+).

Therefore, equation (18) is equivalent to the abstract Hammerstein equation

$$u = Gv \quad \text{and} \quad v + \mathcal{F}oGv = 0. \tag{19}$$

To solve the equations (19), we shall employ the Berkovits topological degree presented in section 2. For this, we first prove that the set

$$B := \Big\{ v \in W^{-1,p'}(\Omega, w^{1-p'}) \setminus \quad v + t\mathcal{F}oGv = 0 \quad \text{for some} \quad t \in [0,1] \Big\}$$

is bounded in $W^{-1,p'}(\Omega, w^{1-p'})$. Indeed, let $v \in B$ and take $u := Gv$.

Combining (2), (A_1), (H_1) and by using the Hardy inequality and the Young's inequality we obtain

$$\begin{aligned}
|||Gv|||^p &= \sum_{i=1}^{N}\int_\Omega |\nabla u|^p\, w_i\, dx \leq \frac{1}{\alpha_1+\alpha_2}\langle \mathcal{L}u,u\rangle = \frac{1}{\alpha_1+\alpha_2}\langle v,Gv\rangle \\
&\leq \frac{t}{\alpha_1+\alpha_2}|\langle \mathcal{F}\circ Gv, Gv\rangle| \\
&\leq \frac{t}{\alpha_1+\alpha_2}\int_\Omega |a(x)||u|^q dx + \frac{t}{\alpha_1+\alpha_2}\int_\Omega |\mathcal{G}(x,u,\nabla u)|\, u\, dx \\
&\leq \frac{\gamma t}{\alpha_1+\alpha_2}\int_\Omega |u|^q\sigma dx + \frac{t}{\alpha_1+\alpha_2}\int_\Omega \varrho\sigma^{1/q}(e(x)+\sigma^{1/q'}|u|^{q/q'} + \sum_{i=1}^{N} w_i^{1/q'}|\nabla u|^{p/q'})udx \\
&\leq \frac{\gamma t}{\alpha_1+\alpha_2}\int_\Omega |u|^q\sigma dx + C_{q'}\int_\Omega (|e(x)|^{q'} + |u|^q\sigma + \sum_{i=1}^{N}|\nabla u|^p w_i)dx + C_q\int_\Omega |u|^q\sigma dx \\
&\leq C_1\sum_{i=1}^{N}\int_\Omega |\partial_i u|^p w_i dx + C_{q'}\int_\Omega |e(x)|^{q'}dx + C_2\sum_{i=1}^{N}\int_\Omega |\partial_i u|^p w_i dx \\
&\quad + C_3\sum_{i=1}^{N}\int_\Omega |\nabla u|^p w_i dx + C_4\sum_{i=1}^{N}\int_\Omega |\partial_i u|^p w_i dx \\
&\leq \max(C_1,C_2,C_3,C_4)|||Gv|||^p + C_{q'}\|e\|_{q'}^{q'} \\
&\leq (Cst+1)|||Gv|||^p,
\end{aligned}$$

where $Cst = \max\left(C_{q'}\|e\|_{q'}^{q'}, \max(C_1,C_2,C_3,C_4)\right)$. For this reason $\left\{Gv \setminus v\in B\right\}$ is bounded.

Since the operator $\mathcal{F}$ is bounded, it is obvious from (19) that the set B is bounded in $W^{-1,p'}(\Omega, w^{1-p'})$. Consequently, there exists a positive constant R such that

$$\|v\|_{W^{-1,p'}(\Omega,w^{1-p'})} < R \quad \text{for all} \quad v\in B.$$

It follows that

$$v + t\mathcal{F}oGv \neq 0 \quad \text{for all} \quad v\in\partial B_R(0) \quad \text{and all} \quad t\in[0,1].$$

Thanks to Lemma 2.5, we have

$$I+\mathcal{F}oG \in \mathcal{F}_T(\overline{B_R(0)}) \quad \text{and} \quad I = \mathcal{F}oG \in \mathcal{F}_T(\overline{B_R(0)}).$$

Since the operators I, $\mathcal{F}$ and G are bounded, then $I+\mathcal{F}oG$ is also bounded. This says that

$$I+\mathcal{F}oG \in \mathcal{F}_{T,B}(\overline{B_R(0)}) \quad \text{and} \quad I \in \mathcal{F}_{T,B}(\overline{B_R(0)}).$$

Let us consider an affine homotopy $\Lambda : [0,1]\times\overline{B_R(0)} \to W^{-1,p'}(\Omega)$ given by

$$\Lambda(t,v) := v + t\mathcal{F}oGv \quad \text{for} \quad (t,v)\in[0,1]\times\overline{B_R(0)}.$$

Applying the homotopy invariance and normalization property of the degree d stated in Theorem 2.8, we obtain

$$d(I+\mathcal{F}oG, B_R(0), 0) = d(I, B_R(0), 0) = 1 \neq 0,$$

consequently, there exists a point $v\in B_R(0)$ such that $v+\mathcal{F}oGv = 0$, what implies that $u = Gv$ is a weak solution of (1). This completes the proof. □

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